1 Introduction

This section is meant to be a supplement to Chapter 10 of the text, but not a replacement for it. Turbulent viscosity modeling is the approached most used in simulating turbulence, especially in applications.

The notation used in the text, and in ME 543, will be followed. For example,

- $\mathbf{U} = [U_1, U_2, U_3]$, the Eulerian velocity
- $\langle \mathbf{U} \rangle$, the average of $\mathbf{U}$, either ensemble, space or time average
- $\mathbf{u} = \mathbf{U} - \langle \mathbf{U} \rangle$, fluctuation about the average; Reynolds decomposition

1.1 The closure problem

Consider a flow satisfying the following incompressible form of the Navier-Stokes equations:

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2}{\partial x_j^2} U_i \,, \; i = 1, 2, 3 \tag{1}$$

$$\frac{\partial U_i}{\partial x_i} = 0 \,. \tag{2}$$

We have 4 equations for the 4 unknowns p, $\mathbf{U}$. But because of the random, chaotic nature of turbulence, to make predictions we usually need to resort to averaging. The approach with no modeling, direct numerical simulation, can only be applied to a restrictive set of problems with moderate Reynolds numbers, reaction rates, etc., and simpler geometries. We will take the average of Equations (1) and (2), and use the following identity:

$$\left\langle U_j \frac{\partial U_i}{\partial x_j} \right\rangle = \left\langle \frac{\partial}{\partial x_j}(U_i U_j) \right\rangle = \left\langle \frac{\partial}{\partial x_j}\Big[(\langle U_j \rangle + u_j)(\langle U_i \rangle + u_i)\Big] \right\rangle = \langle U_j \rangle \frac{\partial}{\partial x_j}\langle U_i \rangle + \frac{\partial}{\partial x_j}\langle u_i u_j \rangle \,. \tag{3}$$

In Equation (3), Equation (2) was used, along with the property that the averaging operator and derivates commute. Taking the average of Equation (1) gives, using Equation (3):

$$\frac{\partial}{\partial t}\langle U_i \rangle + \langle U_j \rangle \frac{\partial}{\partial x_j}\langle U_i \rangle = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\langle p \rangle + \nu \frac{\partial^2}{\partial x_j^2}\langle U_i \rangle - \frac{\partial}{\partial x_j}\langle u_i u_j \rangle \,. \tag{4}$$

Averaging Equation (2) gives

$$\frac{\partial}{\partial x_i}\langle U_i \rangle = 0 \,. \tag{5}$$

After averaging Equations (1) and (2) to obtain Equations (4) and (5), we have again 4 equations, but now we have in general 10 unknows, i.e., $\langle \mathbf{U} \rangle$, $\langle p \rangle$, and $\langle u_i u_j \rangle$, since $\langle u_i u_j \rangle$ has in general 6 independent components. This is the first sign of the closure problem, i.e., we now have more unknowns than equations, and is due to the nonlinearity of the equations. In general problems involving nonlinear stochastic equations (of which Equations (1) and (2) are an example) have no deductive solutions except for the following two special cases.

- the process is Markovian. This implies that the stochastic process has a very limited type of memory, which turbulence does not have. But some models, e.g., PDF models, assume a Markovian process when, for example, using Brownian motion as an example of turbulent transport.

- weak nonlinearities. Small amplitudes are assumed, and perturbation methods are employed. This is especially useful in considering, e.g., weakly nonlinear waves. Turbulence, however, is a strongly nonlinear process, and such perturbation methods are not of use.

The Reynolds stresses, $-\rho\langle u_i u_j\rangle$ are the new unknowns. (Sometimes $\langle u_i u_j\rangle$ are called the Reynolds stresses, which will often be done here.) The question becomes, how do we deal with the Reynolds stresses? There are two general approaches to handle this.

- One approach is to derive an equation for the $\langle u_i u_j\rangle$. This approach is discussed in Chapter 11 of the text. Unfortunately, developing the equations for the $\langle u_i u_j\rangle$ leads to many more unknown, e.g., $\langle u_i u_j u_k\rangle$, $\left\langle p\dfrac{\partial u_i}{\partial x_j}\right\rangle$.

- The second approach is to assume a functional relationship between the $\langle u_i u_j\rangle$ and the mean strain rate, $\bar{S}_{ij} = \dfrac{1}{2}\left\{\dfrac{\partial\langle U_i\rangle}{\partial x_j} + \dfrac{\partial\langle U_j\rangle}{\partial x_i}\right\}$. This is the area of turbulent viscosity modeling. We will see from examining the equation for $\langle u_i u_j\rangle$ that the terms that generate the Reynolds stress contain $\bar{S}_{ij}$, motivating this choice.

2 Turbulence Viscosity Modeling

It will be useful to work with the deviatoric, anisotropic form of the Reynolds stresses, defined by

$$a_{ij} = \langle u_i u_j\rangle - \frac{2}{3}k\delta_{ij}\,, \text{ with } k = \frac{1}{2}\langle u_k u_k\rangle\,, \text{ the turbulent kinetic energy.} \tag{6}$$

Note that, for isotropic turbulence, with $\langle u_1^2\rangle = \langle u_2^2\rangle = \langle u_3^2\rangle$, and $\langle u_1 u_2\rangle = \langle u_1 u_3\rangle = \langle u_2 u_3\rangle = 0$, then

$$\langle u_i u_j\rangle = \langle u_1^2\rangle\delta_{ij} = \frac{1}{3}\langle u_k u_k\rangle\delta_{ij} = \frac{2}{3}k\delta_{ij}\,.$$

Therefore, from the definition of a_{ij}, Equation (6), $a_{ij} = 0$ for each i, j for isotropic turbulence. So a non-zero component of a_{ij} corresponds to a non-isotropic component of $\langle u_i u_j\rangle$. Furthermore, the trace of a_{ij} is given by

$$a_{ii} = \langle u_i u_i\rangle - \frac{2}{3}k\underbrace{\delta_{ii}}_{=3} = 2k - 2k = 0\,.$$

A second order tensor which has 0 trace is referred to as deviatoric. Finally the normalized anisotropic form of the Reynolds stresses is defined as:

$$b_{ij} = \frac{a_{ij}}{2k} = \frac{\langle u_i u_j\rangle}{\langle u_k u_k\rangle} - \frac{1}{3}\delta_{ij}\,. \tag{7}$$

Note that writing $\langle u_i u_j\rangle$ in the mean momentum equation (Equation (4)) in terms of a_{ij} gives the result:

$$\frac{\partial}{\partial t}\langle U_i\rangle + \langle U_j\rangle\frac{\partial}{\partial x_j}\langle U_i\rangle = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\Big[\langle p\rangle + \frac{2}{3}\rho k\Big] + \nu\frac{\partial^2}{\partial x_j^2}\langle U_i\rangle - \frac{\partial}{\partial x_j}a_{ij}\,. \tag{8}$$

The role of $(2/3)\rho k$ is to modify the mean pressure $\langle p \rangle$. Also, if $a_{ij} = 0$ for all i, j, there would be no affect of the Reynolds stresses on $\langle U_i \rangle$. The mathematical problem would be the same as that for U_i itself, but with a modified pressure. It is clear that a_{ij} represents the effect part of the Reynolds stresses.

