Advanced Fluid Turbulence

Problem 1. Problem 11.3, page 393 in the text.

1. Derive Equation (11.26) in the text for the case of homogeneous, anisotropic turbulence with no production.

   From the definition of b_{ij}, i.e., $b_{ij} \equiv \dfrac{\langle u_i u_j \rangle}{\langle u_i u_i \rangle} - \dfrac{1}{3}\delta_{ij} = \dfrac{a_{ij}}{2k}$, then

$$\langle u_i u_j \rangle = 2kb_{ij} + \frac{2}{3}k\delta_{ij} \, .$$

   Plugging this for $\langle u_i u_j \rangle$ into Equation (11.21) in the text, then

$$\frac{d}{dt}\{2kb_{ij} + \frac{2}{3}k\delta_{ij}\} = \mathcal{R}_{ij}^{(s)} - \epsilon_{ij} \text{ , or}$$

$$2b_{ij} \underbrace{\frac{dk}{dt}}_{-\epsilon} + 2k\frac{db_{ij}}{dt} + \frac{2}{3}\delta_{ij} \underbrace{\frac{dk}{dt}}_{-\epsilon} = \mathcal{R}_{ij}^{(s)} - \epsilon_{ij} \text{ , or}$$

$$-2\epsilon b_{ij} + 2k\frac{db_{ij}}{dt} - \frac{2}{3}\epsilon\delta_{ij} = \mathcal{R}_{ij}^{(s)} - \epsilon_{ij} \text{ , or}$$

$$\frac{db_{ij}}{dt} = \frac{1}{2k}\{2\epsilon b_{ij} + \frac{2}{3}\epsilon\delta_{ij} + \mathcal{R}_{ij}^{(s)} - \epsilon_{ij}\} \text{ , or, finally}$$

$$\frac{db_{ij}}{dt} = \frac{\epsilon}{k}\left\{b_{ij} + \frac{\mathcal{R}_{ij}^{(s)}}{2\epsilon} + \frac{1}{3}\delta_{ij} - \frac{\epsilon_{ij}}{2\epsilon}\right\} . \qquad (1)$$

2. Assuming the isotropic form for ϵ_{ij}, i.e., $\epsilon_{ij} = (2/3)\epsilon\delta_{ij}$, and using Equation (1), derive Equation (11.23) in the text.

   With the isotropic form for ϵ_{ij}, the Equation (1) becomes

$$\frac{db_{ij}}{dt} = \frac{\epsilon}{k}\left\{b_{ij} + \frac{\mathcal{R}_{ij}^{(s)}}{2\epsilon} + \frac{1}{3}\delta_{ij} - \underbrace{\frac{2}{3}\epsilon\delta_{ij}\frac{1}{2\epsilon}}_{\delta_{ij}/3}\right\} \text{, or}$$

$$\frac{db_{ij}}{dt} = \frac{\epsilon}{k}\left\{b_{ij} + \frac{\mathcal{R}_{ij}^{(s)}}{2\epsilon}\right\} .$$

3. If instead of isotropy, ϵ_{ij} is taken proportional to $\langle u_i u_j \rangle$, show that Equation (11.27) results.

   First properly normalize the proportionality as $\dfrac{\langle u_i u_j \rangle}{2k} = \dfrac{\epsilon_{ij}}{2\epsilon}$ since, upon contraction of i and j, $\dfrac{\langle u_i u_i \rangle}{2k} = 1$ and $\dfrac{\epsilon}{2\epsilon} = \dfrac{1}{2\epsilon}(2\tilde{\epsilon}) = \dfrac{\epsilon}{\epsilon} = 1$ since $\tilde{\epsilon} = \nu\langle \dfrac{\partial u_i}{\partial x_j}\dfrac{\partial u_i}{\partial x_j}\rangle = \epsilon$ for homogeneous flows. Then plugging this into Equation (1) gives

$$\frac{db_{ij}}{dt} = \frac{\epsilon}{k}\left\{\underbrace{b_{ij} + \frac{1}{3}\delta_{ij}}_{\langle u_i u_j \rangle/2k} + \frac{\mathcal{R}_{ij}^{(s)}}{2\epsilon} - \underbrace{\frac{\epsilon_{ij}}{2\epsilon}}_{\langle u_i u_j \rangle/2k}\right\} = \frac{\mathcal{R}_{ij}^{(s)}}{2\epsilon} \text{ , i.e.,}$$

$$\frac{db_{ij}}{dt} = \frac{\mathcal{R}^{(s)}_{ij}}{2\epsilon} \,. \tag{2}$$

4. Finally if Rotta's model, $\mathcal{R}^{(s)}_{ij} = -2C_R\,\epsilon\, b_{ij}$ is used in Equation (2), show that Equation (11.25) follows with $(C_R - 1)$ replaced by C_R. Plugging Rotta's model into Equation (2) gives

$$\frac{db_{ij}}{dt} = \frac{\mathcal{R}^{(s)}_{ij}}{2\epsilon} = -\frac{2C_R\,\epsilon\, b_{ij}}{2k} = -C_R\frac{\epsilon}{k}b_{ij} \text{ as desired.}$$

Problem 2. Problem 11.6, page 396 of the text.

