1 Introduction

This section is meant to be a supplement to Chapter 11 of the text, but not a replacement for it.

In the handout on 'Turbulence Viscosity Modeling', an outline of the derivation of the Reynolds stress equations was provided. This is also addressed by Problems 7.23, 7.24, and 7.25, page 319, and Equation (7.178) and ff, page 315 in the text. For an incompressible, uniform density flow of a Newtonian fluid, the dynamic equations for $\langle u_i u_j \rangle$ are the following.

$$\begin{aligned} \frac{\bar{D}}{Dt}\langle u_i u_k \rangle &= \underbrace{-\langle u_i u_j \rangle \frac{\langle U_k \rangle}{\partial x_j} - \langle u_k u_j \rangle \frac{\partial \langle U_i \rangle}{\partial x_j}}_{\mathcal{P}_{ik}} - \frac{1}{\rho}\Big[\langle u_k \frac{\partial p}{\partial x_i}\rangle + \langle u_i \frac{\partial p}{\partial x_k}\rangle\Big] \\ &- \frac{\partial}{\partial x_j} \underbrace{\langle u_j u_i u_k \rangle}_{\mathcal{T}^{(u)}_{jik}} - \frac{\partial}{\partial x_j} \underbrace{\left[-\nu \frac{\partial}{\partial x_j}\langle u_i u_k \rangle\right]}_{\mathcal{T}^{(\nu)}_{jik}} - \underbrace{2\nu \left\langle \frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_j}\right\rangle}_{\epsilon_{ik}} , \end{aligned} \tag{1}$$

where $\frac{\bar{D}}{Dt} = \frac{\partial}{\partial t} + \langle U_j \rangle \frac{\partial}{\partial x_j}$ is the substantial derivative following the mean flow, and $i, k = 1, 2, 3$. Note that, if we use these equations, we will have to have models for the second, third, and fifth terms on the right-hand side.

The second term on the right-hand side of Equation (1) is usually written as follows (this is not a unique decomposition):

$$\begin{aligned} -\frac{1}{\rho}\Big[\langle u_k \frac{\partial p}{\partial x_i}\rangle + \langle u_i \frac{\partial p}{\partial x_k}\rangle\Big] &= -\frac{1}{\rho}\Big[\frac{\partial}{\partial x_i}\langle u_k p\rangle + \frac{\partial}{\partial x_k}\langle u_i p \rangle\Big] + \frac{1}{\rho}\Big[\Big\langle p \underbrace{\Big(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\Big)}_{2s_{ki}}\Big\rangle\Big] \\ &= -\frac{\partial}{\partial x_j}\underbrace{\left[\frac{1}{\rho}\langle u_k p\rangle \delta_{ij} + \frac{1}{\rho}\langle u_i p \rangle \delta_{kj}\right]}_{\mathcal{T}^{(p)}_{jki}} + \underbrace{\frac{1}{\rho}\Big\langle p \Big(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\Big)\Big\rangle}_{\mathcal{R}_{ik}} . \end{aligned} \tag{2}$$

Therefore, Equation (1) can finally be written as:

$$\frac{\bar{D}}{Dt}\langle u_i u_k \rangle = -\frac{\partial}{\partial x_j} T_{jki} + \mathcal{P}_{ik} + \mathcal{R}_{ik} - \epsilon_{ik} , \tag{3}$$

with $T_{jki} = T^{(u)}_{jki} + T^{(\nu)}_{jki} + T^{(p)}_{jki}$. These latter terms are the conservative terms, whose effect is to move $\langle u_i u_k \rangle$ in space without changing the overall amount of $\langle u_i u_k \rangle$. They will be modeled as turbulent gradient transport (diffusion) terms.

Note the following about Reynolds stress equation modeling.

- The philosophy behind this general approach is that it allows more physics to be included and thereby, hopefully, giving better modeling.
- But in doing so, many more assumptions are required, with many more constants to be determined.
- Also, the numerical simulation becomes significantly more difficult, e.g, , in the number of equations, the stiffness of the equations, etc.

- Whether the method is used depends on the problem at hand, what types of questions are being asked, etc. For example, although the k-ϵ model does fairly well for axisymmetric, turbulent jets, it does poorly for swirling jets, as new physics enters the problem. There is a need to introduce more physics into the modeling, and this can be done in this case by using Reynolds stress equation modeling.

- There is now some tendency to use k-ϵ modeling, possibly with algebraic stress models, and then go directly to large-eddy simulation if more complexity in physics is required. But large-eddy simulation requires significantly more resources than RANS approaches.

1.1 Reynolds number similarity

In ME543 it was argued that, for high Reynolds turbulent number flows, the statistical properties that depend on the large-scale motions, e.g., $\langle U_i \rangle$, $\langle u_i u_k \rangle$, ϵ, should be independent of the Reynolds number. The dynamics are controlled by the large-scale motions, which are inviscid. Even the kinetic energy viscous dissipation rate ϵ, which occurs at the Kolmogorov scale, is controlled by the nonlinear transfer of energy to small scales, a process controlled by the large-scale motions. Most models (e.g., k-ϵ, Reynolds stress equation modeling) include the assumption of Reynolds number similarity. The model constants are usually taken to be Reynolds number independent, and the effects of the Reynolds number only enter in terms explicitly containing viscosity. This is generally true except for the following:

- Turbulent flows near boundaries, where $\langle u^2 \rangle^{1/2}$ and the energy-containing scale, say ℓ, become small as the boundary is approached, so that the Reynolds number $\langle u^2 \rangle^{1/2} \ell / \nu$ becomes small, and Reynolds number similarity cannot be expected to hold.

- Turbulent flows at low Reynolds numbers.

1.2 Realizable, Realizability

The concept of Realizability comes up sometimes in Reynolds stress equation modeling. It first arose when researchers were developing closure models for the equations for the kinetic energy spectrum $E(k,t)$ for isotropic turbulence. It was found that some of the principal theories at the time (e.g., Millionshekov's quasi-normal approximation) did not assure that $E(k,t) \geq 0$. Because of this, 'realizable' models were formulated, i.e., models that, although not exactly describing turbulence (i.e., satisfying the Navier-Stokes equations) described some hypothetical dynamical system which could in theory be realized. For example, equations could be written for the dynamical system which, when averaged, gave the modeling equations. This would guarrantee that a number of constraints of the type $E(k,t) \geq 0$ could be satisfied.

For the Reynolds stress equations, Schumann (*Phys. Fluids*, **20**:721-725, 1977) pointed out that the realizability constraints are the following:

1. $\det(\langle u_i u_j \rangle) \geq 0$ (see the proof in Schumann's paper).

