Advanced Fluid Turbulence

Problem 11.20, page 424 in the text.

1. Consider homogeneous shear flow at large times, where the Reynolds stress obeys [Equation (11.130)]
$$\frac{d}{dt}\langle u_i u_j\rangle = \frac{\langle u_i u_j\rangle}{k}\frac{dk}{dt} = \frac{\langle u_i u_j\rangle}{k}(\mathcal{P}-\epsilon)\,. \tag{1}$$
But also [Equation (11.128)],
$$\frac{d}{dt}\langle u_i u_j\rangle = \mathcal{P}_{ij} + \mathcal{R}_{ij} - \frac{2}{3}\epsilon\delta_{ij}\,, \tag{2}$$
where isotropic dissipation rate has been assumed for the third term on the right-hand side. Furthermore, modeling the pressure/strain-rate term as
$$\mathcal{R}_{ij} = -C_R\frac{\epsilon}{k}\left(\langle u_i u_j\rangle - \frac{2}{3}k\delta_{ij}\right) - C_2\left(\mathcal{P}_{ij} - \frac{2}{3}\mathcal{P}\delta_{ij}\right). \tag{3}$$
Prove Equations (11.131) and (11.132) in the text.

Combining Equations (1), (2), and (3), eliminating $\langle u_i u_j\rangle$,
$$\frac{\langle u_i u_j\rangle}{k}(\mathcal{P}-\epsilon) = \mathcal{P}_{ij} - C_R\frac{\epsilon}{k}\left(\langle u_i u_j\rangle - \frac{2}{3}k\delta_{ij}\right) - C_2\left(\mathcal{P}_{ij} - \frac{2}{3}\mathcal{P}\delta_{ij}\right) - \frac{2}{3}\epsilon\delta_{ij}\,,$$
or, solving for $\langle u_i u_j\rangle/k$,
$$\begin{aligned}\frac{\langle u_i u_j\rangle}{k}\{(\mathcal{P}-\epsilon)+C_R\,\epsilon\} &= \mathcal{P}_{ij} + \frac{2}{3}C_R\epsilon\delta_{ij} - C_2(\mathcal{P}_{ij} - \frac{2}{3}\mathcal{P}\delta_{ij}) - \frac{2}{3}\epsilon\,\delta_{ij}\\ &= (1-C_2)(\mathcal{P}_{ij} - \frac{2}{3}\mathcal{P}\delta_{ij}) + \frac{2}{3}\mathcal{P}\delta_{ij} + \frac{2}{3}(C_R-1)\epsilon\,\delta_{ij}\\ &= (1-C_2)(\mathcal{P}_{ij} - \frac{2}{3}\mathcal{P}\delta_{ij}) + \frac{2}{3}\delta_{ij}(\mathcal{P}-\epsilon+C_R\epsilon)\,.\end{aligned}$$
So solving for $\langle u_i u_j\rangle/k$,
$$\frac{\langle u_i u_j\rangle}{k} = \frac{1-C_R}{\mathcal{P}+(C_R-1)\epsilon}(\mathcal{P}_{ij} - \frac{2}{3}\mathcal{P}\delta_{ij}) + \frac{2}{3}\delta_{ij}\,,\ \text{or}$$
$$\frac{\langle u_i u_j\rangle}{k} = \frac{2}{3}\delta_{ij} + \Theta\frac{(\mathcal{P}_{ij} - \frac{2}{3}\mathcal{P}\delta_{ij})}{\mathcal{P}}\,,\ \text{with} \tag{4}$$
$$\Theta = \frac{(1-C_2)(\mathcal{P}/\epsilon)}{C_R - 1 + (\mathcal{P}/\epsilon)}\,.$$

2. Obtain the results for b_{ij} given by Equations (11.133) and (11.134).

From the definition of b_{ij} and using Equation (4),
$$b_{ij} = \frac{\langle u_i u_j\rangle}{2k} - \frac{1}{3}\delta_{ij} = \Theta\frac{(\mathcal{P}_{ij} - \frac{2}{3}\mathcal{P}\delta_{ij})}{2\mathcal{P}}\,,\ \text{so with}$$

$$\mathcal{P}_{ij} = -\langle u_i u_k \rangle \frac{\partial}{\partial x_k}\langle U_j \rangle - \langle u_j u_k \rangle \frac{\partial}{\partial x_k}\langle U_i \rangle \text{ , then}$$

$$\mathcal{P}_{11} = -\langle u_1 u_2 \rangle \frac{\partial}{\partial x_2}\langle U_1 \rangle - \langle u_1 u_2 \rangle \frac{\partial}{\partial x_2}\langle U_1 \rangle = -2\langle u_1 u_2 \rangle \frac{\partial}{\partial x_2}\langle U_1 \rangle$$

$$\mathcal{P}_{22} = \mathcal{P}_{33} = 0$$

$$\mathcal{P} = \frac{1}{2}\mathcal{P}_{ii} = \frac{1}{2}\mathcal{P}_{11} = -\langle u_1 u_2 \rangle \frac{\partial}{\partial x_2}\langle U_1 \rangle$$

$$\mathcal{P}_{12} = -\langle u_2^2 \rangle \frac{\partial}{\partial x_2}\langle U_1 \rangle$$

