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The Representation of Small-Scale Turbulence in Numerical Simulation Experiments

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1. Introduction and History

Simulation is the name frequently applied to direct numerical time integration of hydrodynamic equations as initial value-boundary value problems. When the pertinent system of equations is highly nonlinear such numerical integrations have become an important link between experiment and observation and analytic theory, sharing certain properties of each. There is extensive literature but relatively few survey articles or books. This is perhaps because most workers in this field felt that the tools and techniques were subsidiary to the physical problem under investigation. With some important exceptions (for example, the work of Fromm and Harlow on vortex streets[1]), numerical simulation has been most frequently and successfully applied in the areas of large scale meteorology and high speed aerodynamics.* The problems in these two fields of hydrodynamics, although vastly different in most respects, share the properties that they are typically two-dimensional, or nearly so, and that turbulence is either unimportant or that it can presumably be treated by fairly crude approximations. These two properties greatly simplify the numerical simulation problem, but they eliminate from consideration most other fluid dynamics problem areas, except that of low Reynolds number flow, for which linear analytic solutions are often obtainable.

* For large scale meteorology, see recent articles by Smagorinsky, *et al.*,[2] Leith,[3] Mintz,[4] and not too recent surveys on numerical forecasting by Phillips[5] and Thompson.[6] Much of the literature on high-speed flow is in unpublished and often classified reports, but see *e. g.* recent articles by Burstein,[7] Haviland,[8] Gentry, *et al.*,[9] and Harlow[10] and an excellent survey of Russian work (in English) by Belotserkovskii and Chushkin.[11]

The correlation between two-dimensionality and the relative unimportance of turbulence is, of course, no accident. In two-dimensional flow, the squared vorticity is a property which is conserved except for viscous dissipation and external torques. Fjørtoft has shown that the non-linear energy "cascade" will then flow principally upward instead of downward in scale.[12] This process will eventually lead to a separation of the energy spectrum from the dissipation scales—that is, all the energy will be in large scales which are scarcely affected by viscosity. In three-dimensional flows, on the other hand, the process of vortex stretching can and does create small-scale circulations and the dissipation of squared vorticity in a stationary flow is therefore much greater than its creation by large-scale torques. It should not be inferred that small-scale motions are always unimportant in such quasi-two-dimensional flows as the atmospheric general circulation. There may be sources of kinetic energy specifically selective to the small scales, whose motions may interact with those of larger scales in various ways. In the fully three-dimensional flows, however, the interaction from the energy producing scales to those of molecular dissipation occurs in a direct and continuous process and there is no convenient scale which one may choose to separate motions of qualitatively different kinds.

If we are to perform simulation experiments on turbulent flows, the apparent conclusion of the above considerations is that we must integrate over a three-dimensional region with sufficient resolution to include both the energy producing and dissipation scales. Corrsin[13] pointed out the futility of such a program, showing that it would require some 10^{12} mesh points to simulate even a fairly low Reynolds number laboratory experiment. Although this number is less preposterous now than it was 5 years ago, it still is well beyond present or early future capabilities. Our interests in geophysical fluid dynamics extend to Reynolds numbers 10^4 greater than those of Corrsin's example, and since the computing requirements go up about as the cube of this parameter, we may dismiss such complete simulation as hopeless. Instead we must examine the actual mechanics of turbulent exchange more critically to see whether it is possible: *1)* to simulate some of the important effects of the third dimension without doing all the computing necessary to completely include it; and *2)* to simulate the profound but indirect effects that molecular viscosity and diffusion exert on a real fluid without computing the motions on all of the interacting scales. In the past, computers have been capable of handling two-dimensional problems with up to 50 or 75 mesh points in each direction, or three-dimensional problems with less than 20. Since a number of investigators, including the author, have performed calculations aimed at simulating the behavior of turbulent fluids by use of a

two-dimensional computing mesh, it is interesting to consider how they encountered these problems.