2. $\langle u_i u_i \rangle \geq 0$, no sum on i (this constraint is obvious).

3. $|\langle u_i u_j \rangle|^2 \leq \langle u_i^2 \rangle \langle u_j^2 \rangle$, no sum on i, j (the Cauchy-Schwartz inequality).

If these conditions are not satisfied by a model, then it cannot, at least in theory, be related to a realizable process. Schumann was ultimately able to express these conditions in terms of 3 separate equations related to the principal invariants of the matrix $\langle u_i u_j \rangle$ (see Appendix B in the text), and to suggest modifications to the models to insure these constraints. The invariants of a matrix are properties of the matrix which are independent of the coordinate system, e.g., trace($\langle u_i u_j \rangle$) = $\langle u_i^2 \rangle$.

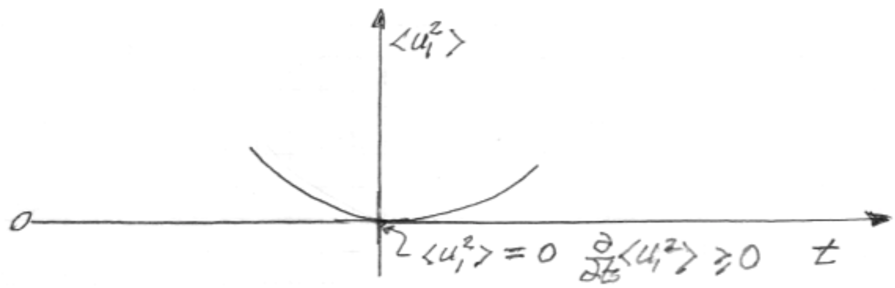

Figure 1: Example of condition needed to maintain realizability.

An example of insuring realizability is the following (see Figure 1). Note that if $\langle u_1^2 \rangle = 0$ at a point, then

$$\left[\frac{\partial}{\partial t} + \langle U_j \rangle \frac{\partial}{\partial x_j}\right] \langle u_1^2 \rangle = \text{modeling} \geq 0$$

for the model to be realizable. If this condition is not met, then negative energy can develop.

Realizable models, formulated so that all the realizability conditions are satisfied, are included as a choice in most of the major commercial computer codes. But note the following.

- Often the imposition of realizability conditions is somewhat complicated, and can demand more computational resources. So sometimes simplified measures, e.g., 'clipping', are used instead so that, e.g., if $\langle u_1^2 \rangle$ becomes negative, its value is set back to 0.

- The realizability methods are much better from a theoretical perspective, e.g, helping to assure the proper behavior of the model under coordinate transformations.

- The realizable models do not always improve the results significantly, however, since the variables are often not near the constraints maintaining realizability.

2 Modeling for $\epsilon_{ik} = 2\nu \left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_j} \right\rangle$

Note that ϵ_{ik} is a quantity that occurs at very small scales, similar to the behavior of the dissipation rate ϵ itself. It has been theoretically argued, and supported by laboratory experiments and field data that, for high Reynolds number flows, the small-scale motions are approximately locally isotropic. Therefore, it is assumed that ϵ_{ik} is isotropic, so that it is modeled as:

$$\epsilon_{ij} \doteq 2\nu \left\langle \frac{\partial u_k}{\partial x_\ell} \frac{\partial u_k}{\partial x_\ell} \right\rangle \frac{1}{3} \delta_{ij} = \frac{2}{3} \left\{ \epsilon - \nu \frac{\partial^2}{\partial x_k \partial x_\ell} \langle u_k u_\ell \rangle \right\} \delta_{ij} \doteq \frac{2}{3} \epsilon \delta_{ij} \, . \tag{4}$$

Note that this expression maintains the same trace as ϵ_{ij}. Furthermore, δ_{ij} is the only second-order isotropic tensor. In addition, the model contains the pseudo-dissipation rate $\tilde{\epsilon} = \nu \left\langle \frac{\partial u_k}{\partial x_\ell} \frac{\partial u_k}{\partial x_\ell} \right\rangle$, which is related to the true dissipation rate ϵ as $\tilde{\epsilon} = \epsilon - \nu \frac{\partial^2}{\partial x_k \partial x_\ell} \langle u_k u_\ell \rangle$. Note that the second term in brackets in Equation (4) can be neglected for

- three-dimensionally homogeneous flows, since $\frac{\partial}{\partial x_i}\langle\cdot\rangle = 0$, or
- for high Reynolds number flows. In this case ϵ scales as u'^3/ℓ while $\nu\frac{\partial^2}{\partial x_i \partial x_j}$ scales as $\nu u'^2/\ell^2$, where u' is an rms velocity, and ℓ an integral scale. So the ratio scales as $(u'^3/\ell)/(\nu u'^2/\ell^2) = u'\ell/\nu = Re_\ell \gg 1$, and the viscous term can be neglect in comparison to ϵ.

Note also that:

- ϵ_{ik} is the easiest term in Equation (1) to model, and with confidence, especially for large Reynolds numbers, and
- the model for ϵ_{ik} implies that an equation for ϵ will be needed in addition to equations for $\langle u_i u_k\rangle$.
- the subtle difference between the dissipation rate ϵ and the pseudo-dissipation rate $\tilde{\epsilon}$.

3 Modeling for $\mathcal{R}_{ik} = \frac{1}{\rho}\Big\langle p\Big(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\Big)\Big\rangle = \frac{2}{\rho}\langle p s_{ik}\rangle$

Recall that, in deriving the equation for $\langle u_i u_k\rangle$, first an equation for u_i was derived by subtracting the equation for $\langle U_i\rangle$ from that for $U_i = \langle U_i\rangle + u_i$, giving

$$\frac{\partial}{\partial t}u_i + u_j\frac{\partial}{\partial x_j}u_i + \langle U_j\rangle\frac{\partial}{\partial x_j}u_i + u_j\frac{\partial}{\partial x_j}\langle U_i\rangle = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2}{\partial x_j^2}u_i + \frac{\partial}{\partial x_j}\langle u_i u_j\rangle\,. \tag{5}$$

Taking the divergence of Equation (5), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\underbrace{\frac{\partial u_i}{\partial x_i}}_{=0} + \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} \quad &+ \quad u_j\frac{\partial}{\partial x_j}\underbrace{\frac{\partial u_i}{\partial x_i}}_{=0} + \frac{\partial\langle U_j\rangle}{\partial x_i}\frac{\partial u_i}{\partial x_j} + \langle U_j\rangle\frac{\partial}{\partial x_j}\underbrace{\frac{\partial u_i}{\partial x_i}}_{=0} + \frac{\partial u_j}{\partial x_i}\frac{\partial\langle U_i\rangle}{\partial x_j} + u_j\frac{\partial}{\partial x_j}\underbrace{\frac{\partial\langle U_i\rangle}{\partial x_i}}_{=0}\\ &= \quad -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i^2} + \nu\frac{\partial^2}{\partial x_j^2}\underbrace{\frac{\partial u_i}{\partial x_i}}_{=0} + \frac{\partial^2}{\partial x_i \partial x_j}\langle u_i u_j\rangle\,, \text{ or}\end{aligned}$$

$$\frac{1}{\rho}\frac{\partial^2}{\partial x_i^2}p = -2\frac{\partial u_i}{\partial x_j}\frac{\partial\langle U_j\rangle}{\partial x_i} - \frac{\partial^2}{\partial x_i\partial x_j}\big(u_i u_j - \langle u_i u_j\rangle\big)\,, \tag{6}$$

i.e., a Poisson equation for the fluctuating pressure p.

