Fluid Turbulence

Problem 1. Problem 10.8, page 378 in text.

- Begin with the k-ω equations given by, for homogeneous turbulence,

$$\frac{dk}{dt} = P - \epsilon \tag{1}$$

$$\frac{d\omega^2}{dt} = -\alpha\omega\left(\omega^2 - \frac{S^2}{\beta^2}\right) \tag{2}$$

where $\omega \equiv \epsilon/k$, and

$$P = -\langle uv\rangle\frac{\partial}{\partial y}\langle U\rangle = -\left(-\nu_T\frac{\partial}{\partial y}\langle U\rangle\right)\frac{\partial}{\partial y}\langle U\rangle = C_\mu\frac{k^2}{\epsilon}S^2\,, \text{ with } S \equiv \frac{\partial}{\partial y}\langle U\rangle\,.$$

Equation (2) can be rewritten as

$$\frac{d\omega}{dt} = -\frac{\alpha}{2}\left(\omega^2 - \frac{S^2}{\beta^2}\right) \tag{3}$$

With $\omega = \epsilon/k$, then Equation (3) becomes

$$\frac{1}{k}\frac{d\epsilon}{dt} - \frac{\epsilon}{k^2}\frac{dk}{dt} = -\frac{\alpha}{2}\left(\omega^2 - \frac{S^2}{\beta^2}\right)\,, \text{ or}$$

$$\frac{d\epsilon}{dt} = \frac{\epsilon}{k}\frac{dk}{dt} - \frac{\alpha}{2}\left(\frac{\epsilon^2}{k^2} - \frac{S^2}{\beta^2}\right)k\,. \tag{4}$$

Plugging Equation (1) into Equation (4) then gives

$$\begin{aligned}\frac{d\epsilon}{dt} &= (P-\epsilon)\frac{\epsilon}{k} - \frac{\alpha}{2}\frac{\epsilon^2}{k} + \frac{\alpha}{2\beta^2 C_\mu}\underbrace{C_\mu\frac{k^2}{\epsilon}S^2}_{P}\frac{\epsilon}{k}\\ &= \left(1 + \frac{\alpha}{2\beta^2 C_\mu}\right)\frac{P\epsilon}{k} - \left(1 + \frac{\alpha}{2}\right)\frac{\epsilon^2}{k} \qquad (5)\end{aligned}$$

as desired.

- Comparing Equation (5) to the standard equation for ϵ, i.e., Equation (10.56) in the text,

$$\frac{d\epsilon}{dt} = C_{\epsilon 1}\frac{P\epsilon}{k} - C_{\epsilon 2}\frac{\epsilon^2}{k}\,, \text{ then}$$

$$C_{\epsilon 2} = \left(1 + \frac{\alpha}{2}\right)\,, \text{ or } \alpha = 2(C_{\epsilon 2} - 1) \doteq 1.84\,, \text{ and}$$

$$C_{\epsilon 1} = 1 + \frac{\alpha}{2\beta^2 C_\mu}\,, \text{ or } \beta = \left[\frac{\alpha}{2C_\mu(C_{\epsilon 1} - 1)}\right]^{1/2} = \left[\frac{C_{\epsilon 2} - 1}{C_\mu(C_{\epsilon 1} - 1)}\right]^{1/2} \doteq 4.27\,,$$

as desired.

Problem 2.

There are several ways to approach the problem of solving the equations and finding the optimum value for $C_{\epsilon 2}$. Here I am going to start by assuming power law solutions for $k(t)$ and $\epsilon(t)$, i.e.,

$$\frac{k}{k_0} = \left(\frac{t}{t_0}\right)^{-n} \qquad \frac{\epsilon}{\epsilon_0} = \left(\frac{t}{t_0}\right)^{-n-1} . \tag{6}$$

Note that, given the power -n for k, then the power for ϵ must be -n-1 to satisfy $dk/dt = \epsilon$. Here are a couple of approaches to find the optimum value for $C_{\epsilon 2}$.

1. Use the curve fit from Comte-Bellot and Corrsin. They found that (see Equation (38), page 284 of their paper): $n = 1.25$. Using Equation (10.60), page 376 of the text, then $C_{\epsilon 2} = \dfrac{n+1}{n} = 1.80$.