The problem of buoyant convection is a central one in the atmospheres of the Earth, Sun, and some other planets and the ocean, and it is not surprising that it has for some time been a prime subject for numerical simulation. Most of the early attempts (Blair *et al.*,[14] Malkus and Witt,[15] Ogura,[16] Lilly[17]) suffered from difficulties associated with numerical differencing techniques, but with the benefit of hindsight we can now look past some of those difficulties to see some of the more significant features. Lilly and Ogura considered the motions produced after release of a warm buoyant semi-cylinder (Lilly) or hemisphere (Ogura) of air at the bottom of a bounded fluid. Figure 1 shows this initial condition. In Lilly's calculations, a nonlinear eddy viscosity proportional to local shear amplitude was used to simulate transfer of energy to scales smaller than the resolving power of the mesh, while Ogura used a constant viscosity. There were numerous other differences in the calculations, but in both cases the results failed to show any structures that could be identified with the irregularity and high amplitude turbulence characteristic of the real world and of laboratory experiments with initial conditions similar to those used in the calculations. In Lilly's calculations, the dissipative damping was too great initially and later became too small. It is not difficult to see how this occurred. In two-dimensional flow the average energy dissipation, $\bar{\epsilon}$ is given by

$$\bar{\epsilon} = \nu\overline{q^2} = \nu\bar{E}/L^2 \tag{1.1}$$

where ν is the kinematic viscosity and $\overline{q^2}$ is the mean squared vorticity. The second equality may be considered a definition for the length scale, L, where $\bar{E}$ is the mean kinetic energy, but the principal point is that L must be of the order of the energy containing scale because of the impossibility of generation of vorticity except by the large-scale torques. In turbulent flow, on the other hand, the well-known relationship between dissipation and gross properties is

$$\bar{\epsilon} = \bar{E}^{3/2}/L \tag{1.2}$$

where L is also nearly equal to the energy containing scale. The ratio between expressions (1.1) and (1.2) is proportional to

$$\frac{\nu}{\bar{E}^{1/2}L} = \frac{1}{R} \tag{1.3}$$

where R is the Reynolds number. In the experiments corresponding to Lilly's calculation, $\bar{E}$ decreased with time after the early organizational

phase but L increased continually, as did the product $\bar{E}^{1/2} L$. Lilly used a variable eddy viscosity coefficient with properties to be described in Section 3 of this paper, with a magnitude given roughly by

$$\nu \propto E^{1/2} h \tag{1.4}$$

where h was the calculation mesh spacing. The ratio (1.3) therefore decreased with time for this case.

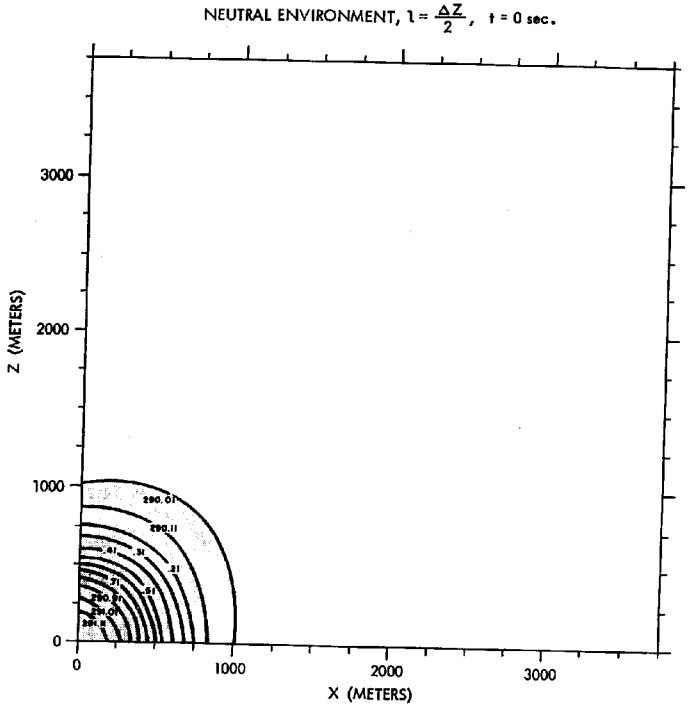

FIGURE 1. Initial conditions for numerical simulation of a two dimensional thermal convection problem.[17] (The isolines are of potential temperature in degrees K. Both lateral boundaries are reflective, so that the left boundary actually represents the middle of a thermal "bubble.")