For modeling purposes, the solution is usually split up as (note that Equation (6) a linear equation in p)

$$p = p^{(r)} + p^{(s)} + p^{(h)}\,, \tag{7}$$

where $p^{(r)}$, the ‘rapid’ component, satisfies:

$$\frac{1}{\rho}\frac{\partial^2}{\partial x_i^2}p^{(r)} = -2\frac{\partial u_i}{\partial x_j}\frac{\partial\langle U_j\rangle}{\partial x_i}\,, \tag{8}$$

related directly to $\langle U_i\rangle$, $p^{(s)}$, the ‘slow’ component, satisfies

$$\frac{1}{\rho}\frac{\partial^2}{\partial x_i^2}p^{(s)} = -\frac{\partial^2}{\partial x_i\partial x_j}\big(u_i u_j - \langle u_i u_j\rangle\big)\,, \tag{9}$$

related to the turbulence, while $p^{(h)}$, the 'harmonic' component, satisfies the homogeneous LaPlace's equation,

$$\frac{1}{\rho}\frac{\partial^2}{\partial x_i^2}p^{(h)} = 0\,. \tag{10}$$

There are some subtleties in making sure that the boundary conditions for these equations are satisfied.

The pressure $p^{(r)}$ is called 'rapid' because it responds immediately to changes in the mean velocity gradients. The harmonic component $p^{(h)}$ is only important near boundaries.

3.1 Modeling the 'rapid' part of the pressure/rate-of-strain correlation

Corresponding to the decomposition of the pressure, there is a corresponding decompositon of the pressure/strain-rate correlation $\mathcal{R}_{ij}$, e.g., for the rapid part,

$$\mathcal{R}_{ij}^{(r)} = \frac{1}{\rho}\Big\langle p^{(r)}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)\Big\rangle\,.$$

Recall from ME 543 (see page 18 and ff in the text) and from the text that, for the Poisson equation,

$$\frac{\partial^2}{\partial x_i^2}f(\mathbf{x}) = S(\mathbf{x})\,,$$

the general solution to this equation is, in an unbounded domain,

$$f(\mathbf{x}) = -\frac{\mathbf{1}}{\mathbf{4\pi}}\iiint_{-\infty}^{\infty}\frac{\mathbf{S(y)}}{|\mathbf{x}-\mathbf{y}|}\mathbf{dy}\,.$$

There is a much simpler expression in Fourier space when Fourier expansions are allowed. Therefore, considering the <u>rapid component</u> for <u>homogeneous</u> shear or strain,

$$\Big\langle\frac{p^{(r)}}{\rho}\frac{\partial u_i}{\partial x_j}\Big\rangle = \Big\langle -\frac{1}{4\pi}\iiint_{-\infty}^{\infty}\left[\frac{-2\overbrace{\frac{\partial\langle U_\ell\rangle}{\partial x_k}}^{\text{constant}}\ \frac{\partial u_k(\mathbf{y})}{\partial y_\ell}\frac{\partial u_i(\mathbf{x})}{\partial x_j}}{|\mathbf{x}-\mathbf{y}|}\right]d\mathbf{y}\Big\rangle\,.$$

But $\Big\langle\frac{\partial u_k(\mathbf{y})}{\partial y_\ell}\frac{\partial u_i(\mathbf{x})}{\partial x_j}\Big\rangle = \frac{\partial^2}{\partial y_\ell\partial x_j}\langle u_k(\mathbf{y})u_i(\mathbf{x})\rangle$, so that

$$\Big\langle\frac{p^{(r)}}{\rho}\frac{\partial u_i}{\partial x_j}\Big\rangle = \frac{1}{2\pi}\frac{\partial\langle U_\ell\rangle}{\partial x_k}\iiint_{-\infty}^{\infty}\frac{\partial^2}{\partial y_\ell\partial x_j}\langle u_k(\mathbf{y})u_i(\mathbf{x})\rangle\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\,.$$

But, for homogeneous turbulence, with $\mathbf{r} = \mathbf{y}-\mathbf{x}$, or $\mathbf{y} = \mathbf{x}+\mathbf{r}$,

$$\langle u_k(\mathbf{y})u_i(\mathbf{x})\rangle \equiv R_{ik}(\mathbf{r}) = \langle u_i(\mathbf{x})u_k(\mathbf{x}+\mathbf{r})\rangle\,,\text{ and}$$

$$\frac{\partial}{\partial y_\ell}R_{ik}(\mathbf{r}) = \frac{\partial}{\partial r_\ell}R_{ik}(\mathbf{r})\frac{\partial r_\ell}{\partial y_\ell}\text{ (no sum)} = \frac{\partial}{\partial r_\ell}R_{ik}(\mathbf{r})\,,\text{ and}$$

$$\frac{\partial}{\partial x_j}R_{ik}(\mathbf{r}) = \frac{\partial}{\partial r_j}R_{ik}(\mathbf{r})\frac{\partial r_j}{\partial x_j}\text{ (no sum)} = -\frac{\partial}{\partial r_j}R_{ik}(\mathbf{r})\,.$$

So finally, changing the variable of integration from $\mathbf{y}$ to $\mathbf{r}$,

$$\left\langle \frac{p^{(r)}}{\rho} \frac{\partial u_i}{\partial x_j} \right\rangle = -\frac{1}{2\pi} \frac{\partial \langle U_\ell \rangle}{\partial x_k} \iiint_{-\infty}^{\infty} \frac{\partial^2}{\partial r_j \partial r_\ell} R_{ik}(\mathbf{r}) \frac{d\mathbf{r}}{|\mathbf{r}|} = 2 \frac{\partial \langle U_\ell \rangle}{\partial x_k} \mathcal{M}_{i\ell jk} \,, \tag{11}$$

$$\mathcal{M}_{i\ell jk} = -\frac{1}{4\pi} \iiint_{-\infty}^{\infty} \frac{\partial^2}{\partial r_j \partial r_\ell} R_{ik}(\mathbf{r}) \frac{d\mathbf{r}}{|\mathbf{r}|} \,. \tag{12}$$

Note that this expression is not closed, since $R_{ik}(\mathbf{r})$ is an additional two-point quantity that is not known. We will see that Equations (11) and (12) will suggest models for the rapid part of the pressure/strain-rate correlation.

