A Brief Review of Stress, Strain, Stress/Strain Transformations, Hooke's Law, and Failure Predictions

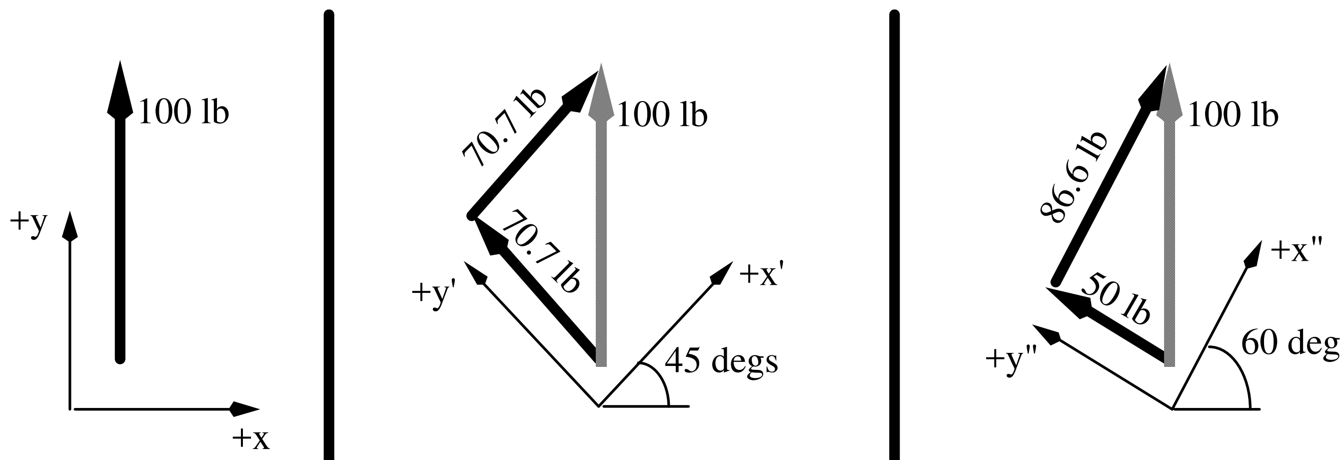
Prof. Mark E. Tuttle
Dept. Mechanical Engineering
M/S 352600
University of Washington
Seattle, WA 98195

tuttle@u.washington.edu

Force Vectors (Tensors)

- A force, $\bar{F}$, is a vector (also called a "1st-order tensor")

- The description of any vector (or any tensor) depends on the coordinate system used to describe the vector:



$\bar{F}$ = 100 lb (y-direction)

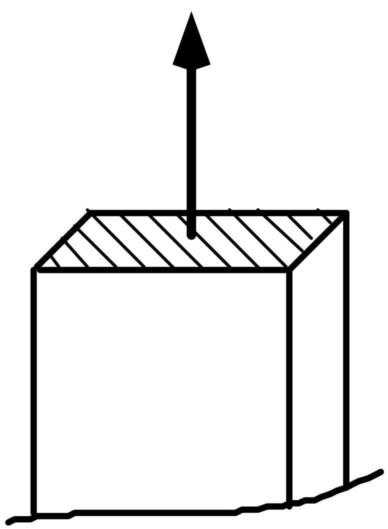
$\bar{F}$ = 70.7 lb (x'-direction) + 70.7 lb (y'-direction)

$\bar{F}$ = 86.6 lb (x"-direction) + 50 lb (y"-direction)

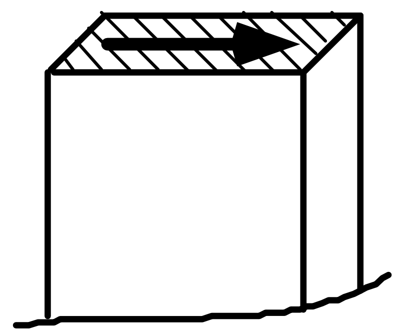
Normal and Shear Forces

- A "normal" force acts perpendicular to a surface
- A "shear" force acts tangent to a surface