$$\mathcal{P}_{13} = \mathcal{P}_{23} = 0 \text{ , so}$$

$$b_{11} = \Theta \frac{\mathcal{P}_{11} - \frac{2}{3}\mathcal{P}}{2\mathcal{P}} = \Theta \frac{2\mathcal{P} - \frac{2}{3}\mathcal{P}}{2\mathcal{P}} = \frac{2}{3}\Theta$$

$$b_{22} = \Theta \frac{(0 - \frac{2}{3}\mathcal{P})}{2\mathcal{P}} = -\frac{1}{3}\Theta = b_{33}$$

$$b_{12} = \Theta \frac{(-\langle u_2^2 \rangle \frac{\partial}{\partial x_2}\langle U_1 \rangle - 0)}{2\mathcal{P}} = \Theta \frac{-\langle u_2^2 \rangle \frac{\partial}{\partial x_2}\langle U_1 \rangle}{-2\langle u_1 u_2 \rangle \frac{\partial}{\partial x_2}\langle U_1 \rangle} = \Theta \frac{\langle u_2^2 \rangle}{2\langle u_1 u_2 \rangle} \text{ .}$$

But $b_{12} = \dfrac{\langle u_1 u_2 \rangle}{2k}$, and $b_{22} = \dfrac{\langle u_2^2 \rangle}{2k} - \dfrac{1}{3} = -\dfrac{1}{3}\Theta$, so that $\langle u_2^2 \rangle = \frac{1}{3}(1-\Theta)2k$, and $\langle u_1 u_2 \rangle = 2kb_{12}$, so

$$b_{12} = \frac{\Theta \frac{2}{3}(1-\Theta)k}{4kb_{12}} \text{ , or}$$

$$b_{12}^2 = \frac{1}{6}\Theta(1-\Theta) \text{ , and}$$

$$b_{12} = \pm\sqrt{\frac{1}{6}\Theta(1-\Theta)} \text{ .}$$

Note that $\mathcal{P} > 0$ for turbulence to exist. Also, from the problem definition, $\dfrac{\partial}{\partial x_2}\langle U_1 \rangle > 0$. So with $\mathcal{P} = -\langle u_1 u_2 \rangle \dfrac{\partial}{\partial x_2}\langle U_1 \rangle > 0$, then $\langle u_1 u_2 \rangle < 0$. So, since $b_{12} = \langle u_1 u_2 \rangle / 2k$, the $b_{12} < 0$ and the minus sign is called for. So,

$$b_{12} = -\sqrt{\frac{1}{6}\Theta(1-\Theta)} \text{ .}$$

Problem 11.31, page 452.

For this case of homogeneous shear flow, we have the following relationships:

$$b_{12} = \frac{\langle u_1 u_2 \rangle}{2k} = -\sqrt{\frac{1}{6}\Theta(1-\Theta)} \text{ from Equation 11.134,} \quad (5)$$

$$\mathcal{P} = \frac{1}{2}\mathcal{P}_{ii} = -\langle u_1 u_2 \rangle \mathcal{S} = -2kb_{12}\mathcal{S} \text{ , with} \quad (6)$$

$$\mathcal{S} = \frac{\partial}{\partial x_2}\langle U_1 \rangle \text{ , and}$$

$$\Theta = \frac{(1 - C_2)\mathcal{P}/\epsilon}{C_R - 1 + \mathcal{P}/\epsilon}\,. \tag{7}$$

Furthermore, assume that

$$-\langle u_1 u_2 \rangle = \frac{C_\mu k^2}{\epsilon}\mathcal{S}\,. \tag{8}$$

Solving Equation (8) for C_μ, and using Equation (5) gives:

$$C_\mu = -\frac{\langle u_1 u_2 \rangle \epsilon}{k^2 \mathcal{S}} = -\frac{2kb_{12}\epsilon}{k^2 \mathcal{S}} = \frac{(2kb_{12})\epsilon(2kb_{12})}{k^2 \mathcal{P}} = \frac{4b_{12}^2 \epsilon}{\mathcal{P}} = \frac{4b_{12}^2}{\mathcal{P}/\epsilon} \text{ since} \tag{9}$$

$$\mathcal{S} = -\frac{\mathcal{P}}{2kb_{12}} \text{ from Equation (6).}$$

Finally, using Equation (5) in (9), with (7) gives:

$$\begin{aligned} C_\mu &= \frac{4b_{12}^2}{\mathcal{P}/\epsilon} = \frac{4}{\mathcal{P}/\epsilon}\frac{1}{6}\Theta(1 - \Theta) = \frac{2}{3}\frac{1}{\mathcal{P}/\epsilon} \cdot \frac{(1 - C_2)\mathcal{P}/\epsilon}{C_R - 1 + \mathcal{P}/\epsilon} \cdot \frac{(C_r - 1 + \mathcal{P}/\epsilon) - (1 - C_2)\mathcal{P}/\epsilon}{C_R - 1 + \mathcal{P}/\epsilon} \\ &= \frac{2}{3} \cdot \frac{(1 - C_2)(C_r - 1 + C_2 \mathcal{P}/\epsilon)}{(C_R - 1 + \mathcal{P}/\epsilon)^2}\,, \end{aligned} \tag{10}$$

as desired.