Ogura's calculations were done assuming a constant eddy viscosity. In the axially symmetric experiments simulated by Ogura, the product $E^{1/2} L$ remains constant so that, from (1.3), R remains constant. A suitably chosen value of ν could then lead to a reasonably successful simulation, as shown by the good agreement of Ogura's results with some laboratory measurements. In a later set of calculations, Lilly[18] temporarily abandoned the variable viscosity in favor of an empirical mixing length approach, in which a spatially constant eddy viscosity was assumed in such a way that the computational Reynolds number (1.3) remained constant in time. In this way a solution was obtained with gross properties comparable to laboratory experiments, although different in various more or less important details. The results were comparable to the applications of mixing length theory to the better known examples of turbulent flow-jets, wakes, mixing zones, *etc.* This is now perhaps generally considered to be a relatively sterile approach and unsatisfactory for adequate simulation, much less explanation, of turbulent mechanics.

Similar partially unsatisfactory results were obtained by Deardorff and Willis[19] and Fromm[20] in two-dimensional numerical simulations of parallel plate convection. The calculations led to steady state or periodically oscillating solutions at Rayleigh numbers for which real three-dimensional flows are always turbulent. The computed heat transfer was considerably larger than observed, due to too high a correlation between vertical motion and temperature. Deardorff later attempted to overcome this difficulty by an interesting artifice.[21] His computational mesh was a sandwich of three planes, one on each side of the central one on which the principal integrations were performed. Variables on the outsides of the sandwich were chosen in such a way that various statistical properties of the flow were axially symmetric. These were then allowed to act on the central plane variables as if they were all parts of the true three-dimensional mesh. Details of the method and results are presented in the original paper but, in general, the method appeared to produce a partially realistic simulation of turbulence and definitely improved the heat transfer statistics. More recently, Elder[22] has reported simulations of a similar flow in which he simply added random turbulence energy to two-dimensional computations and obtained similar improvements in the results. Since there are many interesting problems (including those discussed above) in which the three-dimensionality is only a property of the transient turbulence but not essential to description of the mean flow, a suitable justification or generalization of such quasi-three-dimensional models would represent a tremendous breakthrough.

Much of modern turbulence theory is based on the notion of an inertial range of the turbulent energy spectrum. This is a scale range supposedly occurring only in high Reynolds number flows in which turbulent energy flows down scale, neither being produced from mean flow or potential energy sources nor being dissipated by molecular viscosity in significant amounts. If the turbulence is also isotropic, as is assumed, then a simple dimensional analysis, first performed by Kolmogoroff,[23] indicates that the scalar energy spectrum function $E(\kappa)$ of the scalar wave number κ should be given by

$$E(\kappa) = \alpha\epsilon^{2/3}\kappa^{-5/3} \tag{1.5}$$

where α is a universal dimensionless constant. The dissipation rate, ϵ, is also the energy transfer rate across a given wave number. In recent years, measurements of geophysical (Grant *et al.*,[24] Pond *et al.*[25]) and laboratory-produced (Gibson[26]) turbulent flows of high Reynolds number have verified this prediction reasonably well, sometimes even when the turbulence was not fully isotropic. Measured values of α are near 1.5, and a similar value has been predicted theoretically by Kraichnan.[27]

We now suggest that the existence and relatively simple properties of the inertial range might be used to greatly truncate the otherwise impossibly large requirements of computer resolution. Let us assume that the simulation equations are integrated for variables defined and resolvable in a scale range which includes most of the kinetic energy, and that the scale of the calculation mesh lies within the inertial range. It should then be possible to fit the explicitly calculated motion fields to the inertial range in a smooth and consistent manner. The fitting conditions would require a continuous removal of energy from the small scale explicit motions such that (1.4) is maintained. It might be possible to devise a procedure in which a Fourier analysis is made of the motion field and the amplitude coefficients reduced and adjusted so that (1.5) is forced to prevail in the high-frequency components. Such a procedure seems rather lacking in physical content, however, and also would involve arbitrary decisions on the actual range of wave numbers to be adjusted. More importantly, it would in effect consider the entire computational region to be homogeneous since all parts of the energy spectrum receive contributions from all physical space. If the motion field actually consisted of one or more turbulent areas embedded in quiescent flow, as for example in the case of isolated cumulus clouds, then the arbitrary Fourier mode adjustment would create Gibbs phenomenon oscillations that would effectively transfer energy spatially in a very nonphysical manner.