3.2 Modeling the 'slow' part of the pressure/rate-of-strain correlation

If we consider the equations for the various terms in the Reynolds stress for the case of homogeneous, turbulent shear flow, we find that they are the following:

$$\begin{aligned}
\frac{d}{dt}\langle u_1^2 \rangle &= \underbrace{-\langle u_1 u_2 \rangle \frac{d\langle U_1 \rangle}{dx_2}}_{\mathcal{P}_{11} > 0} + \underbrace{\frac{2}{\rho}\left\langle p \frac{\partial u_1}{\partial x_1} \right\rangle}_{\mathcal{R}_{11}} - \frac{2}{3}\epsilon \\
\frac{d}{dt}\langle u_2^2 \rangle &= 0 + \underbrace{\frac{2}{\rho}\left\langle p \frac{\partial u_2}{\partial x_2} \right\rangle}_{\mathcal{R}_{22}} - \frac{2}{3}\epsilon \\
\frac{d}{dt}\langle u_3^2 \rangle &= 0 + \underbrace{\frac{2}{\rho}\left\langle p \frac{\partial u_3}{\partial x_3} \right\rangle}_{\mathcal{R}_{33}} - \frac{2}{3}\epsilon
\end{aligned}$$

The model suggested in the previous subsection has been used for ϵ_{11}, ϵ_{22}, and ϵ_{33}. Note that the energy is transferred from the mean flow to the turbulence, through the production term $\mathcal{P}_{11}$, only in the $\langle u_1^2 \rangle$ component. Also note that, if the equations are summed to give an equation for the total turbulence kinetic energy, the pressure/rate-of-strain term drops out, since it reduces to $\frac{2}{\rho}\left\langle p \frac{\partial u_i}{\partial x_i} \right\rangle = 0$. This implies that, since generally $\langle u_2^2 \rangle$ and $\langle u_3^2 \rangle$ are appreciable fractions of $\langle u_1^2 \rangle$, then

$$\frac{2}{\rho}\left\langle p \frac{\partial u_1}{\partial x_1} \right\rangle < 0 \,, \quad \frac{2}{\rho}\left\langle p \frac{\partial u_2}{\partial x_2} \right\rangle > 0 \,, \quad \frac{2}{\rho}\left\langle p \frac{\partial u_3}{\partial x_3} \right\rangle > 0 \,.$$

The effect of the pressure/strain-rate terms in this case is to transfer energy from the more energetic components to the less energetic ones.

If we consider again the homogeneous strain experiments (see Figure 2), in the contraction section the flow becomes non-isotropic but, according to the experimental data, in the straight section it 'relaxes back towards isotropy'. In the contraction section with the mean strain-rate, the flow is becoming non-isotropic due mainly to the rapid component of the pressure/rate-of-strain correlation. In the downstream straight section, however, where there is no mean strain-rate, so that the rapid part of the pressure/rate-of-strain correlation is 0, the flow can relax back towards isotropy only due to the slow component.

Consider the equation for the Reynolds stress for the case of decaying, homogeneous, non-isotropic turbulence with no mean flow, as in the straight section downstream of the contraction.

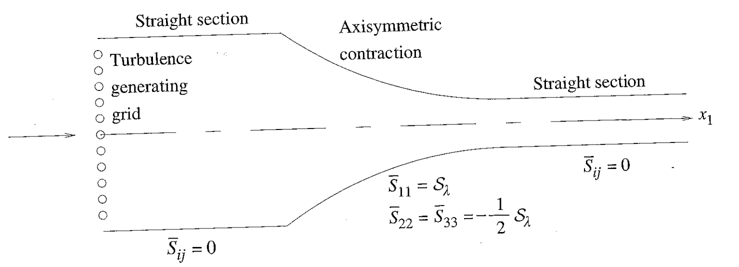

Fig. 10.1. A sketch of an apparatus, similar to that used by Uberoi (1956) and Tucker (1970), to study the effect of axisymmetric mean straining on grid turbulence.

Figure 1: Figure 10.1, page 360.

Figure 2: Sketch of homogeneous strain experiment.

The equation for $\langle u_i u_j \rangle$ reduces to:

$$\frac{d}{dt}\langle u_i u_j \rangle = \mathcal{R}_{ij} - \epsilon_{ij} \,. \tag{13}$$

The rapid part of $\mathcal{R}_{ij}$ is 0 because there is no mean strain-rate, and the harmonic part is 0 because the flow is approximately homogeneous so that boundary effects are not important. This flow then isolates the effect of $\mathcal{R}_{ij}^{(s)}$.

A useful model for this term is the 'return-to-isotropy' model of Rotta,

$$\mathcal{R}_{ij}^{(s)} = -C_R \frac{\epsilon}{k}(\langle u_i u_j \rangle - \frac{2}{3}k\delta_{ij}) \,. \tag{14}$$

Here C_R is the 'Rotta' constant, and $\epsilon/k = 1/\mathcal{T}$ is an inverse turbulence time scale. Note that if $\langle u_1^2 \rangle > \frac{1}{3}(\langle u_1^2 \rangle + \langle u_2^2 \rangle + \langle u_3^2 \rangle)$, i.e., if $\langle u_1^2 \rangle$ is greater than the average of the three mean square velocities, then $\mathcal{R}_{11}^{(s)} < 0$, and energy would be lost from $\langle u_1^2 \rangle$, whereas if $\langle u_1^2 \rangle < \frac{1}{3}(\langle u_1^2 \rangle + \langle u_2^2 \rangle + \langle u_3^2 \rangle)$, then energy would be gained in $\langle u_1^2 \rangle$. So the Rotta model has the proper qualitative physics. Furthermore, $\mathcal{R}_{ii}^{(s)} = -C_R(\epsilon/k)(2k - 2k) = 0$ (i.e., the model is deviatoric) so that it conserves energy as required.

