Advanced Fluid Turbulence

Problem 13.18, page 583 in the text.

1. With the decomposition $\mathbf{U} = \bar{\mathbf{U}} + \mathbf{u}'$, show that the residual stress tensor can be decomposed as

$$\tau_{ij}^R = L_{ij} + C_{ij} + R_{ij} \text{ , where}$$
$$L_{ij} = \overline{\bar{U}_i \bar{U}_j} - \bar{U}_i \bar{U}_j \text{ ,}$$
$$C_{ij} = \overline{\bar{U}_i u_j'} + \overline{u_i' \bar{U}_j} \text{ , and}$$
$$R_{ij} = \overline{u_i' u_j'} \text{ .}$$

From the definition of τ_{ij}^R,

$$\begin{aligned} \tau_{ij}^R &= \overline{U_i U_j} - \bar{U}_i \bar{U}_j = \overline{(\bar{U}_i + u_i')(\bar{U}_j + u_j')} - \bar{U}_i \bar{U}_j \\ &= \overline{\bar{U}_i \bar{U}_j} + \overline{\bar{U}_i u_j'} + \overline{\bar{U}_j u_i'} + \overline{u_i' u_j'} - \bar{U}_i \bar{U}_j \\ &= \underbrace{\overline{\bar{U}_i \bar{U}_j} - \bar{U}_i \bar{U}_j}_{L_{ij}} + \underbrace{\overline{\bar{U}_j u_i'} + \overline{\bar{U}_i u_j'}}_{C_{ij}} + \underbrace{\overline{u_i' u_j'}}_{R_{ij}} \text{ .} \end{aligned}$$

2. For the sharp spectral filter, show that $\tau_{ij}^R - \tau_{ij}^k = L_{ij}$, where $\tau_{ij}^k = \overline{U_i U_j} - \overline{\bar{U}_i \bar{U}_j}$. With $\tau_{ij}^R = \overline{U_i U_j} - \bar{U}_i \bar{U}_j$, then

$$\tau_{ij}^R - \tau_{ij}^k = \overline{U_i U_j} - \bar{U}_i \bar{U}_j - (\overline{U_i U_j} - \overline{\bar{U}_i \bar{U}_j}) = \overline{\bar{U}_i \bar{U}_j} - \bar{U}_i \bar{U}_j = L_{ij}$$

as desired. Notice that, in the sharp spectral filtered case, the velocity computed is represented as a truncated Fourier series, i.e., $\bar{U}_i$. But the momentum equation is also filtered giving, for the nonlinear term, $\dfrac{\partial}{\partial x_j} \overline{U_i U_j}$. But what is actually computed for the nonlinear term is, since the velocity has been represented by a truncated Fourier series, $\dfrac{\partial}{\partial x_j} \overline{\bar{U}_i \bar{U}_j}$. Therefore, the appropriate residual stress is τ_{ij}^k as given above.