A more physically acceptable procedure is suggested by consideration of the local interactions between the explicit scale motions and those of the submesh length scales. The latter cannot be known in detail, but certain statistical probabilities can be established with the aid of the Kolmogoroff spectrum function. In the following, I will describe a first and second order theory for the interactions. Most of the detailed derivations and analysis are available in an unpublished report.[28] For simplicity, the results are presented here for the case of an incompressible constant density fluid.

2. MEAN FLOW AND TURBULENCE EQUATIONS

If F is any hydrodynamic variable, a function of space and time, then we define the spatial mesh cube average to be

$$\bar{F}(x_1, x_2, x_3, t) = \frac{1}{h^3}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} F(x_1 + y_1, x_2 + y_2, x_3 + y_3, t)\, dy_1\, dy_2\, dy_3 \tag{2.1}$$

where x_1, x_2, x_3, and t are the spatial Cartesian coordinates and time, and h is the mesh separation distance. The Navier–Stokes and continuity equations for an incompressible fluid case may be written in tensor notation

$$\frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k} + \frac{\partial}{\partial x_i}\left(\frac{p}{\rho}\right) - \nu\frac{\partial^2 u_i}{\partial x_k^2} = 0 \tag{2.2}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2.3}$$

Upon forming the mesh box average of these, we obtain

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_k\frac{\partial \bar{u}_i}{\partial x_k} + \frac{\partial}{\partial x_i}\left(\frac{\bar{p}}{\rho} + \tfrac{2}{3}E\right) - \nu\frac{\partial^2 \bar{u}_i}{\partial x_k^2} = \frac{\partial \tau_{ik}}{\partial x_k} \tag{2.4}$$

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \tag{2.5}$$

where τ_{ik} is a Reynolds stress given by

$$\tau_{ik} = -(\overline{u_i u_k} - \bar{u}_i\bar{u}_k) + \tfrac{2}{3}\delta_{ik}E \tag{2.6}$$

E is the small scale kinetic energy,

$$E = (\overline{u_s^2} - \bar{u}_s^2)/2 \tag{2.7}$$

and δ_{ik} is the Kronecker delta.

The separation of a gradient part from the stress tensor derivative is done so that the sum $\tau_{ii} = 0$; thus normal stresses vanish in isotropic turbulence. Equations (2.4) and (2.5) form the system that must be

integrated in a simulation problem with limited spatial resolution, and the central problem is to evaluate the Reynolds stresses in terms of the averaged quantitites. This problem may be attacked directly by forming the conservation equations for the stress components, which are

$$\frac{\partial \tau_{ij}}{\partial t} + \bar{u}_k \frac{\partial \tau_{ij}}{\partial x_k} - \nu\left\{\frac{\partial^2 \tau_{ij}}{\partial x_k^2} + 2\overline{\left(\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}\right)} - \frac{\partial \bar{u}_i}{\partial x_k}\frac{\partial \bar{u}_j}{\partial x_k}\right.$$
$$\left. - \tfrac{2}{3}\delta_{ij}\left[\overline{\left(\frac{\partial u_s}{\partial x_k}\right)^2} - \left(\frac{\partial \bar{u}_s}{\partial x_k}\right)^2\right]\right\} = \tfrac{2}{3}E\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \left[\overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}\right.$$
$$\left. - \frac{\bar{p}}{\rho}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\right] - \left[\tau_{ik}\frac{\partial \bar{u}_j}{\partial x_k} + \tau_{jk}\frac{\partial \bar{u}_i}{\partial x_k} - \tfrac{1}{3}\delta_{ij}\tau_{kk}\left(\frac{\partial \bar{u}_k}{\partial x_s} + \frac{\partial \bar{u}_s}{\partial x_k}\right)\right]$$
$$+ \frac{\partial}{\partial x_k}\left[\overline{u_i u_j u_k} - \bar{u}_i\overline{u_j u_k} - \bar{u}_j\overline{u_i u_k} - \bar{u}_k\overline{u_i u_j} + 2\bar{u}_i\bar{u}_j\bar{u}_k\right.$$
$$- \frac{\delta_{ij}}{3}(\overline{u_s^2 u_k} - 2\bar{u}_s\overline{u_s u_k} - \bar{u}_k\overline{u_s^2} + 2\bar{u}_s^2\bar{u}_k) + \delta_{ik}\left(\overline{u_j\frac{p}{\rho}} - \bar{u}_j\frac{\bar{p}}{\rho}\right)$$
$$\left. + \delta_{jk}\left(\overline{u_i\frac{p}{\rho}} - \bar{u}_i\frac{\bar{p}}{\rho}\right) - \tfrac{2}{3}\delta_{ij}\left(\overline{u_k\frac{p}{\rho}} - \bar{u}_k\frac{\bar{p}}{\rho}\right)\right] \tag{2.8}$$