Because $\mathcal{R}_i^{(s)}$ does not affect the trace of $\langle u_i u_j \rangle$, i.e, the kinetic energy, it can be more helpful to work with the deviator part of $\langle u_i u_j \rangle$, i.e. a_{ij}, or its normalized version

$$b_{ij} = \frac{a_{ij}}{2k} = \frac{\langle u_i u_j \rangle}{\langle u_k u_k \rangle} - \frac{1}{3}\delta_{ij} \,. \tag{15}$$

Assuming homogeneous flow with no mean shear nor mean strain, the equation for b_{ij} can be obtained as follows.

$$\begin{aligned} \frac{d}{dt}b_{ij} &= \frac{d}{dt}\Big[\frac{\langle u_i u_j \rangle}{2k} - \frac{1}{3}\delta_{ij}\Big] = \frac{1}{2k}\frac{d}{dt}\langle u_i u_j \rangle + \frac{\langle u_i u_j \rangle}{2}\frac{(-1)}{k^2}\frac{dk}{dt} \\ &= \frac{1}{2k}\big[\mathcal{R}_{ij}^{(s)} - \epsilon_{ij}\big] + \frac{\langle u_i u_j \rangle}{2k}\frac{\epsilon}{k} \quad (\text{with } \frac{dk}{dt} = -\epsilon) \\ &= \frac{\epsilon}{k}\Big[\underbrace{\frac{\langle u_i u_j \rangle}{2k} - \frac{\delta_{ij}}{3}}_{b_{ij}} + \frac{\delta_{ij}}{3} + \frac{\mathcal{R}_{ij}^{(s)}}{2\epsilon} - \frac{\epsilon_{ij}}{2\epsilon}\Big] \,. \end{aligned} \tag{16}$$

Assuming isotropy, i.e., $\epsilon_{ij} \doteq (2/3)\epsilon\delta_{ij}$, then b_{ij} satisfies

$$\frac{d}{dt}b_{ij} = \frac{\epsilon}{k}\Big(b_{ij} + \frac{\mathcal{R}_{ij}^{(s)}}{2\epsilon}\Big) . \tag{17}$$

But the Rotta model in terms of b_{ij} is

$$\mathcal{R}_{ij}^{(s)} = -C_R\frac{\epsilon}{k}(\langle u_i u_j\rangle - \frac{2}{3}k\delta_{ij}) = -2C_R\epsilon\Big(\frac{\langle u_i u_j\rangle}{2k} - \frac{\delta_{ij}}{3}\Big) = -2C_R\epsilon b_{ij} ,$$

so that, with this model, the equation for b_{ij} becomes

$$\frac{d}{dt}b_{ij} = \frac{\epsilon}{k}\Big(b_{ij} - 2C_R\frac{\epsilon b_{ij}}{2\epsilon}\Big) = -(C_R - 1)\frac{\epsilon}{k}b_{ij} . \tag{18}$$

Therefore Rotta's model is a linear return to isotropy model. Clearly $C_R > 1$ is required for return to isotropy to occur according to the model.

Several 'return to isotropy' experiments have been carried out, and the Rotta model is found to be somewhat deficient (as are, to some extent, all models). One generalization is to allow $\mathcal{R}_{ij}^{(s)}$ to be an arbitrary function of the tensor b_{ij}. This implies that it must depend upon (see Appendix B in the text for a discussion of these issues):

1. various powers of b_{ij}, and
2. the invariants of b_{ij} in the coefficients of the powers of b_{ij}.

The invariants of b_{ij} are: $I_B = \mathrm{trace}(b_{ij}) = 0$, $II_b = b_{ij}b_{ji}$, $III_b = b_{ij}b_{jk}b_{ki}$, where $b_{ij}b_{jk}$ and $b_{ij}b_{jk}b_{k\ell}$ are 3x3 matrices.

The Cayley-Hamilton theorem says that the powers of b_{ij} greater than 2 are directly related to b_{ij} and $b_{ij}^2 = b_{ij}b_{jk}$. So only the first two powers of b_{ij} need to be considered. Since $\mathcal{R}_{ij}^{(s)}$ is symmetric and deviatoric, the result is:

$$\mathcal{R}_{ij}^{(s)} = \epsilon f_1(II_b, III_b, Re)b_{ij} + \epsilon f_2(II_b, III_b, Re)(b_{ik}b_{lj} - \frac{1}{3}b_{\ell k}b_{k\ell}\delta_{ij}) , \tag{19}$$

where f_1 and f_2 are two non-dimensional, unknown functions that depend on the Reynolds number, $Re = uL/\nu = k^2/\epsilon\nu$, and the remaining two principal invariants of the matrix b_{ij}. Note that for Rotta's model, $f_2 = 0$ and $f_1 = -2C_R$. A model of this type was suggested by Sarkar and Speziale (*Phys. Fluids A*, **2**(1): 84-93, 1990), in which they chose: $f_1 = -2C_R \doteq -3.4$ and $f_2 = 6(C_R - 1) \doteq 4.2$. This model with the chosen values is consistent with realizability.

A test of the model for $\mathcal{R}_{ij}^{(s)}$ can be performed by considering a homogeneous, non-isotropic turbulent flow with no mean shear nor strain, and solving Equation (16) (or (18)) for b_{ij}. This was done by Sarkar and Speziale with comparisons with the experiments of Choi and Lumley (1984) who generated the non-isotropic flow using planar strain. Comparisons were made of the time development of the second and third invariants of b_{ij}, II_b and III_b. Figures (3) and (4) give comparisons for II_b and III_b, respectively, of laboratory data with model predictions. The dashed curves give the predictions of the Rotta model, while the solid curve gives the predictions of Sarkar and Speziale's model. The improvements provided by the more general model are apparent.

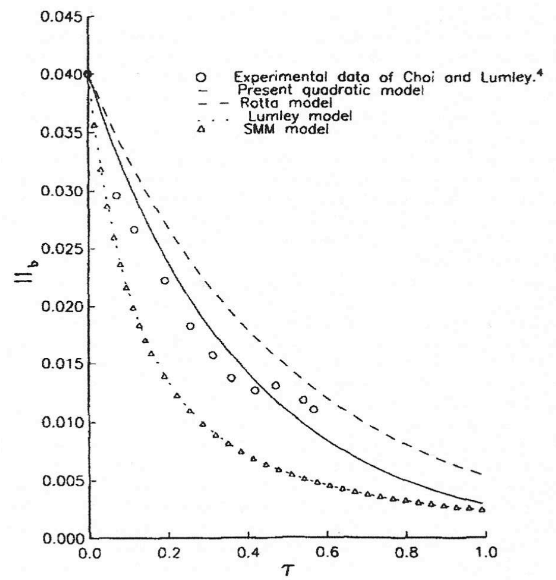

Figure 3: Temporal decay of the second invariant of b_{ij}: comparisons with experiments of Choi and Lumley with the Rotta model and the Sarkar/Speziale quadratic model.