2. Use linear, least squares curve fit of the data. Taking the log of Equation (6) gives:

   $$\ln(k/k_0) = -n(t/t_0)\,,\ \ln(\epsilon/\epsilon_0) = -(n+1)\ln(t/t_0)\,.$$

   When the values for k, ϵ, and t from Table 4, page 299 of the paper are used, the following results are obtained using least squares fits:

   $M = 5.08$:

   $$\begin{aligned} \text{curve for } k\text{:} \quad n &= 1.2956,\,,\ C_{\epsilon 2} = 1.77 \\ \text{curve for } \epsilon\text{:} \quad n+1 &= 2.3597\,,\ C_{\epsilon 2} = 1.74\,. \end{aligned}$$

   $M = 2.54$:

   $$\begin{aligned} \text{curve for } k\text{:} \quad n &= 1.319\,,\ C_{\epsilon 2} = 1.758 \\ \text{curve for } \epsilon\text{:} \quad n+1 &= 2.3573\,,\ C_{\epsilon 2} = 1.74\,. \end{aligned}$$

   Note that these values for $C_{\epsilon 2}$ are somewhat less that the values obtained from the Comte-Bellot/Corrsin fit, and significantly below the standard value of 1.92. Some of the difference between these results and those of Comte-Bellot/Corrsin could be due to their use of a virtual origin (this could easily be checked), and a different non-dimensionalization for t.

3. Plotted curves.

   $M = 5.08$

   $$\begin{aligned} k &= k_0(t/t_0)^{-n_1}\,,\ k_0 = 7.39 \cdot 10^2 \mathrm{cm}^2/\mathrm{s}^2\,,\ U_0 t_0/M = 42\,,\ n_1 = 1.2956 \\ \epsilon &= \epsilon_0(t/t_0)^{-n_2}\,,\ \epsilon_0 = 4740 \mathrm{cm}^2/\mathrm{s}^3\,,\ n_2 = 2.3597 \end{aligned}$$

   $M = 2.54$

   $$\begin{aligned} k &= k_0(t/t_0)^{-n_1}\,,\ k_0 = 630 \mathrm{cm}^2/\mathrm{s}^2\,,\ U_0 t_0/M = 45\,, n_1 = 1.319 \\ \epsilon &= \epsilon_0(t/t_0)^{-n_2}\,,\ \epsilon_0 = 7540 \mathrm{cm}^2/\mathrm{s}^3\,,\ n_2 = 2.3573 \end{aligned}$$

   See the plots below.