P = Normal Force

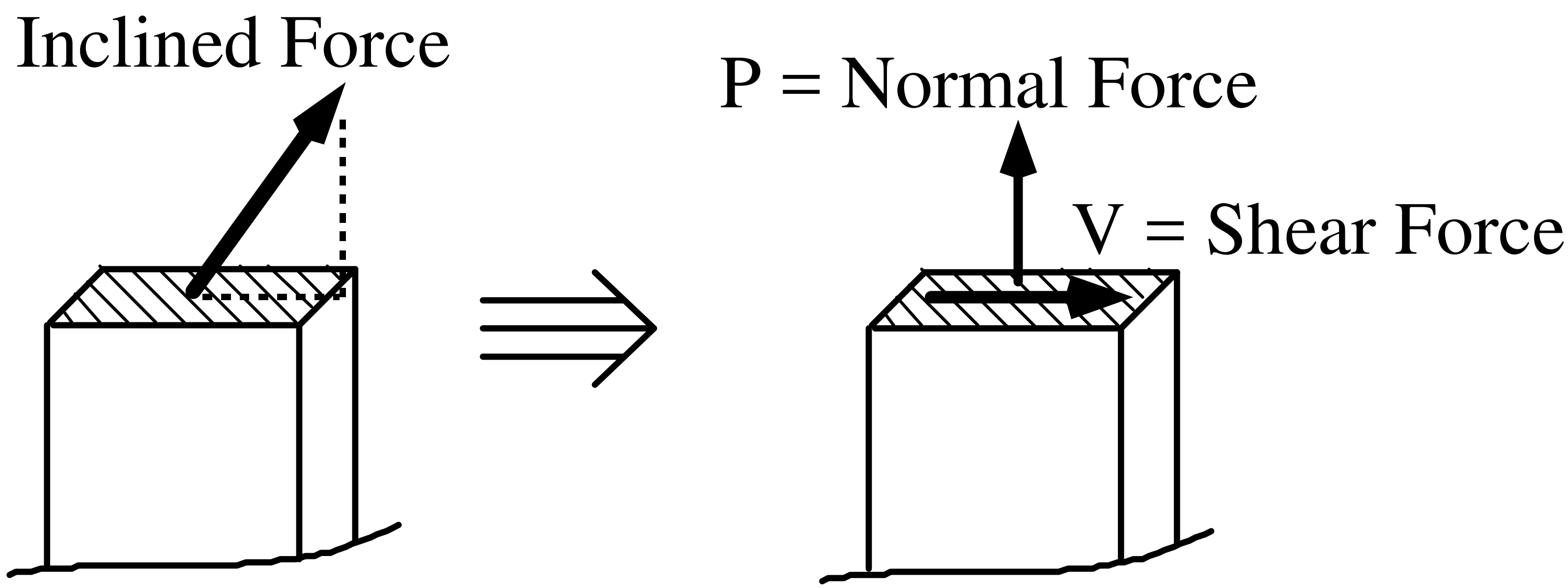


V = Shear Force



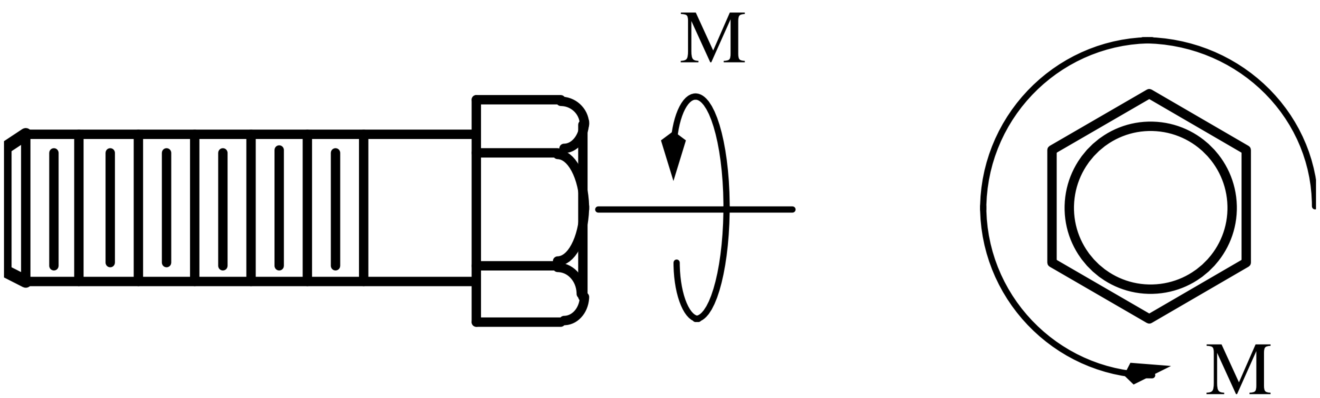
Forces Inclined to a Plane

- A force inclined to a plane can always be described as a combination of normal and shear forces