2.1 Closure assumptions

The general statement to close the equations is that a_{ij} is a known functional of $\bar{S}_{ij}$ (an approach taken in continuum mechanics). This implies that $a_{ij}(\mathbf{x}, t)$ could depend upon:

- the past history of $\bar{S}_{ij}$, as well as upon
- the values of $\bar{S}_{ij}$ in the vicinity of $\mathbf{x}$ (not only on $\mathbf{x}$ itself).

Some people would say that if this functional relationship could be determined, then the closure problem would be solved.

Note that it might be thought that a_{ij} could also depend on the mean vorticity tensor, $\bar{\Omega}_{ij} = \frac{1}{2}\left\{ \frac{\partial \langle U_i \rangle}{\partial x_j} - \frac{\partial \langle U_j \rangle}{\partial x_i} \right\}$. For the fluid stresses τ_{ij}, this is thought to not be the case, based upon the *principle of material frame indifference*. For turbulence, however, it is thought that a_{ij} should depend in general upon $\bar{\Omega}_{ij}$, since larger-scale fluid rotation can affect the turbulence and hence the Reynolds stresses. This point will come up again in Chapter 11.

To close the equations, and produce a relationship between a_{ij} and $\bar{S}_{ij}$, three main *ad hoc* (unjustifiable) assumptions are made.

1. $a_{ij}(\mathbf{x}, t)$ depends only on the values of $\bar{S}_{ij}$ at $(\mathbf{x}, t)$. This assumes that the turbulence has no memory, e.g., it does not act like a visco-elastic fluid. It also assumes that there is no local spatial influence. This assumption results in $a_{ij}(\mathbf{x}, t) = \mathcal{F}[\bar{S}_{ij}(\mathbf{x}, t)]$, where $\mathcal{F}(\cdot)$ is a function to be determined.

2. a_{ij} is a linear, isotropic function of $\bar{S}_{ij}$, i.e.,

$$a_{ij}(\mathbf{x}, t) = -2\nu_T \bar{S}_{ij}(\mathbf{x}, t)\,. \tag{9}$$

   Here ν_T is call the turbulent viscosity, and needs to be determined.

3. Various assumed methods to determine ν_T. We will consider each of these points separately. (We will find in Chapter 11 that similar assumptions are used to model new terms which appear in the equation for $\langle u_i u_j \rangle$, for example $\langle u_i u_j u_k \rangle$.)

2.1.1 Assumption 1: $a_{ij}(\mathbf{x}, t) = \mathcal{F}[\bar{S}_{ij}(\mathbf{x}, t)]$.

One might think that $a_{ij}(\mathbf{x}, t)$ would depend on the past history of $\bar{S}_{ij}$. For example, in analogy with a visco-elastic fluid with memory, a_{ij} would be expected to have a form of the following type:

$$a_{ij}(\mathbf{x}, t) = \mathcal{C} \int_{-\infty}^{t} \exp[-b(t-\tau)] \bar{S}_{ij}(\mathbf{x}, t-\tau) d\tau\,,$$

where $\mathcal{C}$ and b are constants. It is assumed, however, that this is not the case.

To understand the implications of this assumption, consider the dynamic equations for the $\langle u_i u_j \rangle$, which can be derived in the following manner (this will be discussed in more depth in Chapter 11; see also Problems 7.23, 7.24, and 7.25, page 319 in the text). Writing Equation (1) in

terms of the Reynolds decompositions of $\mathbf{U}$ and p, and subtracting Equation (4) gives the following equation for $\mathbf{u}$:

$$\frac{\partial}{\partial t}u_i + u_j\frac{\partial}{\partial x_j}\langle U_i\rangle + \langle U_j\rangle\frac{\partial}{\partial x_j}u_i + u_j\frac{\partial}{\partial x_j}u_i = -\frac{1}{\rho}\frac{\partial}{\partial x_i}p + \nu\frac{\partial^2}{\partial x_j^2}u_i + \frac{\partial}{\partial x_j}\langle u_i u_j\rangle\,. \tag{10}$$

Writing this equation for u_k, multiplying the resulting equation by u_i, multiplying Equation (10) by u_k, and adding the resulting two equations gives the following:

$$\underbrace{\frac{\partial}{\partial t}\langle u_i u_k\rangle + \langle U_j\rangle\frac{\partial}{\partial x_j}\langle u_i u_k\rangle}_{\text{time rate of change of } \langle u_i u_k\rangle \text{ following mean flow}} = \underbrace{-\langle u_i u_j\rangle\frac{\partial\langle U_k\rangle}{\partial x_j} - \langle u_k u_j\rangle\frac{\partial\langle U_i\rangle}{\partial x_j}}_{\text{production}} - \underbrace{\frac{\partial}{\partial x_j}\langle u_j u_i u_k\rangle}_{\text{turbulent diffusion}}$$

$$- \underbrace{\frac{1}{\rho}\left[\left\langle u_k\frac{\partial p}{\partial x_i}\right\rangle + \left\langle u_i\frac{\partial p}{\partial x_k}\right\rangle\right]}_{\text{velocity/pressure gradient correlation}} + \underbrace{\nu\frac{\partial^2}{\partial x_j^2}\langle u_i u_k\rangle}_{\text{molecular diffusion}} - \underbrace{2\nu\left\langle\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_j}\right\rangle}_{\text{dissipation rate}}\,. \tag{11}$$

In deriving this, the conservation of mass for u_i has been used. Furthermore, it can be easily shown that:

$$\left\langle u_i\frac{\partial^2}{\partial x_j^2}u_k + u_k\frac{\partial^2}{\partial x_j^2}u_i\right\rangle = \nu\frac{\partial^2}{\partial x_j^2}\langle u_i u_k\rangle - 2\nu\left\langle\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_j}\right\rangle.$$

In examining Equation (11), it is seen that the generation of $\langle u_i u_k\rangle$ is due to terms such as $\langle u_i u_j\rangle\dfrac{\partial\langle U_k\rangle}{\partial x_j}$, where $\dfrac{\partial\langle U_k\rangle}{\partial x_j}$ can be written in terms of $\bar{S}_{ij}$ and $\bar{\Omega}_{ij}$ as $\dfrac{\partial\langle U_k\rangle}{\partial x_j} = \bar{S}_{ij} + \bar{\Omega}_{ij}$. This provides some motivation for relating $\langle u_i u_j\rangle$ to $\bar{S}_{ij}$ (and possibly also to $\bar{\Omega}_{ij}$).

