ME 544 Homework Solutions 1 May 2018

**Advanced Fluid Turbulence**

Problem 13.1, page 562 in the text.

- Consider: $\frac{\partial}{\partial t}\bar{\mathbf{U}}$:

$$\begin{aligned}\frac{\partial}{\partial t}\bar{\mathbf{U}} &= \frac{\partial}{\partial t}\int G(\mathbf{r},\mathbf{x})\mathbf{U}(\mathbf{x}-\mathbf{r},t)d\mathbf{r} \text{ (Leibnitz rule)} \quad (1)\\ &= \int G(\mathbf{r},\mathbf{x})\frac{\partial}{\partial t}\mathbf{U}(\mathbf{x}-\mathbf{r},t)d\mathbf{r} = \overline{\frac{\partial \mathbf{U}}{\partial t}}\,. \quad (2)\end{aligned}$$

  So the operations of filtering and differentiating with respect to time commute.

- Consider: $\overline{\langle \mathbf{U}\rangle}$:

$$\begin{aligned}\overline{\langle \mathbf{U}\rangle} &= \int G(\mathbf{r},\mathbf{x})\langle \mathbf{U}(\mathbf{x}-\mathbf{r},t)\rangle d\mathbf{r} \text{ (the average commutes with the integral)} \quad (3)\\ &= \left\langle \int G(\mathbf{r},\mathbf{x})\mathbf{U}(\mathbf{x}-\mathbf{r},t)d\mathbf{r}\right\rangle = \langle \bar{\mathbf{U}}\rangle\,. \quad (4)\end{aligned}$$

  Therefore the operation of probability averaging and filtering commute.

- Next consider: $\frac{\partial}{\partial x_j}\bar{U}_i$:

$$\begin{aligned}\frac{\partial}{\partial x_j}\bar{U}_i &= \frac{\partial}{\partial x_j}\int G(\mathbf{r},\mathbf{x})U_i(\mathbf{x}-\mathbf{r},t)d\mathbf{r} \quad (5)\\ &= \int G(\mathbf{r},\mathbf{x})\frac{\partial}{\partial (x_j-r_j)}U_i(\mathbf{x}-\mathbf{r},t)d\mathbf{r} - \int \frac{\partial G(\mathbf{r},\mathbf{x})}{\partial x_j}U_i(\mathbf{x}-\mathbf{r},t)d\mathbf{r} \quad (6)\\ &= \overline{\frac{\partial U_i}{\partial x_j}} + \int \frac{\partial G(\mathbf{r},\mathbf{x})}{\partial x_j}U_i(\mathbf{x}-\mathbf{r},t)d\mathbf{r}\,. \quad (7)\end{aligned}$$

  So the operations of filtering and differentiating with respect to space do not commute unless the filter is homogeneous (independent of $\mathbf{x}$).

Problem 13.7, page 570 in the text.

1. Show that Equation (13.39) in the text is satisfied.

   The two-point velocity correlation $R(r)$ is defined by

$$R(r) = \langle u(x+r)u(x)\rangle\,.$$

   The two-point correlation function of the filtered velocity is defined by

$$\begin{aligned}\bar{R}(r) &\equiv \left\langle \int_{-\infty}^{\infty} G(y)u(x+r-y)dy \int_{-\infty}^{\infty} G(z)u(x-z)dz\right\rangle\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(y)G(z)\langle u(x+r-y)u(x-z)\rangle dydz\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(y)G(z)R(r+z-y)dydz\end{aligned}$$

   using the definition of $R(r)$.

2. Show that the spectrum of $\bar{u}(x)$ can be written as in Equation (13.40) in the text.

   From its definition in terms of $\bar{R}(r)$, the filtered spectrum is given by

$$\begin{aligned}
\bar{E}_{11}(k) &\equiv \frac{1}{\pi}\int_{-\infty}^{\infty} \bar{R}(r)e^{-ikr}dr \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(y)G(z)R(r+z-y)\,dy\,dz\,e^{-ikr}dr \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(y)e^{-iky}G(z)e^{ikz}R(r+z-y)e^{-ik(r+z-y)}\,dy\,dz\,dr \qquad (8)
\end{aligned}$$

   as desired.

3. From the definition of the energy spectrum $E_{11}(k)$,

$$E_{11}(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} R(r)e^{-ikr}dr = \frac{1}{\pi}\int_{-\infty}^{\infty} R(s+z-y)e^{-ik(s+z-y)}ds \qquad (9)$$

   with the change of dummy variables of integration $r \to s+z-y$.

4. Finally, using Equation (9) in (8) gives:

$$\begin{aligned}
\bar{E}_{11}(k) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(y)e^{-iky}G(z)e^{ikz}R(r)e^{-ik(r)}\,dy\,dz\,dr \\
&= \frac{1}{\pi}\left\{\int_{-\infty}^{\infty} G(y)e^{-iky}dy\right\}\left\{\int_{-\infty}^{\infty} G(z)e^{ikz}dz\right\}\left\{\int_{-\infty}^{\infty} R(r)e^{-ikr}dr\right\} \\
&= \hat{G}(k)\hat{G}^*(k)E_{11}(k) = |\hat{G}(k)|^2E_{11}(k)
\end{aligned}$$

   as desired.