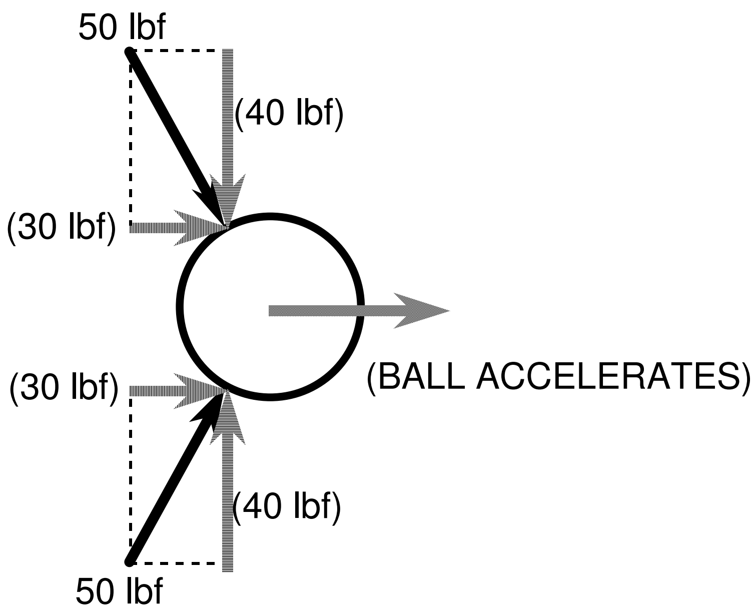
Moments

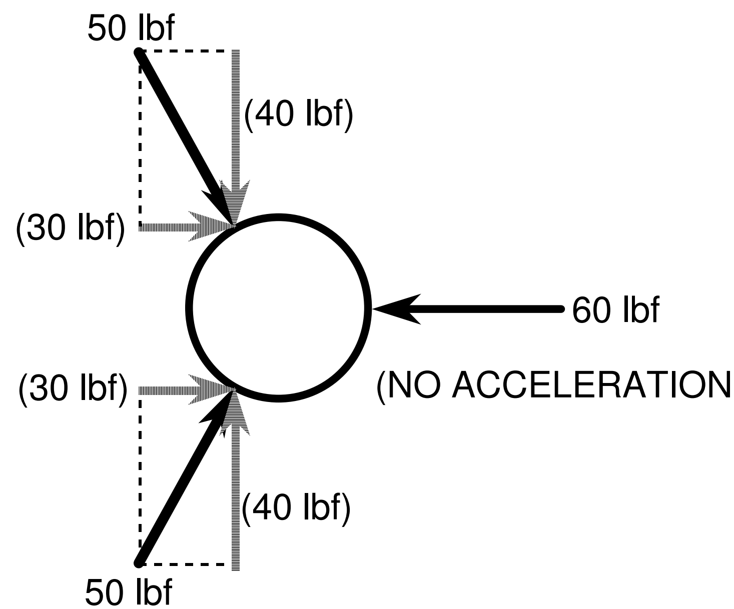
- A moment (also called a "torque" or a "couple") is a force which tend to cause rotation of a rigid body
- A moment is also vectoral quantity...



Static Equilibrium

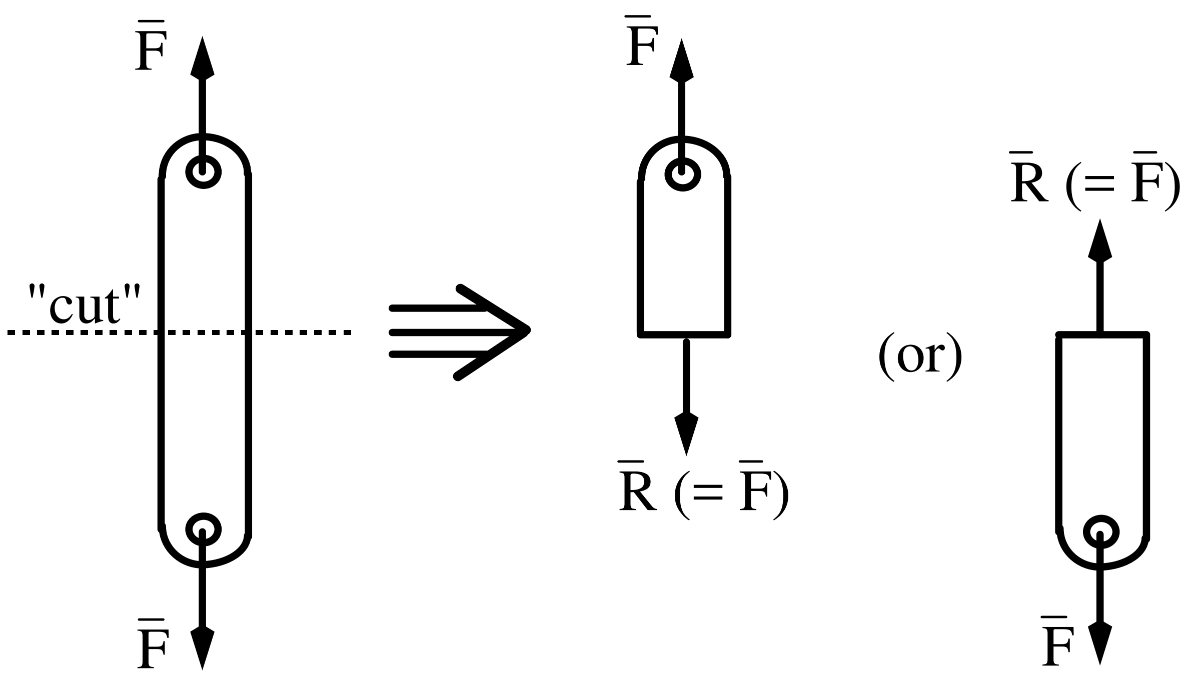
- A rigid solid body is in "static equilibrium" if it is:
  - at rest, or
  - moves with a constant velocity
- Static equilibrium exists if: $\sum \overline{F} = 0$ and $\sum \overline{M} = 0$