Since $\tau_{ij} = \tau_{ji}$, (2.8) represents only six separate equations, and since $\tau_{ii} = 0$, only five of these are independent. A sixth independent quadratic equation is that for the small-scale energy E,

$$\frac{\partial E}{\partial t} + \bar{u}_k\frac{\partial E}{\partial x_k} - \nu\left[\frac{\partial^2 E}{\partial x_k^2} - \overline{\left(\frac{\partial u_i}{\partial x_k}\right)^2} + \left(\frac{\partial \bar{u}_i}{\partial x_k}\right)^2\right] = \tfrac{1}{2}\tau_{ij}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$
$$- \frac{\partial}{\partial x_k}\left(\overline{\frac{u_k u_i^2}{2}} - \frac{\bar{u}_k\overline{u_i^2}}{2} - \bar{u}_i\overline{u_i u_k} + \bar{u}_i^2\bar{u}_k + \overline{\frac{u_k p}{\rho}} - \frac{\bar{u}_k\bar{p}}{\rho}\right) \tag{2.9}$$

As is well known in turbulence theory, the quadratic equations do not succeed in closing the system, since they introduce the triple products $u_i u_j u_k$, *etc.* In addition, the pressure velocity products are difficult to evaluate. Pressure is not considered an independent variable, since it may be obtained from (2.2) and (2.3) by solution of the Poisson equation

$$\frac{\partial^2(p/\rho)}{\partial x_k^2} = -\frac{\partial^2(u_i u_j)}{\partial x_i\,\partial x_j} \tag{2.10}$$

This shows that the pressure field derivatives are of the order of, and determined by, the nonlinear terms, but are not local since the solution of a Poisson equation at a given point depends on the forcing function throughout all physical space. The usefulness of (2.8)–(2.9) depends on our obtaining useful approximations to the triple product and pressure velocity product terms.

3. FIRST ORDER THEORY

About the simplest reasonable closure assumption is the eddy viscosity hypothesis. The Reynolds stresses are assumed proportional to the rates of strain, or deformation, of the large-scale flow, *i.e.*,

$$\tau_{ij} = K\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) = KD_{ij} \tag{3.1}$$

where K is the eddy viscosity, assumed to be always positive and a function of the averaged flow variables, and D_{ij} is a component of the mean flow deformation tensor. There seems to be no definite experimental data available in the literature which would establish the accuracy of this hypothesis in its present form. Simple energy arguments suffice to show that the stress and mean deformation tensors must be positively correlated if an inertial subrange is maintained, and (2.8) shows that the mean flow deformations tend to generate parallel stresses from the first term on the right, but the correlation may be weak.

Smagorinsky[29] suggested a form for the eddy viscosity coefficient:

$$K = (kh)^2 D/\sqrt{2} \tag{3.2}$$

where h is the mesh separation distance, D the deformation tensor amplitude ($D = |D_{ij}|$), and k is a universal constant, somewhat analogous to the Karman constant of boundary layer theory. It can be shown that this formula is consistent with the existence of the Kolmogoroff spectrum function (1.4) for scales near h, provided that

$$k \approx .23\alpha^{-3/4} \tag{3.3}$$

The essence of the proof is simple. In the case of steady state and homogeneous turbulence the energy equation, (2.9), takes the simple form

$$\epsilon = \tfrac{1}{2}\tau_{ij}D_{ij} \tag{3.4}$$

where ϵ, the dissipation rate, is equal to the sum of the last two terms in the square brackets on the left side of (2.9). Upon substitution of (3.1) and (3.2) this may be written

$$\epsilon = (kh)^2(D^2/2)^{3/2} \tag{3.5}$$

An exact expression for D^2 involves some rather intricate integrals. To a fairly good approximation, however, it is equal to twice the vorticity spectrum function integrated over wave numbers less than π/h, that is

$$D^2 \approx 4\int_0^{\pi/h} \kappa^2 E(\kappa)\,d\kappa = 3\alpha\epsilon^{2/3}\left(\frac{\pi}{h}\right)^{4/3} \tag{3.6}$$

where the equality is obtained by substitution of the Kolmogoroff spectrum function, (1.4), for $E(\kappa)$ and subsequent integration. π/h is the largest wave number unambiguously representable on a finite difference mesh. Substitution of (3.6) into (3.5) yields the result, (3.3).