3.3 Rapid Distortion 'Theory'

In order to obtain more information about the rapid component of the pressure/rate-of-strain correlation, it is useful to consider Rapid Distortion 'Theory', or RDT. The approach was first introduced by Taylor and Prandtl in the 1930's, but was popularized by Batchelor, Townsend and others in the 1950's, and is now sometimes used in problems where a linear mechanism (e.g., strong shear, wave motion, etc.) might exist. (In addition to Section 11.4 in the text, see also A. M. Savill (1987, "Recent developments in rapid-distortion theory", *Annu. Rev. Fluid Mech.*, **19**:531-575).) This theory can provide some useful insights into turbulence dynamics, and sometimes useful models.

To motivate the approach, consider the homogeneous shear flow again. With the mean velocity gradient $S = \dfrac{d\langle U_1\rangle}{dx_2}$, then S^{-1} gives a time scale characterizing the mean flow. On the other hand, k/ϵ gives a time scale characterizing the turbulence. The ratio is:

$$\frac{\text{turbulence time}}{\text{mean flow time}} \sim \frac{k/\epsilon}{S^{-1}} = \frac{Sk}{\epsilon}\,.$$

In the homogeneous shear flow experiments, $\dfrac{Sk}{\epsilon} \doteq \text{constant} \doteq 6$, indicating that the mean flow time scale is shorter than the turbulence time scale. RDT considers the limit as $\dfrac{Sk}{\epsilon} \to \infty$ (whereas in the previous discussions of the 'slow' component of the pressure/rate-of-strain correlation, the consideration is that $\dfrac{Sk}{\epsilon} \to 0$).

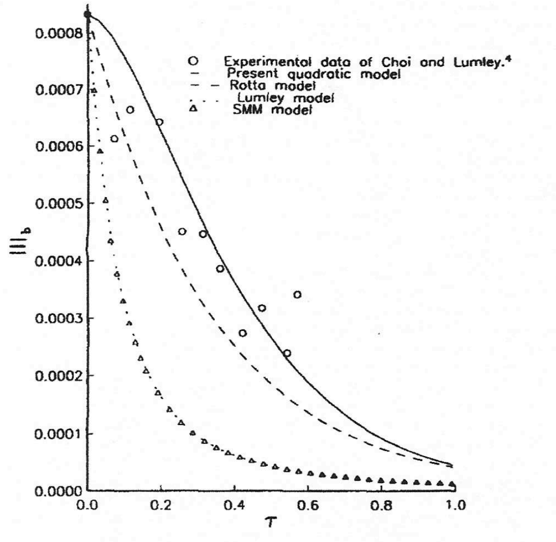

Figure 4: Temporal decay of the third invariant of b_{ij}: comparisons with experiments of Choi and Lumley with the Rotta model and the Sarkar/Speziale quadratic model.

Consider again the equation for u_i, now for homogeneous shear or strain,

$$\frac{\bar{D}}{Dt}u_i = -u_j\frac{\partial}{\partial x_j}\langle U_i\rangle - \underbrace{u_j\frac{\partial}{\partial x_j}u_i}_{\text{nonlinear}} + \nu\nabla^2 u_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \underbrace{\frac{\partial}{\partial x_j}\langle u_i u_j\rangle}_{\substack{=0 \\ \text{homogeneity}}} \quad , \tag{20}$$

with $p = p^{(r)} + p^{(s)}$, and the corresponding Poisson equation for pressure:

$$\frac{1}{\rho}\nabla^2(p^{(r)} + p^{(s)}) = -2\frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial u_j}{\partial x_i} - \underbrace{\frac{\partial^2}{\partial x_i \partial x_j}u_i u_j}_{\text{nonlinear}} \,. \tag{21}$$

In RDT, the equations are linearized (the nonlinear terms are dropped), arguing that:

$$\left|u_j\frac{\partial}{\partial x_j}\langle U_i\rangle\right| \gg \left|u_j\frac{\partial}{\partial x_j}u_i\right| ,$$

$$\left|\frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right| = \left|\frac{\partial^2}{\partial x_i \partial x_j}(\langle U_i\rangle u_j)\right| \gg \left|\frac{\partial^2}{\partial x_i \partial x_j}u_i u_j\right| .$$

The effect of the mean shear is taken to be much stronger than the turbulence self-interactions. This is not a valid approximation for turbulence, but it can give some insight into the dynamics, and can enable the direct computation of $\mathcal{R}^{(r)}$. The resulting RDT equations are:

$$\frac{\bar{D}}{Dt}u_i = -u_j\frac{\partial}{\partial x_j}\langle U_i\rangle - \frac{1}{\rho}\frac{\partial p^{(r)}}{\partial x_i} \text{ , and} \tag{22}$$

$$\frac{1}{\rho}\nabla^2 p^{(r)} = -2\frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial u_j}{\partial x_i} \,. \tag{23}$$

neglecting viscous effects, i.e., examining only the large-scale dynamics. The RDT is almost always applied to three-dimensionally homogeneous flows with constant $\dfrac{\partial\langle U_i\rangle}{\partial x_j}$, that is, with uniform shear or strain.

Since the RDT equations are linear, analytical solutions can be obtained, and the parameter $\mathcal{M}_{ijkl}$ in Equation (12) can be evaluated, either analytically or numerically, giving insight as to how to model the rapid part of the pressure/rate-of-strain correlation. The result is a suggestion for models of $\mathcal{R}^{(r)}_{ik}$ of the following form:

$$\mathcal{R}^{(r)}_{ik} = C_r\big(\mathcal{P}_{ik} - \frac{2}{3}\mathcal{P}\delta_{ik}\big)\,, \tag{24}$$

where $\mathcal{P}_{ik}$ is the production term (see Equation (1)), and $\mathcal{P}$ is its trace; so the model has been constructed so that the trace of $\mathcal{R}^{(r)}_{ik}$ is 0, i.e., that is, it is deviatoric.

3.4 General pressure/rate-of-strain models

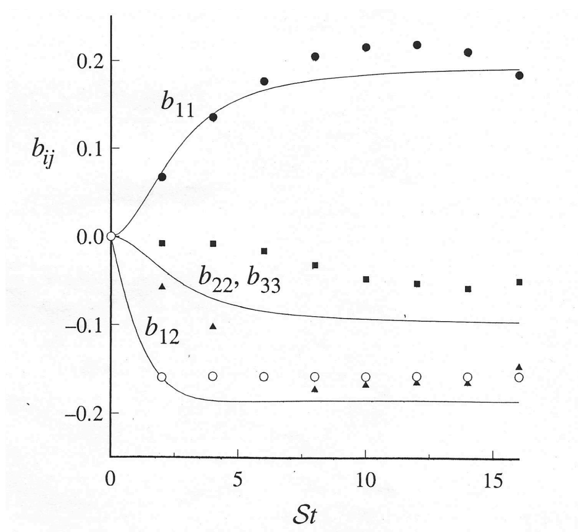

Figure 5: Comparisons of DNS data for b_{ij} with predictions using Equation (25).