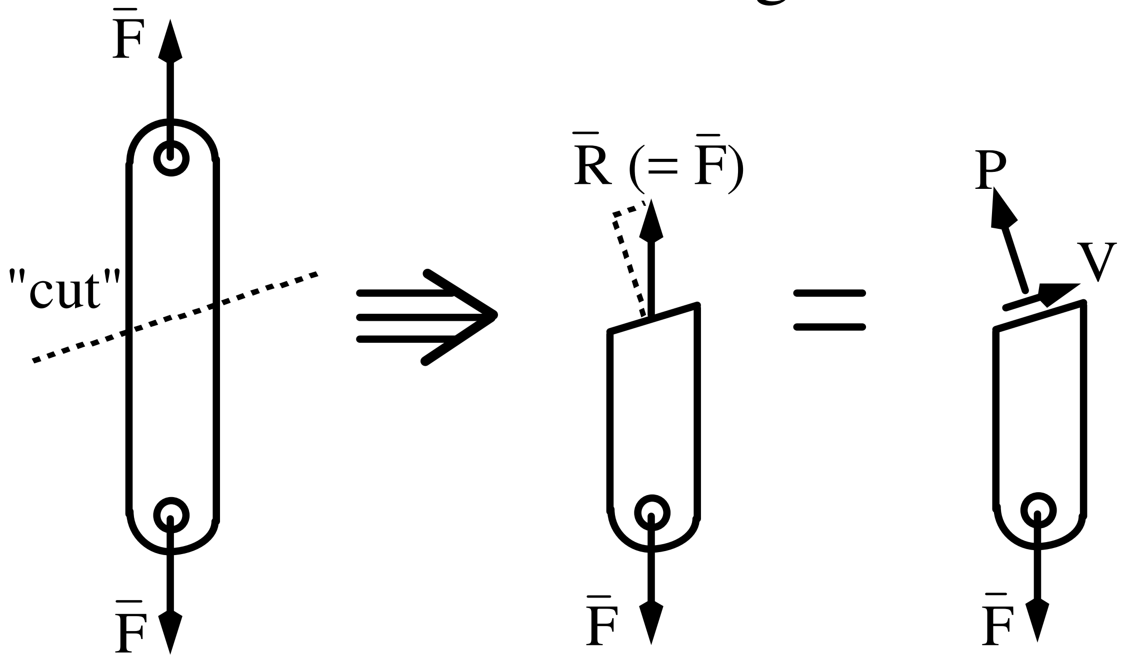
Free Body Diagrams and "Internal Forces"

- An imaginary "cut" is made at plane of interest
- Apply $\sum \overline{F} = 0$ and $\sum \overline{M} = \underline{0}$ to either half to determine internal forces, $\overline{R}$



Free Body Diagrams and "Internal Forces"

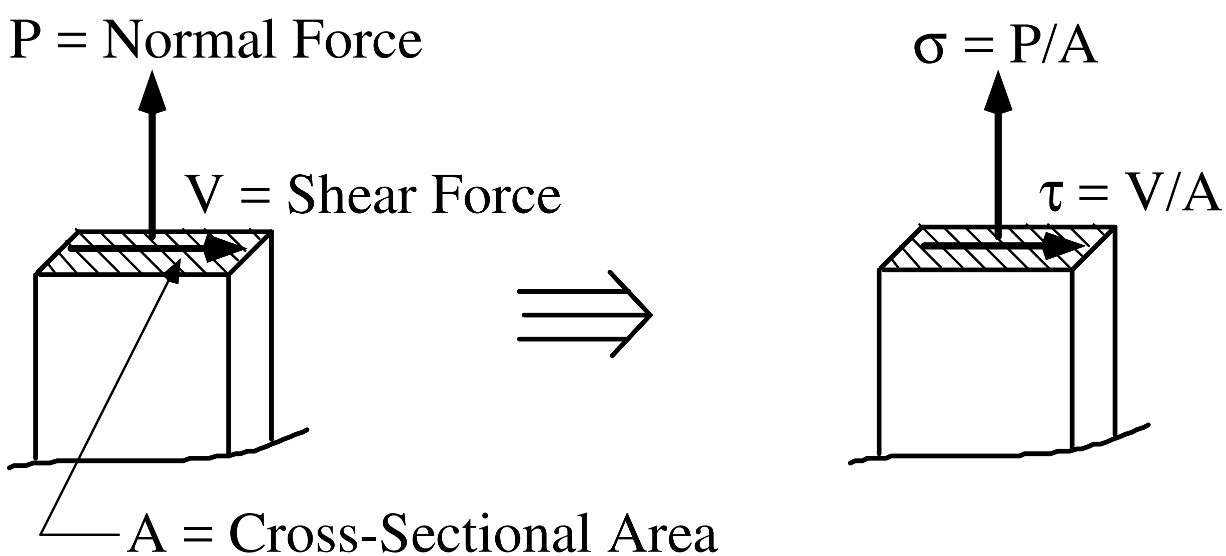
- The imaginary cut can be made along an arbitrary plane
- Internal force $\overline{R}$ can be decomposed to determine the normal and shear forces acting on the arbitrary plane