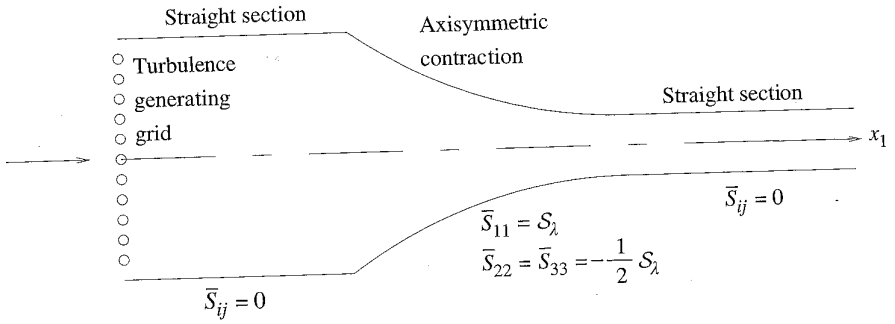

Fig. 10.1. A sketch of an apparatus, similar to that used by Uberoi (1956) and Tucker (1970), to study the effect of axisymmetric mean straining on grid turbulence.

Figure 1: Figure 10.1, page 360.

Figure 1: Axisymmetric contraction, taken from the text.

Now consider the flow through an axisymmetric contraction, as shown in Figure 1 taken from the text. In the argument presented here, a statistically planar flow is assumed for simplicity. Before entering the contraction, the turbulent flow is approximately homogeneous and isotropic, so

that a_{ij} and $\bar{S}_{ij}$ are both 0 for each component i, j. In the contraction, Equation (11) simplifies to:

$$\Big[\langle U_1\rangle\frac{\partial}{\partial x_1} + \langle U_2\rangle\frac{\partial}{\partial x_2}\Big]\langle u_1^2\rangle = -2\langle u_1^2\rangle \underbrace{\frac{\partial\langle U_1\rangle}{\partial x_1}}_{\alpha>0} - 2\langle u_1 u_2\rangle\frac{\partial\langle U_1\rangle}{\partial x_2} + \dots \tag{12}$$

$$\Big[\langle U_1\rangle\frac{\partial}{\partial x_1} + \langle U_2\rangle\frac{\partial}{\partial x_2}\Big]\langle u_2^2\rangle = -2\langle u_2^2\rangle \underbrace{\frac{\partial\langle U_2\rangle}{\partial x_2}}_{-\alpha<0} - 2\langle u_1 u_2\rangle\frac{\partial\langle U_2\rangle}{\partial x_1} + \dots \tag{13}$$

Since $\langle u_1 u_2\rangle$ is 0 entering the contraction, and $\langle U_2\rangle$ would be small compared to $\langle U_1\rangle$, then it is expected that the second terms on the LHS of each equation, and the second terms are the RHS of each equation should be small compared to the first terms. Furthermore, $\dfrac{\partial\langle U_1\rangle}{\partial x_1} = \alpha = -\dfrac{\partial\langle U_2\rangle}{\partial x_2}$ using Equation (5), where α is a positive constant. Therefore, $\langle u_1^2\rangle$ would be expected to decrease in the contraction, and $\langle u_2^2\rangle$ would be expected to increase. As the flow enters the contraction, a_{11} is 0. Since, from Equation (6) defining a_{ij},

$$a_{11} = \langle u_1^2\rangle - \frac{1}{3}\Big[\langle u_1^2\rangle + \langle u_2^2\rangle + \langle u_3^2\rangle\Big].$$

So as $\langle u_1^2\rangle$ decreases and $\langle u_2^2\rangle$ increases, a_{11} would be expected to which become negative. Furthermore, a_{22}, which also enters the contraction as 0, would be expected to increase, becoming positive.

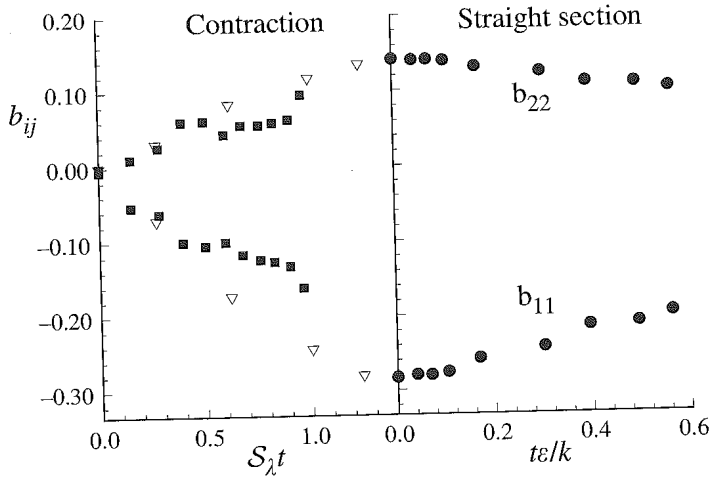

Fig. 10.2. Reynolds-stress anisotropies during and after axisymmetric straining. Contraction: experimental data of Tucker (1970), $\mathcal{S}_\lambda k/\varepsilon = 2.1$; △ DNS data of Lee and Reynolds (1985), $\mathcal{S}_\lambda k/\varepsilon = 55.7$; the flight time t from the beginning of the contraction is normalized by the mean strain rate $\mathcal{S}_\lambda$. Straight section: experimental data of Warhaft (1980); the flight time from the beginning of the straight section is normalized by the turbulence timescale there.

Figure 2: Figure 10.2, page 361.

Figure 2: Flow through the axisymmetric contraction and then the straight section.

It is seen in Figure 2, which contains results from both laboratory experiments and direct numerical simulations, that as the flow enters the contraction section, both $b_{11} = a_{11}/2k$ and $b_{22} = a_{22}/2k$ are 0 as expected. Then as the flow moves through the contraction, b_{11} becomes negative and b_{22} becomes positive as discussed above. As the flow enters the straight section after the contraction, $\bar{S}_{ij}$ becomes 0 again. From the turbulent viscosity model, it would then be expected that $a_{ij} = -\nu_T \bar{S}_{ij}$ would also be 0. In the straight section, however, b_{11} and b_{22} only slowly relax back towards 0, through both viscous dissipation and intercomponent energy transfer, which will be discussed in Chapter 11: the turbulent exhibits memory of the upstream flow.