For the general cases where $\dfrac{Sk}{\epsilon}$ is neither 0 nor infinity, both aspects of the pressure/rate-of-strain modeling are needed (both the slow and the fast parts). One approach that is often used is a combination of the Rotta model for $\mathcal{R}^{(s)}_{ik}$, and the model discussed in the previous section for $\mathcal{R}^{(r)}_{ik}$; this latter model is attributed to Naot, Shavit and Wolfstein (1970). (For example, this is the basic model used in Fluent, although one can switch to a non-linear model for $\mathcal{R}^{(s)}_{ik}$.) The full model is then:

$$\mathcal{R}_{ij} = -C_R\frac{\epsilon}{k}\big(\langle u_i u_j\rangle - \frac{2}{3}k\delta_{ij}\big) - C_2\big(\mathcal{P}_{ij} - \frac{2}{3}\mathcal{P}\delta_{ij}\big)\,. \tag{25}$$

Normally, $C_R \doteq 1.8$ and $C_2 \doteq 3/5$. The model for $\mathcal{R}^{(r)}_{ij}$ is call the 'isotropization of production' (IP) model, since it acts to counteract the effect of production to increase anisotropy. Typical predictions are given in Figure (5) for a case of homogeneous shear. (The comparison is with the DNS data of Rogers and Moin (1987).) The predictions of the model give the proper trends, although there is significant error (note that b_{ij} is a normalized quantity). Unfortunately the agreement with k is much worse, since the ratio $\mathcal{P}/\epsilon$, which is set by the constants in the equation for ϵ, is somewhat off. From the DNS data, $(\mathcal{P}/\epsilon) \doteq 1.7$, whereas the model prediction is $(\mathcal{P}/\epsilon) \doteq 2.1$. There are several more general models for $\mathcal{R}_{ij}$ discussed in the text. These depend on various symmetric, deviatoric combinations of $\bar{S}_{ij}$, $\bar{\Omega}_{ij}$, and b_{ij}.

4 Transport terms

As discussed in Section 1, the transport term can be split into three parts,

$$T_{jki} = T^{(u)}_{jki} + T^{(\nu)}_{jki} + T^{(p)}_{jki} \text{ , where}$$

$$\begin{aligned} T^{(u)}_{jki} &= \langle u_j u_k u_i \rangle \quad \text{turbulent transport} \\ T^{(\nu)}_{jki} &= -\nu \frac{\partial}{\partial x_j} \langle u_i u_k \rangle \quad \text{viscous diffusion} \\ T^{(p)}_{jki} &= \frac{1}{\rho} \big[\langle u_k p \rangle \delta_{ij} + \langle u_i p \rangle \delta_{jk} \big] \quad \text{pressure transport} \end{aligned}$$

Note that the decomposition of the pressure term discussed in Section 1 is not unique in general, although it is in homogeneous cases. The viscous term, which does not need to be modeled, is usually very small, and will not be discussed further. The pressure term is usually of the same form as the turbulent transport term, but smaller and of opposite sign. This is not true, however, near boundaries, which have a strong effect on pressure. With $T'_{jki} = T^{(u)}_{jki} + T^{(p)}_{jki}$, a typical model for T'_{jki} is (Daley and Harlow, 1970):

$$T'_{jki} = -C_s \frac{k}{\epsilon} \langle u_i u_\ell \rangle \frac{\partial}{\partial x_\ell} \langle u_i u_k \rangle \, .$$

Note that the turbulent viscosity in this expression is given by $\nu_{ij} = C_s (k/\epsilon) \langle u_j u_\ell \rangle$, and is now non-isotropic, thought to be an important feature. The model also retains the same tensor symmetries as does $T^{(u)}_{jki}$ and $T^{(p)}_{jki}$.

5 Kinetic energy dissipation-rate equation

In the k-ϵ model, the equation for the kinetic energy dissipation rate ϵ is, with $\nu_T = C_\mu k^2/\epsilon$:

$$\frac{\bar{D}}{Dt}\epsilon = \underbrace{\frac{\partial}{\partial x_i}\Big(\frac{\nu_T}{\sigma_\epsilon} \frac{\partial}{\partial x_i} \epsilon \Big)}_{\text{turbulent transport}} + \underbrace{C_{\epsilon 1} \frac{\mathcal{P}\epsilon}{k}}_{\text{production}} - \underbrace{C_{\epsilon 2} \frac{\epsilon^2}{k}}_{\text{dissipation rate}} \, .$$

In Reynolds stress modeling, with $\langle u_i u_j \rangle$ now being directly computed instead of being modeled, this is slightly generalized to be (with non-isotropic diffusivitiy):

$$\frac{\bar{D}}{Dt}\epsilon = \frac{\partial}{\partial x_i} \Big[C_\epsilon \frac{k}{\epsilon} \langle u_i u_j \rangle \frac{\partial}{\partial x_j} \epsilon \Big] + C_{\epsilon 1} \frac{\mathcal{P}\epsilon}{k} - C_{\epsilon 2} \frac{\epsilon^2}{k} \, .$$

Here $\mathcal{P} = (1/2)\mathcal{P}_{ii}$ is directly computed from $\langle u_i u_j \rangle$ and $\langle U_i \rangle$ (no further modeling is needed), and the diffusion coefficent is again non-isotropic. The new constant is usually given as $C_\epsilon = 0.15$.