Stress

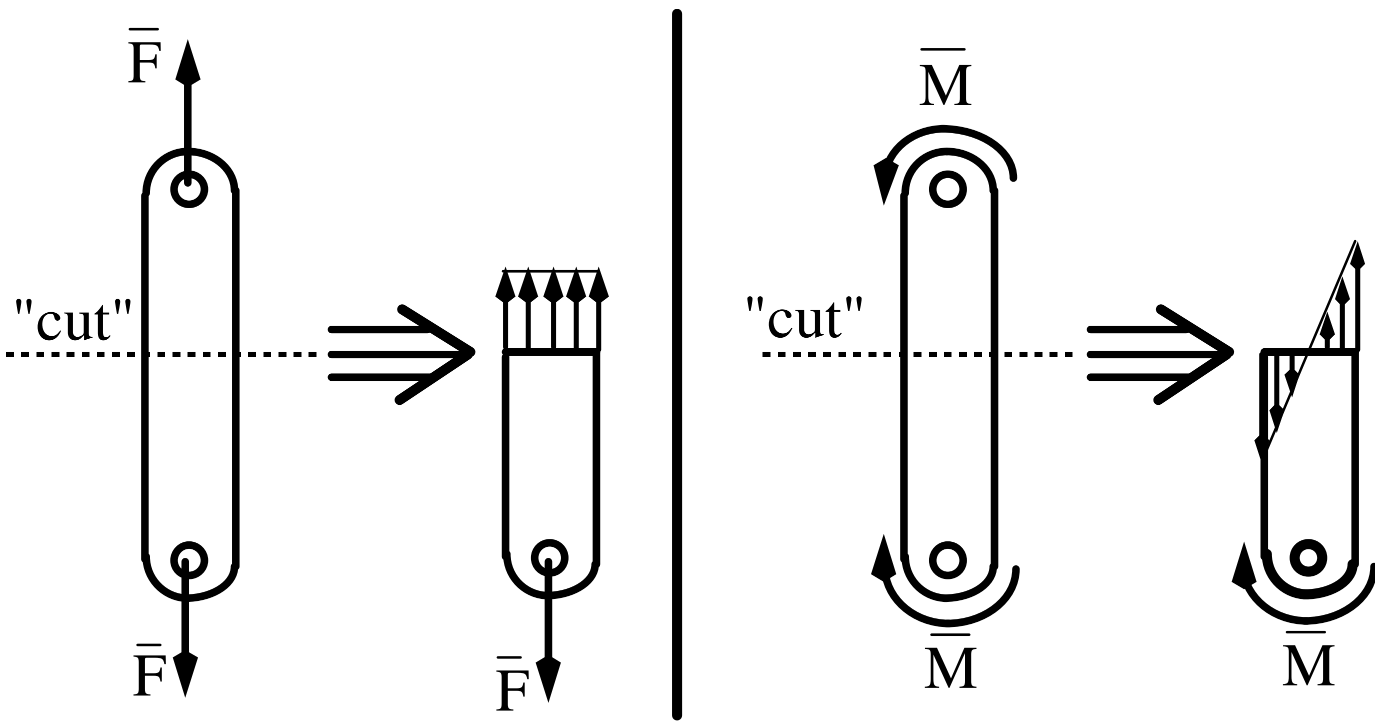
Fundamental Definitions

- Two "types" of stress:
  - normal stress = σ = P/A
  - shear stress = τ = V/A
  - where P and V must be *uniformly distributed* over A



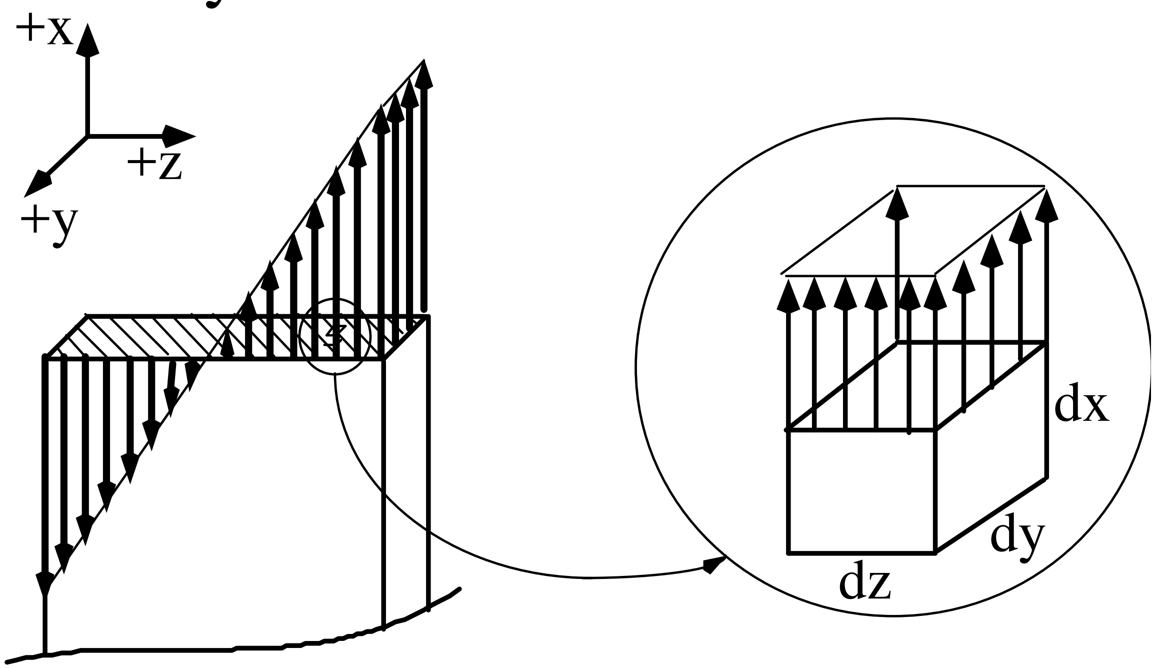
Distribution of Internal Forces

- Forces are distributed over the internal plane... they may or may not be uniformly distributed