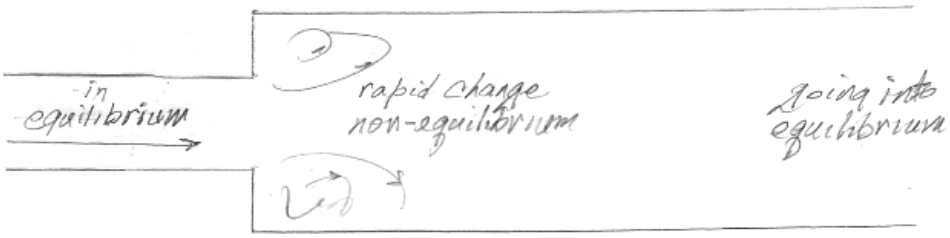

Figure 3: Flow through a sudden contraction.

Clearly turbulence can have some 'memory', and Assumption 1 will be a good approximation only in special circumstances. (We will see that the k-ϵ modeling will bring in some aspects of memory into the modeling, but not into the relationship between a_{ij} and $\bar{S}_{ij}$.) In some free shear flows, for example the jets, wakes, and shear layers discussed in Chapter 5, away from the near field $\langle \mathbf{U} \rangle$ is approximately self-similar, or 'in equilibrium', and memory effects are not important. On the other hand for the flow through a sudden expansion (see the sketch in Figure 3) or a sudden contraction, where the mean flow is changing very rapidly, the effects of memory may be very important. In such flows the turbulence relaxes back to equilibrium much more slowly than does the mean flow.

2.1.2 Assumption 2. $a_{ij}(\mathbf{x}, t)$ is a linear isotropic function of $\bar{S}(\mathbf{x}, t)$, i.e.,

$$a_{ij}(\mathbf{x}, t) = -2\nu_T \bar{S}_{ij}(\mathbf{x}, t)\,. \tag{14}$$

Here the constant of proportionality, ν_T , is the turbulence viscosity. (Note that nonlinear functions of $\bar{S}_{ij}$, as well as dependence on $\bar{\Omega}_{ij}$, will be discussed later in the course, in particular in discussing algebraic stress models.)

Note that for simple gases, the kinetic theory of gases can be used to derive a relationship between the stress tensor τ_{ij} and the rate-of-strain S_{ij}. The assumptions used to derive this are:

1. $\lambda/\ell \ll 1$, where λ is the mean free path of the molecules, and ℓ is a length scale of the motion. For example, for a boundary layer flow ℓ could be taken to be the boundary layer thickness. In most cases of interest, $\lambda/\ell \sim 10^{-8}$ or less, so that the assumption is justified. The assumption is related to the continuum approximation.

2. $\dfrac{\lambda/a}{T} \ll 1$, where a is the rms velocity of the molecules, related to the sound speed, λ/a is a time scale associated with molecular interactions, and T is a time scale of the fluid motions. The molecular interactions are what is causing τ_{ij}. Again this assumption is justified for most flows of interest.

Now consider this argument applied to turbulence to determine $\langle u_i u_j \rangle$, or a_{ij}, as a function of $\bar{S}_{ij}$. Here the turbulence itself is causing the stresses $\langle u_i u_j \rangle$. The length scale of the turbulence interactions, say L, is related to the integral scale of the turbulence. With ℓ taken to be a length scale of the mean flow, then usually $L \sim \ell$, so that the first assumption is not valid for turbulence. A similar argument for the time scales shows that the second assumption is also not valid for turbulence. Taken together these imply that there is not a theoretical justification for assuming that a_{ij} should be a linear function of $\bar{S}_{ij}$.

The most general linear relationship between τ_{ij} and S_{ij} is, for a fluid:

$$\tau_{ij} = B_{ijk\ell} S_{k\ell}$$

where a sum on (k, ℓ) is implied, and B is a fourth-order tensor. Most common fluids are isotropic, i.e., there are no preferred directions in the molecular interactions, i.e., the same relationship should hold irregardless of rotations of the reference frame. In this case, using the most general, isotropic form for B, the result is

$$\tau_{ij} = 2\mu S_{ij}$$

for the viscous part of the stress for a incompressible fluid.

Referring back to Equation (14), this implies that a_{ij} and -$2\nu_T \bar{S}_{ij}$ have the same tensorial properties. Note that both a_{ij} and $\bar{S}_{ij}$ are symmetric tensors (matrices). One property of a symmetric tensor is that it can be diagonalized in its principal axis frame of reference. From Problem 4.5, page 60 in the text, it was found that the principal axis of a mean flow with shear, i.e., $\dfrac{\partial \langle U_1 \rangle}{\partial x_2} \neq 0$ is 45^o. It was also found, however, that the principal axis of the Reynolds stresses, and therefore a_{ij}, was 22^o. So unfortunately the principal axes of a_{ij} and $\bar{S}_{ij}$ do not align, another weakness in the assumption for a_{ij}, Equation (14).

2.1.3 Assumption 3. The determination of ν_T.

The modeling of ν_T generally uses the analogy between turbulence and the kinetic theory of gases. The kinetic theory predicts that, for the kinematic viscosity of a fluid, $\nu = c_1 a \lambda$, where a is again the speed of sound in the gas, λ is the mean free path between molecules, and c_1 is a constant of order 1. In analogy with this, it is assumed that the turbulent viscosity is given by

$$\nu_T = c_2 \mathcal{U} \mathcal{L} \,. \tag{15}$$

The issue then is how to determine the constant c_2 and the velocity scale $\mathcal{U}$ and length scale $\mathcal{L}$. There are a number of methods to do this. Note that the kinematic viscosity ν is a property of the fluid. It is a thermodynamic variable, and for example expressed as $\nu = \nu(T, p)$, where T is the temperature and p the pressure. On the other hand ν_T is a property of the flow, and is expected to vary in space and time, even if T and p are constant.