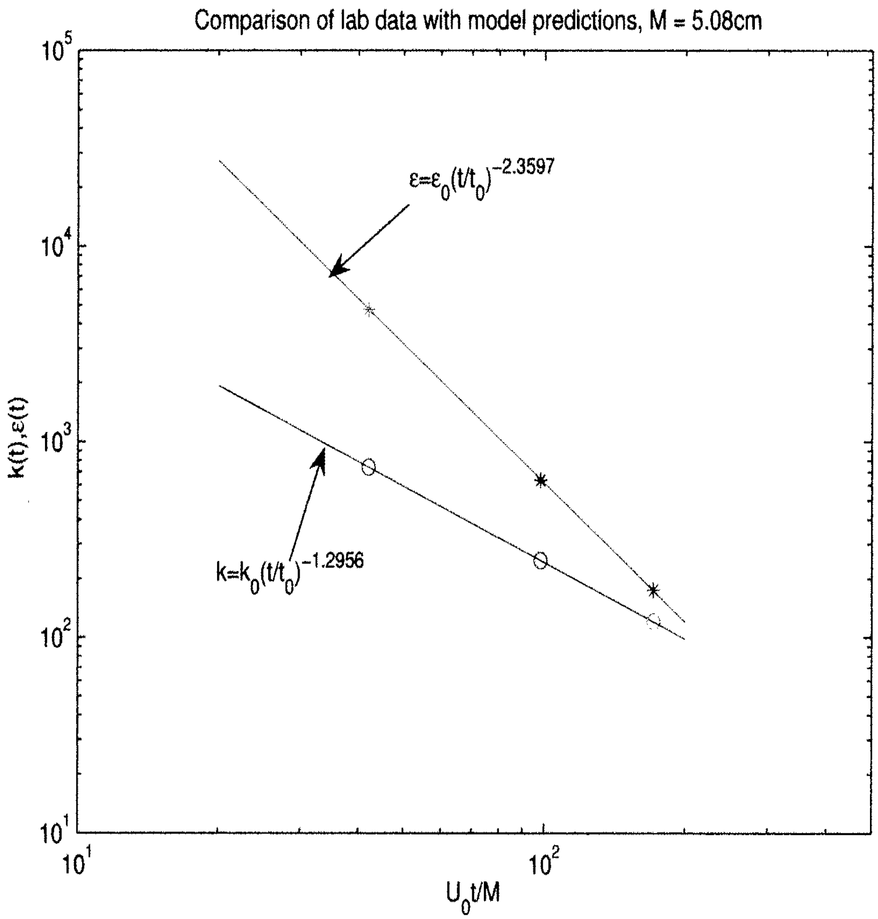

Figure 1: Comparison for $M = 5.08$ cm.

4. Some idea of the differences between the recommended value of $C_{\epsilon 2}$, 1.92, and the values obtained by curve fitting the data can be obtained by comparing the results obtained using the recommended values to the experimentally obtained values. Since $C_{\epsilon 2} = (n+1)/n$, then the recommended value for n would be

$$nC_{\epsilon 2} = n + 1\,, \text{ or } n = \frac{1}{C_{\epsilon 2} - 1} = \frac{1}{0.92} \doteq 1.087\,.$$

With this value of n, the predicted final values for the data are the following.

(a) For $M = 5.08\,\text{cm}$,

$$u_1'(171) = 22.2(171/42)^{-1.087/2} = 10.35 \text{ (compared to 8.95)}.$$

$$\epsilon(171) = 4740(171/42)^{-2.087} = 253.1 \text{ (compared to 174)}.$$

(b) For $M = 2.54\,\text{cm}$,

$$u_1'(385) = 20.5(385/45)^{-1.087/2} = 6.38 \text{ (compared to 5.03)}.$$

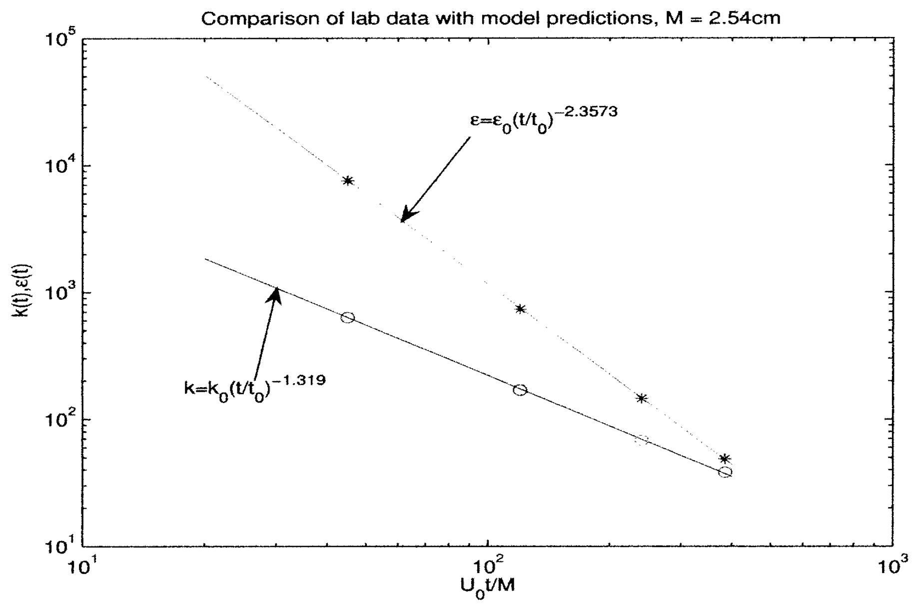

Figure 2: Comparison for $M = 2.54$cm.

$$\epsilon(385) = 7540(385/45)^{-2.087} = 85.5 \text{ (compared to 48.5).}$$

It appears that the recommended value for n predicts too high a value for for both u_1' and for ϵ. The value of n has probably been adjusted to better fit other data.