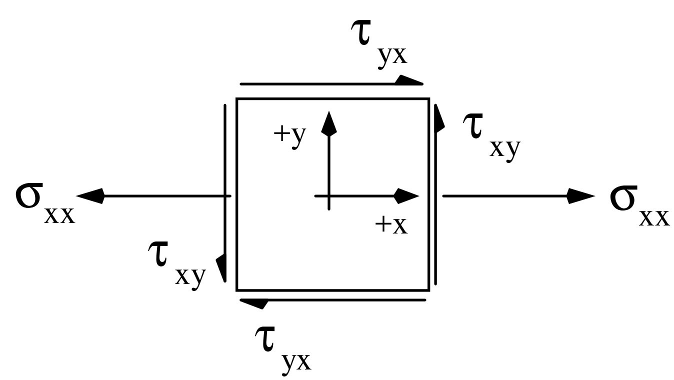
Infinitesimal Elements

- A free-body diagram of an "infinitesimal element" is used to define "stress at a point"
- Forces can be considered "uniform" over the infinitesimally small elemental surfaces

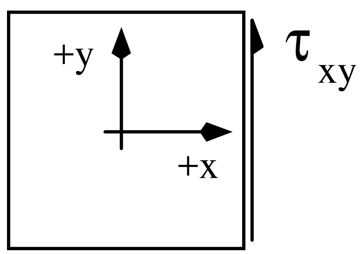


Labeling Stress Components

- Two subscripts are used to identify a stress component, e.g., "σ_{xx}" or "τ_{xy}" (note: for convenience we sometimes write $\sigma_x = \sigma_{xx}$ or $\sigma_{xy} = \tau_{xy}$)

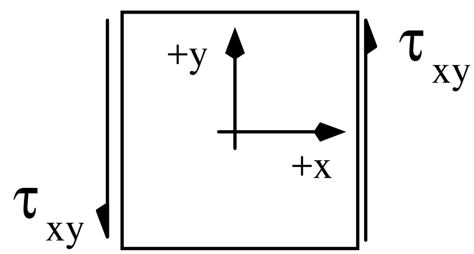


Admissable Pure Shear Stress States



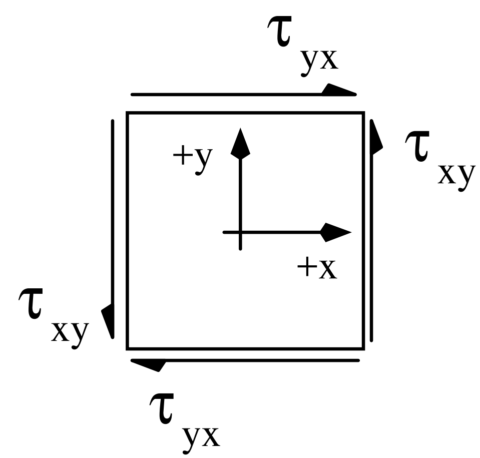
$$\sum \overline{F} \neq 0$$
$$\sum \overline{M} \neq 0$$

(inadmissable)



$$\sum \overline{F} = 0$$
$$\sum \overline{M} \neq 0$$

(inadmissable)



$$\text{If}: \left|\tau_{yx}\right| = \left|\tau_{xy}\right|$$

then :

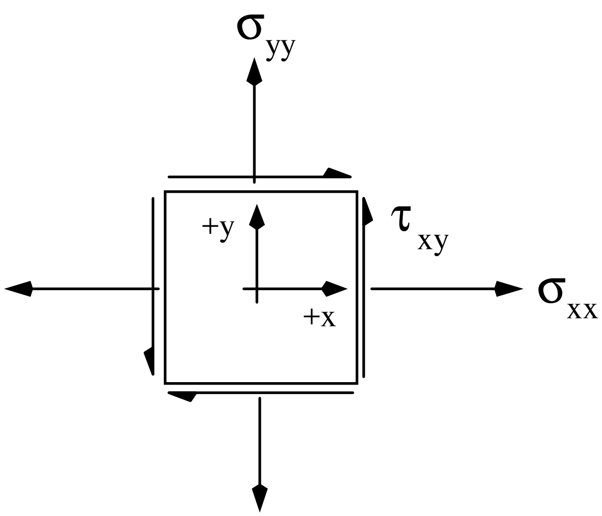
$$\sum \overline{F} = 0$$
$$\sum \overline{M} = 0$$

(admissable)

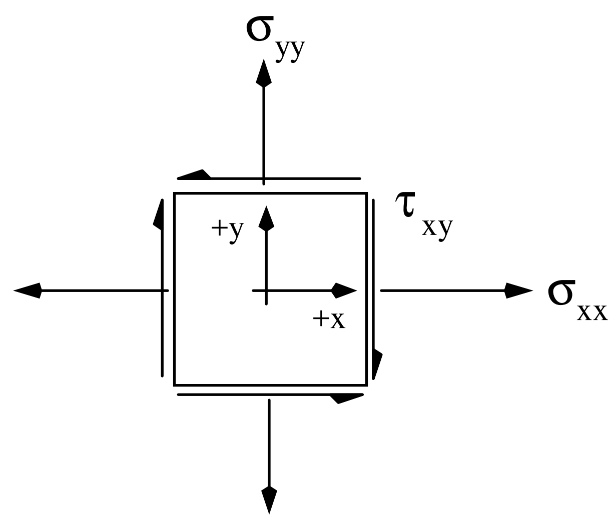
Stress Sign Conventions

- The algebraic sign of an element face is "positive" of the outward-pointing unit normal to the face "points" in a positive coordinate direction
- A stress component is positive if:
  - stress component acts on a positive face and "points" in a positive coordinate direction, or
  - stress component acts on a negative face and "points" in a negative coordinate direction