Scheme 1. For 'equilibrium' shear flows, e.g., approximately self-similar jets, wakes, mixing layers and boundary layers, $\mathcal{U}$ and $\mathcal{L}$ are determined from the mean velocity $\langle \mathbf{U} \rangle$ itself. For example, for an axisymmetric jet (see Figure 4), the velocity scale $\mathcal{U}$ at a downstream location x can be defined as the the peak mean velocity at that position, i.e, $\mathcal{U} = \langle U \rangle|_{\max}$. On the other hand the length scale $\mathcal{L}$ at a downstream location x is often defined as the half-width of the jet, $\mathcal{L} = \delta(x)$, where $\delta(x)$ is defined by $\langle U \rangle(x, \delta(x)) = (1/2)\mathcal{U}$. The constant c_2 is determined for each flow by fitting the predictions with the data. Note that, for these models for equilibrium flows:

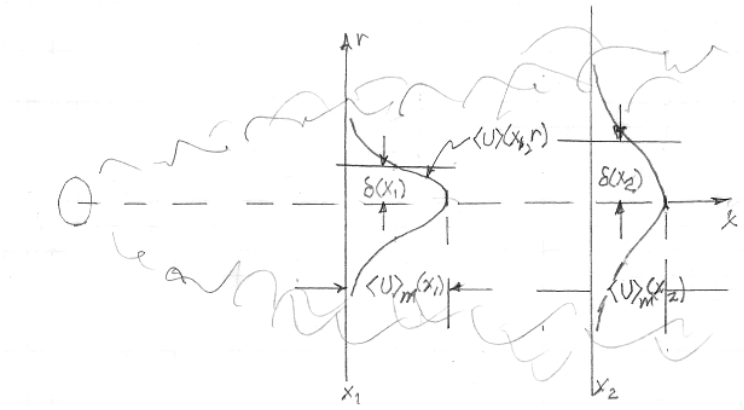

Figure 4: Sketch of an axisymmetric turbulent jet.

- the model predicts the equilibrium flows pretty well, if the constant c_2 is adjusted for each case.
- In the self-similar regime, the flows are in 'equilibrium'.
- There is usually only one key component of both a_{ij} and $\bar{S}_{ij}$. For example, for an axisymmetric jet, the non-zero components of a_{ij} are a_{xr} and $\bar{S}_{xr}$.
- The turbulent viscosity ν_T is independent of y or r, depending only on x.
- There is not generality to the model; it is only useful for very simple flows.

Scheme 2. The next more complex modeling uses a 'mixing length', say ℓ_m, often used for simpler flows, such as boundary layer flows. Here $\mathcal{L} = \ell_m$ is the length scale used and $\mathcal{U} = \ell_m \left| \dfrac{\partial \langle U \rangle}{\partial y} \right|$ is the velocity scale, which can depend on the local coordinates. So $\nu_T = c_3 \ell_m \cdot \ell_m \left| \dfrac{\partial \langle U \rangle}{\partial y} \right|$, where c_3 is a constant determined by fitting the computational results with data. Now ℓ_m needs to be specified, but not $\mathcal{U}$. For example, when used in the boundary layer, often $\ell_m \propto y$, the distance from the boundary. This approach works well in boundary layers which are not too complicated, e.g., no acceleration or deceleration, no density stratification nor rotation. Also there can be a problem when $\dfrac{\partial \langle U \rangle}{\partial y} = 0$, since then $\nu_T = 0$, which is in error.

Scheme 3. Kinetic energy equation modeling. In this approach, the equation for the turbulent kinetic energy is solved, and the velocity scale is defined by $\mathcal{U} = c_4 k^{1/2}$, where k is the turbulent kinetic energy. This is sometimes called a 'one-equation model'. Still ℓ_m needs to be specified, for example as a mixing length. The equation for the turbulent kinetic energy can be obtained from the equation for the Reynolds stresses, Equation (11) by setting $k = i$, summing on i, and dividing

by 2. The result is:

$$\underbrace{\frac{\partial}{\partial t}k + \langle U_j\rangle\frac{\partial}{\partial x_j}k}_{\substack{\text{time rate of change of } k\\ \text{following mean flow}}} = -\underbrace{\langle u_iu_j\rangle\bar{S}_{ij}}_{\text{production}} - \underbrace{\frac{\partial}{\partial x_j}\langle\Big(\frac{1}{2}\langle u_i^2u_j\rangle\Big)}_{\text{turbulent diffusion}}$$

$$-\underbrace{\frac{1}{\rho}\frac{\partial}{\partial x_i}\langle pu_i\rangle}_{\text{pressure diffusion}} + \underbrace{2\nu\frac{\partial}{\partial x_j}\langle u_is_{ij}\rangle}_{\text{viscous diffusion}} - \underbrace{2\nu\langle s_{ij}s_{ij}\rangle}_{\substack{\text{viscous dissipation}\\ \text{rate}}}\,, \tag{16}$$

where $s_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)$, the strain-rate of the fluctuating velocity, and the fact that $\bar{\Omega}_{ij}\langle u_iu_j\rangle = 0$ was utilized in writing the production term.

Now $\mathcal{L}$ still needs to be specified, e.g., as a mixing length ℓ_m, and $\mathcal{U}$ is defined as $c_4k^{1/2}$, so that $\nu_T = c_4k^{1/2}\ell_m$. Then $\langle u_iu_j\rangle$ can be determined from the turbulence viscosity model, k can be determined by solving the turbulence kinetic energy equation, and $\langle\mathbf{U}\rangle$ and $\langle p\rangle$ can be found by solving the mean momentum balance and mean continuity equations. But there is now a need to model the turbulent diffusion term and the turbulent dissipation rate term. They are usually treated as follows.

I. The transport terms are usually lumped together and modeled as:

$$-\frac{1}{2}\langle u_i^2u_j\rangle - \frac{1}{\rho}\langle pu_j\rangle + 2\nu\langle u_is_{ij}\rangle = -\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial x_j}\,.$$

This modeling is very similar to the modeling for a_{ij}, and contains all the same reservations. The new parameter σ_k is the turbulent Prandtl number for k and is defined by $\sigma_k = \nu_T/D_k$, where D_k is the turbulent diffusivity of k. So $\sigma_k = \dfrac{\text{turbulent viscosity}}{\text{turbulent diffusivity}}$. Note that:

- this is a gradient diffusion assumption, i.e., it corresponds to a flux of k down the gradient;
- the turbulent Prandtl numbers are all expected to be near 1, since the turbulent diffusion of any quantity (e.g., heat, kinetic energy, chemical species, etc.) is expected to occur at about the same rate as the turbulent diffusion of momentum (given in terms of ν_T). It is the same mechanism, the turbulent eddies, that does the diffusing.