6 Algebraic stress models

Recall that a major weakness in the turbulence viscosity modeling is the assumption that $a_{ij} = \langle u_i u_j \rangle - (2/3)k\delta_{ij}$ is a linear, isotropic function of $\bar{S}_{ij}$. On the other hand the problem with Reynolds stress modeling is the size and complexity inherent in the simulations. Both of these weaknesses can be avoided, to some extent, by algebraic stress models. This approach starts from the Reynolds stress model equations

$$\mathcal{D}_{ij} \equiv \underbrace{\frac{\bar{D}}{Dt}\langle u_i u_j \rangle + \frac{\partial}{\partial x_k} T_{kij}}_{\text{differential terms}} = \underbrace{\mathcal{P}_{ij} + \mathcal{R}_{ij} - \frac{2}{3}\epsilon\delta_{ij}}_{\text{algebraic terms}} \,. \tag{26}$$

Note that

$$\frac{1}{2}\mathcal{D}_{\ell\ell} = \mathcal{P} - \epsilon \left(= \frac{\bar{D}}{Dt}k + \frac{\partial}{\partial x_\ell}\big[\langle u_\ell \frac{u_i^2}{2}\rangle + \frac{1}{\rho}\langle p u_\ell \rangle\big]\right) . \tag{27}$$

Furthermore, consider

$$\frac{\bar{D}}{Dt}\langle u_i u_j \rangle = k\frac{\bar{D}}{Dt}\Big(\frac{\langle u_i u_j \rangle}{k}\Big) + \frac{\langle u_i u_j \rangle}{k}\frac{\bar{D}}{Dt}k \quad \text{(exact)}.$$

If it is assumed that $\langle u_i u_j \rangle$ changes mainly due to changes in k, and not due to changes in the normalized $\dfrac{\langle u_i u_j \rangle}{k}$, a 'weak equilibrium' assumption, then

$$\frac{\bar{D}}{Dt}\langle u_i u_j \rangle \doteq \frac{\langle u_i u_j \rangle}{k}\frac{\bar{D}}{Dt}k \,.$$

A similar argument can be made for the diffusion term, when a turbulence model is used for it. The result is that Equation (26) can be approximated (modeled) as:

$$\mathcal{D}_{ij} \doteq \frac{\langle u_i u_j \rangle}{k}\Big\{\frac{\bar{D}}{Dt}k - \frac{\partial}{\partial x_k}\Big(\frac{C_s k^2}{\epsilon}\frac{\langle u_k u_\ell \rangle}{k}\frac{\partial}{\partial x_\ell}k\Big\} \doteq \mathcal{P}_{ij} + \mathcal{R}_{ij} - \frac{2}{3}\epsilon\delta_{ij} \,. \tag{28}$$

But from Equation (27), the term in brackets on the LHS of Equation (28) is just $\mathcal{P} - \epsilon$. So finally we have

$$\frac{\langle u_i u_j \rangle}{k}(\mathcal{P} - \epsilon) = \mathcal{P}_{ij} + \mathcal{R}_{ij} - \frac{2}{3}\epsilon\delta_{ij} \,. \tag{29}$$

With appropriate models for $\mathcal{R}_{ij}$, these give 5 algebraic equations for $\langle u_i u_j \rangle$. In addition, equations need to be solved for k and ϵ. Note that the value of k gives the needed sixth component of $\langle u_i u_j \rangle$.

Note that for homogeneous shear flow, and introducing the Rotta and Naot et al. models for $\mathcal{R}_{ij}$, then (see Problem 11.20, page 424 of the text),

$$b_{ij} = \frac{\langle u_i u_j \rangle}{2k} - \frac{1}{3}\delta_{ij} = \frac{\frac{1}{2}(1 - C_2)}{C_R - 1 + \mathcal{P}/\epsilon} \cdot \frac{\mathcal{P}_{ij} - \frac{2}{3}\mathcal{P}\delta_{ij}}{\epsilon} \,. \tag{30}$$

Noting that $\mathcal{P}_{ij} = -\langle u_i u_k \rangle \dfrac{\partial}{\partial x_k}\langle U_j \rangle - \langle u_j u_k \rangle \dfrac{\partial}{\partial x_k}\langle U_i \rangle$, then Equation (30) gives a very complex, algebraic set of equations relating $\langle u_i u_j \rangle$ to $\bar{S}_{ij}$. Furthermore, for a simple shear flow, $\langle \mathbf{U} \rangle = [\langle U_1 \rangle (x_2), 0, 0]$, with $\langle u_1 u_2 \rangle = -C_\mu \dfrac{k^2}{\epsilon}\dfrac{\partial}{\partial x_2}\langle U_1 \rangle$, then the $\langle u_1 u_2 \rangle$ component of Equation (30) can be solved for C_μ, giving:

$$C_\mu = \frac{\frac{2}{3}(1 - C_2)(C_R - 1 + C_2(\mathcal{P}/\epsilon))}{(C_R - 1 + (\mathcal{P}/\epsilon))^2} \,,$$

suggesting that C_μ is not constant, and has a strong dependence of $(\mathcal{P}/\epsilon)$.

Algebraic stress models are sometimes used as a compromise between k-ϵ and Reynolds stress models. These models result in nonlinear, non-isotropic relationships between $\langle u_i u_j \rangle$ and $\bar{S}_{ij}$.

7 Some Alternative Approaches

There are a number of other approaches to developing modeling for the Reynolds stress equations. Some of these are the following.

- Harlow (Los Alamos) One of the earliest modelers using Reynolds stress equations, very good work.

- Mellor & Yamada (1974, *J. Atmos. Sci.*, **31**:1791; 1982, *Rev. Geophys. Space Phys.*, **20**:851). Includes the effects of buoyancy and the rotation of the earth. Equations are included for the potential temperature Θ, with $\Theta = \langle \Theta \rangle + \theta$, i.e.,

$$\rho \frac{\bar{D}}{Dt}\langle \Theta \rangle = \frac{\partial}{\partial x_k}\Big(-\rho \langle u_k \theta \rangle\Big)$$

$$\frac{\bar{D}}{Dt}\langle \theta^2 \rangle = \text{etc.},$$

  along with a buoyancy term in the vertical component of the momentum equation. This approach was originally developed for the atmospheric boundary layer, and includes a hierarchy of models; the user chooses the appropriate one.

  - Level 4 full Reynolds stress model (semi-systematic approach); expand equations in terms of b_{ij}, neglect non-isotropic parts. No equation for ϵ, but equations for $\langle u^2 \rangle L$, $\langle u_j \theta \rangle$, and $\langle \theta^2 \rangle$.
  - Level 3 Algebraic stress model for $\langle u_i u_j \rangle$ and $\langle u_j \theta \rangle$, some additional simplifying assumptions, equations for k, θ^2.
  - Level 2 1/2 the most popular; same as Level 3, but algebraic equation for $\langle \theta^2 \rangle$, scalar production and dissipation in balance.
  - Level 2 Sames as Level 2 1/2, but algebraic equation for k, kinetic energy production and dissipation rate are in balance.
  - Level 1 not in use?

- Lumley introduced a systematic procedure, following the approach used in continuum mechanics; functional expansions, neglect higher order terms in (i) anisotropy, and (ii) ratios of length, time scales.

- RNG by Yakhot and Orszag; originally RNG stood for Renormalization Group, which it is not; the initials have been kept, and the approach adds some terms to k-ϵ and Reynolds stress equation modeling which can be useful.

- Most modeling approaches produce similar results, which Lumley has interpreted as good.