4. SECOND ORDER THEORY

In the second order theory, a direct stress-strain proportionality is not assumed. Instead, equations (2.8) and (2.9) are used in nontrivial but substantially simplified forms such that, in a steady state homogeneous turbulent flow, they reduce to the first order system, (3.1)–(3.3). The simplified equations are:

$$\frac{\partial \tau_{ij}}{\partial t} + \bar{u}_k \frac{\partial \tau_{ij}}{\partial x_k} = \frac{4}{15} E D_{ij} - \frac{4}{15} \frac{E^{1/2}}{k_1 h} \tau_{ij} + \frac{\partial}{\partial x_k}\left(K_\tau \frac{\partial \tau_{ij}}{\partial x_k}\right) \tag{4.1}$$

$$\frac{\partial E}{\partial t} + \bar{u}_k \frac{\partial E}{\partial x_k} = \tfrac{1}{2}\tau_{ij} D_{ij} - c\frac{E^{3/2}}{h} + \frac{\partial}{\partial x_k}\left(K_\tau \frac{\partial E}{\partial x_k}\right) \tag{4.2}$$

where K_τ is an eddy diffusive coefficient for the covariances, given by the formula

$$K_\tau = k_2 h E^{1/2} \tag{4.3}$$

and c, k_1, and k_2 are dimensionless constants. The terms of (4.1) are identified with those of (2.8) as follows: the first two on the left are identical; the molecular viscosity terms in (2.8) are neglected; the first term and part of the second term on the right of (2.8) are identified with the first in (4.1), and the remainder of the second and the third of (2.8) with the second in (4.1); the last term in (4.1) is considered an approximation to the last in (2.8). In (2.9), the molecular dissipation term is not neglected, but is approximated by the second term on the right of (4.2); the last terms on the right are to be identified with each other, and all others are exactly identical. In the remainder of this section we attempt to justify the above approximations.

The dissipation term in (4.2) arises directly from the assumption that the mesh interval h lies within the inertial subrange. To a similar approximation as that used for evaluation of D^2 in (3.6), we may express the turbulent energy, E, as the integral of the Kolmogoroff spectrum from $\kappa = \pi/h$ to infinity, *i.e.*,

$$E \approx \int_{\pi/h}^{\infty} \alpha \epsilon^{2/3} \kappa^{-5/3}\, d\kappa = \tfrac{3}{2}\alpha \epsilon^{2/3}\left(\frac{h}{\pi}\right)^{2/3} \tag{4.4}$$

The expression for the dissipation rate follows directly, where

$$c \approx \pi\left(\frac{2}{3\alpha}\right)^{3/2} \approx .93 \tag{4.5}$$

In steady state homogeneous turbulence, the terms on the left of (4.1) and (4.2) and the diffusion terms vanish. The remaining terms may be solved to yield

$$E = \frac{k_1}{2c} h^2 D^2 \tag{4.6}$$

$$\tau_{ij} = k_1 \sqrt{\frac{k_1}{2c}}\, h^2 D D_{ij} \tag{4.7}$$

Equation (4.7) is clearly identical to (3.1)–(3.2), provided that

$$k_1 = c^{1/3} k^{4/3} \approx .094 \tag{4.8}$$

The first term on the right of (4.1) is obtained from consideration of the effects of application of a sudden strain on an initially isotropic and homogeneous field of turbulence. The analysis leading to this term follows that of Batchelor[30] and Townsend,[31] which stem from earlier work by Taylor.[32] The result shows that the effect of the pressure velocity product (second term on the right of 2.8) is to reduce by 60 percent (2/3 to 4/15) the stress generation by the velocity terms alone (third term on the right). This result is exact, however, only at the initial instant when the turbulence is isotropic. The second term on the right of (4.1) was chosen in form and magnitude so that in the steady state homogeneous case, the system would reduce to the first order theory. By itself, however, it is a term which tends to reduce the stress components in a formally similar way to the dissipation term in (4.2), although the processes it represents probably have little or nothing to do with molecular dissipation.