II. The dissipation rate, $\epsilon = 2\nu\langle s_{ij}s_{ij}\rangle$ in the turbulent kinetic energy equation is usually modeled as:

$$\epsilon = c_5\frac{k}{\ell_m/k} = c_5\frac{k^{3/2}}{\ell_m}\,,$$

with $T = \ell_m/k^{1/2}$ the time scale of the turbulence. This modeling is constructed so that $\epsilon \sim u^3/\ell$, as discussed in ME 543.

Note that, with this one equation model, k can now, to some extent, evolve on its own time scale. It is not completely tied to $\langle\mathbf{U}\rangle$, as in previous models. So this model for Scheme 3 generally works better than that for Scheme 2, the mixing length model. But still ℓ_m has to be specified.

Scheme 4. k-ϵ modeling, a two equation model.

In this approach, $\mathcal{U}$ is found from k through solving the equation for the turbulence kinetic energy, as in Scheme 3, but $\mathcal{L}$ is determined from k and ϵ as:

$$\mathcal{L} = k^{3/2}/\epsilon\,,$$

so that the model for the turbulent viscosity is:

$$\nu_T = C_\mu k^2/\epsilon\,.$$

What is needed now is an equation for ϵ. This can be accomplished in several ways. The first is to formally derive the equation for ϵ from the equation for u_i, starting by taking the gradients of the equation for u_i, then forming the equation for s_{ij}, multiplying this equation by s_{ij}, and taking the average. This approach is very complex, but straight-forward, and leads to many terms to be modeled. A second approach is to write the equation for ϵ as:

$$\Big[\frac{\partial}{\partial t} + \langle U_j\rangle\frac{\partial}{\partial x_j}\Big]\epsilon = \left\{\begin{matrix}\text{turbulent}\\ \text{diffusion}\end{matrix}\right\} + \{\text{production}\} - \left\{\begin{matrix}\text{dissipation}\\ \text{rate of }\epsilon\end{matrix}\right\}.$$

Then models need to be proposed for the three terms on the right-hand side of the equation. Another alternative is to use the fact that, for high Reynolds number flows, $\epsilon \doteq \nu\langle\omega_i^2\rangle$ (see the notes from ME 543), with $\langle\omega_i^2\rangle$ the mean square of the fluctuating vorticity, which is accurate to $\mathcal{O}(R_\ell^{1/2})$. So we can use the equation for the mean square vorticity, multiplied by ν. The result is:

$$\begin{aligned}\Big[\frac{\partial}{\partial t} + \langle U_j\rangle\frac{\partial}{\partial x_j}\Big]\epsilon = &- \underbrace{\frac{\partial}{\partial x_j}\langle u_j\nu\omega_i^2\rangle}_{\text{turbulent diffusion}}\\ + \underbrace{\langle s_{ij}\nu\omega_i\omega_j\rangle}_{\substack{\text{production}\\ \text{stretching}}} &- \underbrace{\nu^2\Big\langle\Big(\frac{\partial\omega_i}{\partial x_j}\Big)\Big\rangle}_{\text{dissipation rate}} + \underbrace{\mathcal{O}\Big(\frac{1}{R_\ell^{1/2}}\Big)}_{\substack{\text{approximation}\\ \text{error}}}\quad,\end{aligned} \tag{17}$$

where terms of the order $R_\ell^{-1/2}$ and higher have been neglected.

For any of these cases, the modeling is:

$$\text{turbulent diffusion of }\epsilon\text{:}\ \ -\langle u_j\nu\omega_i^2\rangle = \frac{\nu_T}{\sigma_\epsilon}\frac{\partial}{\partial x_j}\epsilon\,,\ \text{and}$$

$$\text{dissipation rate of }\epsilon\text{:}\ \ \nu^2\Big\langle\Big(\frac{\partial x_i}{\partial x_j}\Big)^2\Big\rangle = -C_{\epsilon 2}\frac{\epsilon^2}{k}\,.$$

Here σ_ϵ is the turbulent Prandtl number for ϵ, and $T = k/\epsilon$ is taken to be the turbulence time scale, so the dissipation rate of ϵ is proportional to $\epsilon/T = \epsilon^2/k$. Finally, the production term is modeled as:

$$\text{turbulence production of }\epsilon\text{:}\ \ \langle s_{ij}\nu\omega_i\omega_j\rangle = C_{\epsilon 1}\frac{\mathcal{P}\epsilon}{k}\,,$$

where $\mathcal{P} = -\langle u_iu_j\rangle\dfrac{\partial\langle U_i\rangle}{\partial x_j}$, the production term in the equation for k. So the production of the dissipation rate is taken to be proportional to $\mathcal{P}/T$, that is, proportional to the 'rate of change of $\mathcal{P}$'. This is difficult to justify, except possibly through intuitive arguments and that fact that it is dimensionally correct. The results of the turbulent viscosity assumption for ν_T, and the modeled equations for k and for ϵ, lead to the following closed set of equations.

$$\frac{\partial}{\partial t}\langle U_i\rangle + \langle U_j\rangle\frac{\partial}{\partial x_j}\langle U_i\rangle = -\frac{1}{\rho}\frac{\partial}{\partial x_i}\langle P\rangle + \nu\frac{\partial^2}{\partial x_j^2}\langle U_i\rangle - \frac{\partial}{\partial x_j}\langle u_iu_j\rangle \tag{18}$$

$$\frac{\partial}{\partial x_i}\langle U_i\rangle = 0 \tag{19}$$

$$\langle u_i u_j\rangle = \frac{2}{3}k\,\delta_{ij} - 2\nu_T\overline{S}_{ij} \tag{20}$$

$$\overline{S}_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial x_j}\langle U_i\rangle + \frac{\partial}{\partial x_i}\langle U_j\rangle\right) \tag{21}$$

$$\nu_T = C_\mu k^2/\epsilon \tag{22}$$

$$\frac{\partial}{\partial t}k + \langle U_j\rangle\frac{\partial}{\partial x_j}k = -\langle u_i u_j\rangle\frac{\partial}{\partial x_j}\langle U_i\rangle + \nu\frac{\partial^2}{\partial x_j^2}k + \frac{\partial}{\partial x_j}\left(\frac{\nu_T}{\sigma_k}\frac{\partial}{\partial x_j}k\right) - \epsilon \tag{23}$$

$$\frac{\partial}{\partial t}\epsilon + \langle U_j\rangle\frac{\partial}{\partial x_j}\epsilon = -C_{\epsilon 1}(\epsilon/k)\langle u_i u_j\rangle\frac{\partial}{\partial x_j}\langle U_i\rangle + \frac{\partial}{\partial x_j}\left(\frac{\nu_T}{\sigma_\epsilon}\frac{\partial}{\partial x_j}\epsilon\right) - C_{\epsilon 2}\epsilon^2/k \tag{24}$$

$$C_\mu \doteq 0.09 \quad C_{\epsilon 1} \doteq 1.44 \quad C_{\epsilon 2} \doteq 1.92 \quad \sigma_k \doteq 1.0 \quad \sigma_\epsilon \doteq 1.3 \tag{25}$$