1. Show that, according to Rotta's model, and assuming that the flow is statistically homogeneous with no mean shear nor mean strain, the invariants b^n_{ii} evolve as Equation (11.37) in the text, i.e., according to:

$$\frac{db^n_{ii}}{dt} = -n(C_R - 1)\frac{\epsilon}{k}b^n_{ii} \,. \tag{3}$$

The Rotta model is given by

$$\frac{db_{ij}}{dt} = -(C_R - 1)\frac{\epsilon}{k}b_{ij} \,. \tag{4}$$

We will prove the result by induction. Note that Equation (3) holds for $n = 1$ by taking $i = j$ in Equation (4) and summing. Next consider the equation for b^2_{ii}. Writing Equation (4) with $i \to j$, $j \to i$, then

$$\frac{db_{ji}}{dt} = -(C_R - 1)\frac{\epsilon}{k}b_{ji} \,. \tag{5}$$

Multiplying Equation (4) by b_{ji}, Equation (5) by b_{ij}, and adding gives:

$$b_{ij}\frac{db_{ji}}{dt} + b_{ji}\frac{db_{ij}}{dt} = \frac{d}{dt}(b_{ij}b_{ji}) = -(C_R - 1)\frac{\epsilon}{k}(b_{ij}b_{ji} + b_{ji}b_{ij}) \,,$$

or, with $b^2_{ii} = b_{ij}b_{ji}$,

$$\frac{db^2_{ii}}{dt} = -2(C_R - 1)\frac{\epsilon}{k}b^2_{ii} \,.$$

So Equation (3) also holds for $n = 2$.

Consider now the general case,

$$b^n_{ii} = \underbrace{b_{ij}}_{1}\ \underbrace{b_{jk}}_{2} \cdots \underbrace{b_{mi}}_{n} \,.$$

Taking the time derivative of this, using the chain rule n times on the right-hand side gives:

$$\begin{aligned}\frac{db^n_{ii}}{dt} &= \frac{d}{dt}(b_{ij}b_{jk}\cdots b_{mi}) \\ &= \left(\frac{db_{ij}}{dt}\right)b_{jk}\cdots b_{mi} + b_{ij}\left(\frac{db_{jk}}{dt}\right)\cdots b_{mi} + \cdots + b_{ij}b_{jk}\cdots\left(\frac{db_{mi}}{dt}\right).\end{aligned}$$

Finally, using Equation (4) n times gives Equation (3) as desired.

2. Show that ξ and η evolve as

$$\frac{d\xi}{dt} = -(C_R - 1)\frac{\epsilon}{k}\xi \tag{6}$$

$$\frac{d\eta}{dt} = -(C_R - 1)\frac{\epsilon}{k}\eta \tag{7}$$

where $6\eta^2 = -2\Pi_b = b_{ii}^2 = b_{ij}b_{ji}$, and $6\xi^3 = b_{ii}^3 = b_{ij}b_{jk}b_{ki}$.

From Equation (3), using the definition of η,

$$\frac{d}{dt}6\eta^2 = -2(C_R - 1)\frac{\epsilon}{k}6\eta^2 \text{, or}$$

$$2\eta\frac{d\eta}{dt} - 2(C_R - 1)\frac{\epsilon}{k}\eta^2 \text{, or, dividing by } 2\eta$$

$$\frac{d\eta}{dt} = -(C_R - 1)\frac{\epsilon}{k}\eta \text{, as desired.} \tag{8}$$

Next, using the definition of ξ in Equation (3), using the definition of ξ gives:

$$\frac{d}{dt}6\xi^3 = -3(C_R - 1)\frac{\epsilon}{k}6\xi^3 \,.$$

$$\text{But } \frac{d}{dt}\xi^3 = 3\xi^2\frac{d\xi}{dt} \text{, so}$$

$$3\xi^2\frac{d\xi}{dt} = -3(C_R - 1)\frac{\epsilon}{k}\xi^3 \text{, or, dividing by } 3\xi^2,$$

$$\frac{d\xi}{dt} = -(C_R - 1)\frac{\epsilon}{k}\xi \text{, as desired.} \tag{9}$$

3. How does the ratio ξ/η evolve? Show that the trajectories in the ξ-η plane are straight lines directed towards the origin.

Note that $\frac{d}{dt}(\xi/\eta) = \frac{1}{\eta}\frac{d\xi}{dt} - \frac{\xi}{\eta^2}\frac{d\eta}{dt}$ Using Equations (8) and (9) in this equation gives

$$\begin{aligned}
\frac{d}{dt}(\xi/\eta) &= \frac{1}{\eta}\{-(C_R - 1)\frac{\epsilon}{k}\xi\} - \frac{\xi}{\eta^2}\{-(C_R - 1)\frac{\epsilon}{k}\eta\} \\
&= -(C_R - 1)\frac{\epsilon}{k}\frac{\xi}{\eta} + (C_R - 1)\frac{\xi}{\eta} = 0 \text{, i.e.,}
\end{aligned}$$

$$\frac{d}{dt}(\xi/\eta) = 0 \,.$$

So $\xi/\eta = \text{constant} = \xi_0/\eta_0$, say, and $\xi = (\xi_0/\eta_0)\eta$, a straight line through the origin. Assuming, for example, that initially ξ and η are both positive, then from Equations (6) and (7), with $C_R > 1$, both would be decreasing functions of time, so they would follow the straight line towards, not away from, the origin. The same can be proven if the initial conditions are in any of the other three quadrants of the (ξ, η) plane.