Stress Sign Conventions



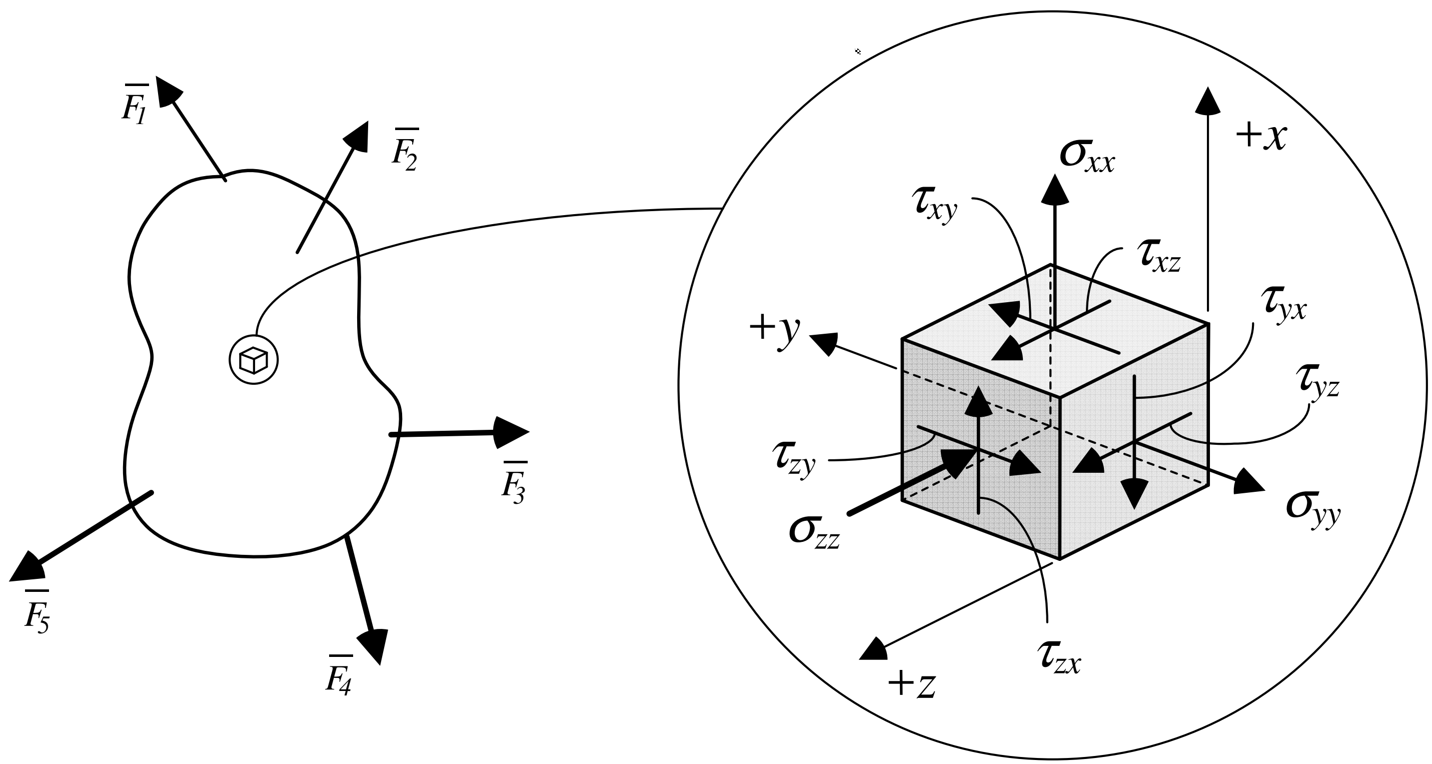
all stresses positive



σ_{xx} and σ_{yy} positive,
τ_{xy} negative

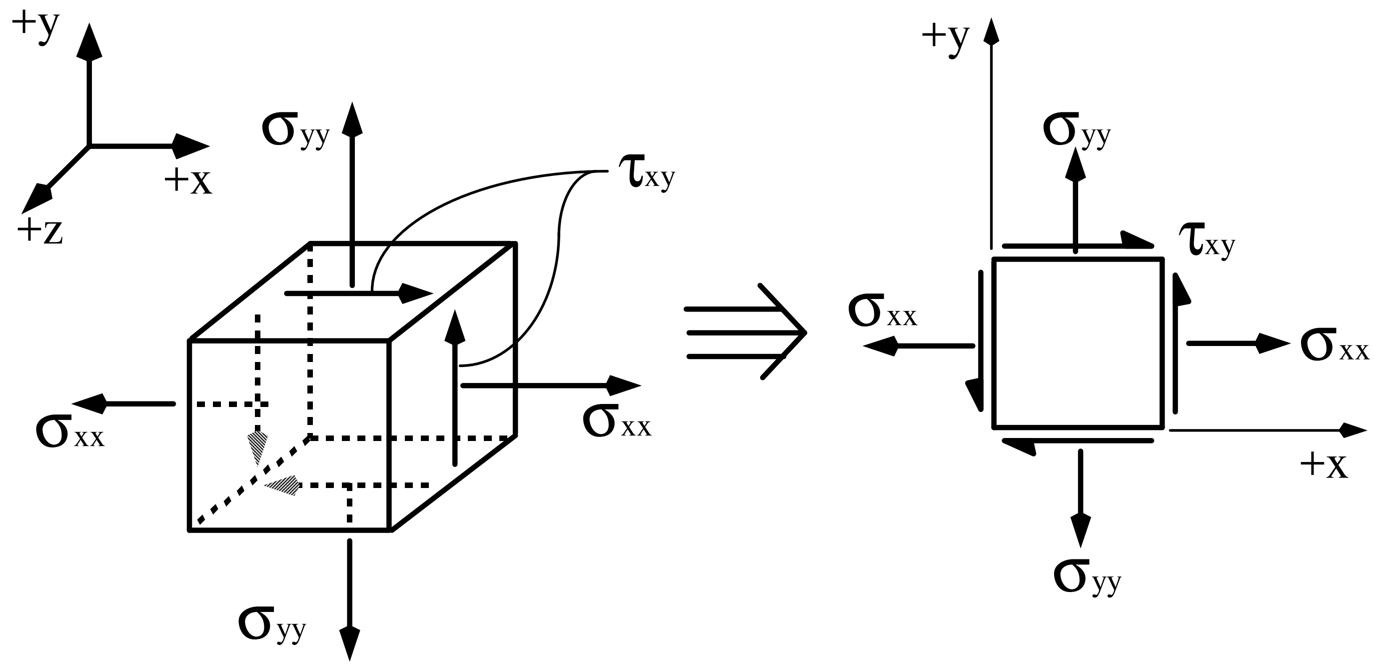
The Stress Tensor

- Stress is a "2nd-order tensor", and in the most general case six components of stress exist "at a point"