Note that, as written, Equation (23) implies that the kinetic energy flux is modeled as:

$$-\frac{1}{2}\langle u_i^2 u_j\rangle - \frac{1}{\rho}\langle p u_j\rangle + 2\nu\langle u_i s_{ij}\rangle - \nu\frac{\partial}{\partial x_j}k = -\frac{\nu_T}{\sigma_k}\frac{\partial}{\partial x_j}k\,. \tag{26}$$

This implies that the term $+\nu\dfrac{\partial}{\partial x_j}k$ has been added to the right side of Equation (23). In the text by Pope, and in several other texts, the term $\nu\dfrac{\partial}{\partial x_j}k$ is not included in Equation (26), so that the third-to-the-last term on the right-hand side of Equation (23) is not included as well.

Also note the following:

- The values for 5 constants are needed, the standard values of which are given above. These are 'optimized' to give better agreement with a range of problems. For a specific problem, however, more 'optimized' values can often be found.
- It is assumed (hoped) that, once determined, the same values apply to all flows. This is not the case, however, as one might expect from a discussion of the modeling weaknesses (and a homework problem).
- There are a number of nonlinear coefficients with divisions by unknowns, e.g., k^2/ϵ, ϵ/k, and ϵ^2/k. These can cause significant mathematical and numerical difficulties. In numerical simulation, much slower convergence is found compared to the simulation of comparable laminar flow problems.
- The turbulence viscosity ν_T now adjusts on the turbulence fluctuation time scale, i.e., that of k and ϵ, and not necessarily on the time scale of $\langle U_i\rangle$.
- However, $\langle u_i u_j\rangle$ is still directly linked to $\bar{S}_{ij}$. This implies, for example, that the problem of homogeneous strain, where $\bar{S}_{ij}$ and $\langle u_i u_j\rangle$ have clearly different principal axes, cannot be modeled properly.

It is straight forward to extend the k-ϵ modeling to passive scalars, e.g., C, whose concentration equation is, after averaging:

$$\frac{\partial}{\partial t}\langle C\rangle + \langle U_j\rangle \frac{\partial}{\partial x_j}\langle C\rangle = \mathcal{D}_c \frac{\partial^2}{\partial x_j^2}\langle C\rangle - \frac{\partial}{\partial x_j}\langle u_j c\rangle\,, \tag{27}$$

using the Reynolds decomposition for C as $C = \langle C\rangle + c$, and $\mathcal{D}_c$ is the molecular diffusivity of C. The turbulence scalar flux term $\langle u_j c\rangle$ is usually modeled as:

$$\langle u_j c\rangle = -\frac{\nu_T}{\sigma_c}\frac{\partial}{\partial x_j}\langle C\rangle\,,$$

where σ_c is the turbulence Prandtl number of C, and assumed to be $\mathcal{O}(1)$.

The weakest points in the k-ϵ modeling are generally considered to be:

- the turbulence viscosity assumptions, for the reasons already discussed, and
- the equation for ϵ, since the modeling of the various terms in the equation for ϵ are not well undersood.

Other turbulence viscosity, two-equation models

There are several two-equation models, with the k-ϵ and the k-ω being the ones most often used. The two-equation models usually retain the equation for k, but then include another equation to give a length or a time scale. For example,

$$\nu_T = c\mathcal{U}\mathcal{L} = c\mathcal{U}(\mathcal{U}\mathcal{T}) = c\mathcal{U}^2\mathcal{T} = \hat{c}\mathcal{U}^2\frac{1}{\omega}\,,$$

where the time scale $\mathcal{T}$ defined as $\mathcal{T} = k/\epsilon$, as in the k-ϵ approach, or in terms of ω as $\mathcal{T} = 1/\omega$ in the k-ω model. Another approach is to define $\mathcal{L}$ as an integral scale, $\mathcal{U}$ as an rms turbulence velocity $\sqrt{\langle u^2\rangle}$, and to derive an equation for $\langle u^2\rangle\mathcal{L}$.

The k-ϵ model (and other related two-equation models) is the simplest 'complete' model, i.e., once the model constants are determined, nothing is in needed to be specified to close the mathematical problem. It is 'fairly' accurate for simpler flows, such as jets and wakes with no strong swirling, boundary layers with no strong acceleration or decelerations, etc., so that no new physics is brought in that is not included in the modeling. It often gives the proper trends, even if somewhat inaccurate. The better the user understands the strengths and weaknesses of the models, the better the use is made of it.

2.2 Setting the model constants

To set the model constants, for specific flows where accurate data (either laboratory or DNS data) are available, simulations are run with the model, the results compared with data, and the constants are adjusted to give the best comparisons of the simulation output with the data. For the k-ϵ model there are 5 constants which are not predicted, but need to be set by comparisons with data. If the modeling were 'exact', the constants could be set once and for all time, and then the model applied to any flow satisfying the assumptions made in the modeling (e.g., incompressible flow). In reality, the values of the constants usually suggested represent a compromise among the comparisons with various flows. For a particular flow, the agreement with data can probably be improved by 'optimizing' the constants for that flow.

In setting the constants, it is usual to start with simpler flows, for which only one or two constants are involved, fixing those constants, and then proceeding to more complex flows, involving more constants. Consider the following sequence.

Homogeneous, turbulence decay. This flow has been studied extensively, and several good sets of laboratory (and now DNS) data are available. For this flow, $\langle U_i \rangle = \text{constant}$, $\langle P \rangle = \text{constant}$, and $\dfrac{\partial}{\partial x_i}\langle \cdot \rangle = 0$, so that the averaged momentum and continuity equations reduce to just $0 = 0$, and do not provide any need for prediction. The k-ϵ model for the k and ϵ equations reduce to, in a coordinate system moving with the mean flow:

$$\frac{dk}{dt} = -\epsilon \,, \text{ and} \tag{28}$$

$$\frac{d\epsilon}{dt} = -C_{\epsilon 2}\frac{\epsilon^2}{k} \,. \tag{29}$$

Therefore, comparison of a prediction with data for this flow should produce a value of the constant $C_{\epsilon 2}$. In the laboratory experiments it is found that, approximately, both k and ϵ exhibit power law decay, i.e.,

$$k(t) = k_0\Big(\frac{t}{t_0}\Big)^{-n} \,,\; \epsilon(t) = \epsilon_0\Big(\frac{t}{t_0}\Big)^{-m} \,.$$

Equations (28) and (29) are satisfied when $m = n + 1$, and $C_{\epsilon 2} = \dfrac{n+1}{n}$, so that $C_{\epsilon 2}$ can be set by determining n. From laboratory data (Comte-Bellot and Corrsin, *J. Fluid Mech.*, **48**(2):273-337, 1971, page 284) , $n \doteq 1.25$, so that $C_{\epsilon 2} \doteq 1.80$, somewhat lower than the standard value of 1.92.