The last terms on the right of (4.1) and (4.2) are rather arbitrarily chosen approximations to the diffusion terms. No justification is offered for the form of these terms except the intuitive notion that turbulence must, on the average, diffuse itself down-gradient. It is assumed that these terms will not generally be of any great importance. The value of the coefficient k_2 is even less justifiable but $k_2 = k_1$ seems a reasonable initial choice. This choice would make $K_\tau = K$.

5. COMPARISON WITH EXPERIMENT

Although there are probably no existing measurements designed or completely suitable for verification of the above model, Townsend's experiments with rapidly distorted turbulence[33] provide a partial test. In these experiments a grid generated field of turbulence in a wind tunnel enters a section of the tunnel in which it is squeezed in one lateral direction and stretched in the other, the total cross-section remaining unchanged. Stresses and anisotropies are rapidly generated and

the field appears to approach a new equilibrium until the strain rate is eventually relaxed. Townsend presents his results in terms of two structure parameters, K_1 and K_2, which in the present notation are equivalent to

$$K_1 = \frac{\tau_{33} - \tau_{22}}{\frac{4}{3}E + \tau_{11}} \tag{5.1}$$

$$K_2 = -\frac{3}{2}\frac{\tau_{11}}{E} \tag{5.2}$$

The equivalence is not exact because of the difference in definition of the turbulence between the experiment and the model. The scale of the experimental turbulence is much smaller than the tunnel dimensions, and if we visualize a computational mesh interval equal to this scale, then the identification seems reasonably close. The experimental measurements are made, however, on the energy containing scale and not on the inertial range. It is doubtful whether an inertial range even exists at the Reynolds number used. The dimensional relations indicated in equations (4.1) and (4.2) may still be valid, but the coefficients are likely to be different. Nevertheless, it is interesting to compare the results of the experiments with calculation from the theory.

For comparison with experiments, it is convenient to scale the steady-state forms of (4.1 and 4.2) by the total external strain parameter, $D_{22} - D_{33}$, and a length scale, h, which in this case must be related to the integral scale of the turbulence. The comparison is hampered by the fact that the integral scale is not constant down the tunnel, but we ignore this complication. The dimensionless equations may then be written

$$\frac{\partial T}{\partial \xi} = \frac{4}{15}\left(4\Sigma - \frac{\epsilon^{1/2}T}{k_1}\right) \tag{5.3}$$

$$\frac{\partial \epsilon}{\partial \xi} = T - c\epsilon^{3/2} \tag{5.4}$$

where

$$T = 16(\tau_{22} - \tau_{33})/h^2(D_{22} - D_{33})^2$$

$$\epsilon = 16E/h^2(D_{22} - D_{33})^2$$

$$\xi = x_1(D_{22} - D_{33})/4\bar{u}_1$$

The diffusion terms in (4.1), (4.2) have been neglected, which is justifiable under the experimental circumstances. Townsend did not show data which would unambiguously determine h, so that the initial conditions on ϵ are not certain. His results showed, however, that the ratio of the initial to final kinetic energy was about 2. Assuming the final value to be close to the steady state solution of (5.3), and (5.4), this suggests that the initial value was near 0.8.

Equations (5.3) and (5.4) were integrated numerically using as initial conditions $T_0 = 0$ and $\epsilon_0 = 0.6$, 0.8, and 1.0, with the coefficients evaluated from (4.5) and (4.8). Figure 2 shows the ratio K_1 for $\epsilon_0 = 0.8$, superimposed upon Townsend's experimental data. The results of the calculations for $\epsilon_0 = 0.6$ and 1.0 are not significantly different. The agreement of the present theory with observations is startling but partly spurious, since a more correct evaluation of c and k_1 leads to about a 30 percent reduction in the asymptotic value of K_1. One is then tempted to arbitrarily make a small change in the coefficients to force the results to agree with experiment, but perhaps a better agreement should not be expected from the nature of the comparison.