Homogeneous turbulent shear flow. This flow is the next in the order of complexity (in the text see section 5.4.5, page 154 and following). The mean velocity now has one non-zero component, say $\langle U_1 \rangle$, which varies linearly in the x_2 direction, say. So $\langle U_1 \rangle = \dfrac{d\langle U_1 \rangle}{dx_2} x_2$, where $\dfrac{d\langle U_1 \rangle}{dx_2} = \text{constant}$. It can be shown that such a mean flow can be consistent with homogeneity in all 3 directions. Laboratory experiments and direct numerical simulations have both been carried out for this flow.

For this homogeneous shear flow, the mean momentum equation reduces to

$$\frac{\partial}{\partial t}\langle U_1 \rangle = 0 \,, \tag{30}$$

and the mean flow is independent of x_1, x_3, and t. Defining $S = \dfrac{d\langle U_1 \rangle}{dx_2}$, then the mean strain-rate tensor is:

$$\bar{S}_{ij} = \begin{bmatrix} 0 & \frac{1}{2}S & 0 \\ \frac{1}{2}S & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \,.$$

With $a_{ij} = -2\nu_T \bar{S}_{ij}$, then

$$\langle u_1 u_2 \rangle = -2\nu_T \bar{S}_{12} = -2\nu_T \frac{1}{2} S = -C_\mu \frac{k^2}{\epsilon} S \,, \text{ so}$$

$$\frac{dk}{dt} = -\underbrace{\langle u_1 u_2 \rangle S}_{\langle u_1 u_2 \rangle \bar{S}_{12} + \langle u_1 u_2 \rangle \bar{S}_{21}} - \epsilon = C_\mu \frac{k^2 S^2}{\epsilon} - \epsilon \,, \text{ and}$$

$$\frac{d\epsilon}{dt} = -C_{\epsilon 1}\langle u_1 u_2\rangle S\frac{\epsilon}{k} - C_{\epsilon 2}\frac{\epsilon^2}{k} = C_\mu C_{\epsilon 1}\frac{k^2}{\epsilon}S^2\frac{\epsilon}{k} - C_{\epsilon 2}\frac{\epsilon^2}{k} = C_\mu C_{\epsilon 1} k S^2 - C_{\epsilon 2}\frac{\epsilon^2}{k}\,.$$

Given $C_{\epsilon 2}$ from the comparisons for homogeneous decay, then comparisons of model predictions with data for this case can be used to set the constants C_μ and $C_{\epsilon 1}$.

Even the simple case of homogeneous shear can reveal some of the strengths and weaknesses of the k-ϵ model. Consider again the assumption given by Equation (14), i.e,

$$a_{ij} = -2\nu_T \bar{S}_{ij}\,.$$

From laboratory experiments (Champagne et al., *J. Fluid Mech.*, **41**:81, 1970), the principal axes of $\bar{S}_{ij}$ in both the experiment and in the k-ϵ model are $\pm 45^o$. The principal axes of $\langle u_i u_j\rangle$ in one laboratory experiment were $(28^o, 62^o)$, and in the other $(20^o, 70^o)$. The principal axes of $\langle u_i u_j\rangle$ in the k-ϵ model are, because of the proportionality to $\bar{S}_{ij}$, again $\pm 45^o$. In addition, in the model, $\langle u_i u_j\rangle = (2/3)k\delta_{ij} - 2\nu_T \bar{S}_{ij}$, so that

$$\langle u_1^2\rangle = (2/3)k\delta_{11} - 2\nu_T \underbrace{\bar{S}_{11}}_{=0} = (2/3)k = \langle u_2^2\rangle = \langle u_3^2\rangle\,.$$

So for the model,

$$\langle u_1^2\rangle/\langle u_2^2\rangle = \langle u_1^2\rangle/\langle u_3^2\rangle = 1\,,$$

while from the laboratory data,

$$\langle u_1^2\rangle/\langle u_2^2\rangle \doteq 3\,,\ \langle u_1^2\rangle/\langle u_3^2\rangle \doteq 2\,.$$

Note that the k-ϵ modeling results lead to, using the standard values for the constants (see Equation (10.62), page 377 in the text),

$$\frac{\mathcal{P}}{\epsilon} \doteq \frac{C_{\epsilon 2} - 1}{C_{\epsilon 1} - 1} \doteq 2.1\,,$$

whereas from the laboratory data of Tavoularis and Corrsin, p. 334,

$$\frac{\mathcal{P}}{\epsilon} = \frac{\dfrac{\langle u_1 u_2\rangle}{u_1' u_2'} u_1' u_2' \dfrac{d\langle U_1\rangle}{dx_2}}{\epsilon} = \frac{0.45\sqrt{0.475}\sqrt{0.165}}{3.42} 48.6 = 1.79\,,$$

so that the model is about 11% in error.

These results suggest that the k-ϵ modeling can be useful, but one cannot expect the results to be very accurate. The modeling can be especially useful in predicting trends (e.g., what happens if the flow rate is increased, etc.), and to make qualitative and semi-quantitative estimates.

2.3 k-ϵ approaches avoiding the turbulence viscosity assumption

There are two approaches where the k and ϵ equations are used, but the simple turbulence viscosity relationship $a_{ij} \propto \bar{S}_{ij}$ is avoided. These are the following.

- Algebraic Reynolds stress models (see section 11.9.1, page 448 of the text). In these, the closure assumption is a system of algebraic equations relating a_{ij}, $\bar{S}_{ij}$, $\bar{\Omega}_{ij}$, k, and ϵ. They are obtained by brutally truncating the modeling equations for the Reynolds stresses.

- Nonlinear models (see section 11.9.2, page 452). Various possible relationships between a_{ij}, $\bar{S}_{ij}$ and $\bar{\Omega}_{ij}$ are developed, while maintaining the deviatoric and symmetric properties of a_{ij}.

These approaches will be discussed at the end of the chapter on the modeling of the Reynolds stress equations.