6. DISCUSSION

The first order system, (2.4)–(2.7), (3.1)–(3.3), consists of four equations in four unknowns. Numerically these involve forward time integration of the three velocity component equations and solution of a Poisson equation to determine the third term of (2.4). The second order system adds six additional time-dependent equations, (4.1), (4.2), to be forward-integrated and six new unknowns. The storage requirements are therefore more than doubled over what already seemed a formidably large problem. If we look at a geophysical problem, atmospheric cloud convection, we see that three new mean variables are added (temperature, humidity, and liquid water) and nine turbulent flux components. In the second order system there would then be 21 time dependent equations in place of 6 in the first order system. Formally, neither system can be fully justified except for nearly homogeneous, nearly isotropic turbulence, but the practical limits of this requirement are unknown. From considerations of their form, the second order equations would seem to be capable of application at lower spatial resolution than the first order system, perhaps at the low wave number end of the inertial range. The results of the experimental comparison suggest that they may even be used within the energy containing range, with small changes in the coefficients.

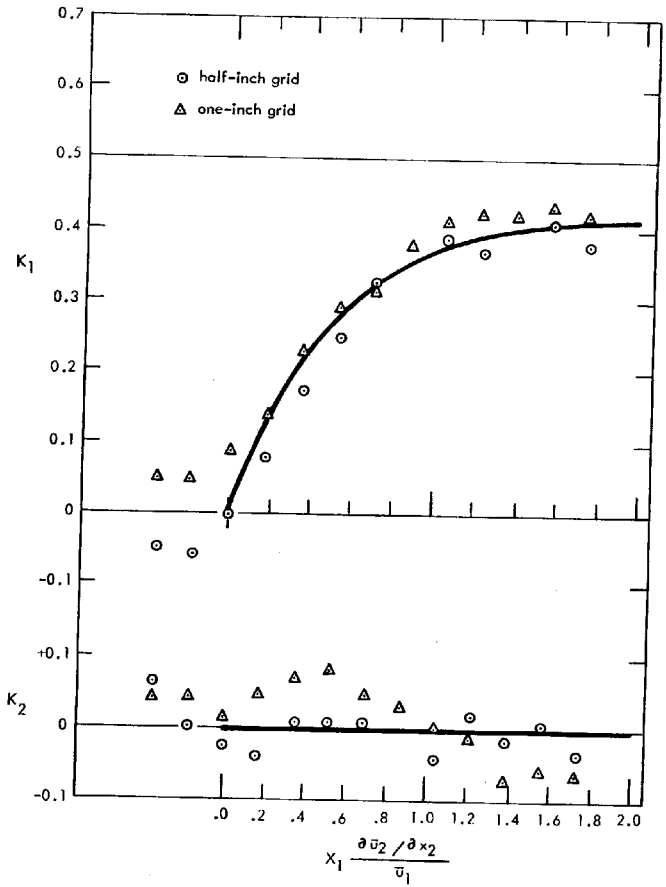

FIGURE 2. Distortion of an initially homogeneous isotropic turbulence field by a plane strain. (The abscissa is proportional to the total strain and the ordinates K_1 and K_2 are proportional to the total stresses generated in strained and unstrained directions, respectively. The plotted points are taken from Townsend[23] and the solid lines are solutions of equations (4.1) and (4.2).)

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Optical Scattering in the Atmosphere

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1. INTRODUCTION

Problems in radiative transfer through media which may scatter, absorb, and emit radiation are of considerable importance to many branches of science. To the astronomer, the nature of the reflected light from planetary atmospheres offers many clues as to the composition and depth of these atmospheres. Transfer theory has been used in the study of neutron diffusion[1], [2] and has been applied to the study of the design and shielding of nuclear reactors.[3] Of more interest to the writer are the many and varied problems of radiative transfer within and through the earth's atmosphere. This latter class of problems comprises a set ranging from the transfer of solar and infrared radiation through the entire atmosphere, a problem of fundamental importance in the study of the general circulation of the earth's atmosphere, to the study of the transfer properties of perhaps 1000 meters of the earth's atmosphere near its surface for determining atmospheric visibility properties. Intermediate problems in the set comprise such topics as the transfer of solar and terrestrial radiation through clouds, the absorption of ultraviolet radiation by the atmospheric ozone layer, and the propagation of microwaves through clouds and precipitation areas, to mention a few. It is, then, no great surprise that considerable effort has been expended in recent years towards solving particular forms of the transfer equation in conjunction with one of the many unsolved problems of radiative transfer.

Radiation, in traversing a medium, may undergo one or more of the processes of scattering, absorption, and emission. In the absence of scattering, the equation of radiative transfer, while still imposing very