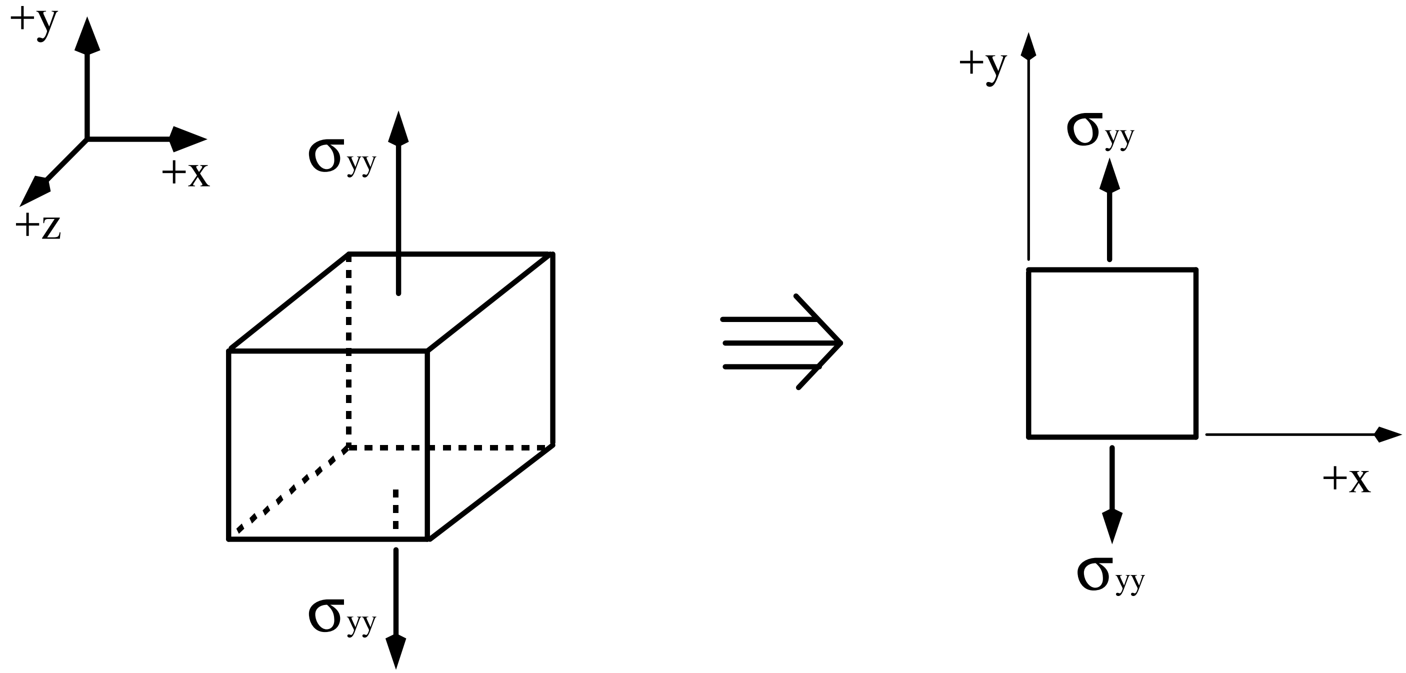
Plane Stress

- If all non-zero stress components exist in a single plane (i.e., if $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$), the state of stress is called "plane stress"

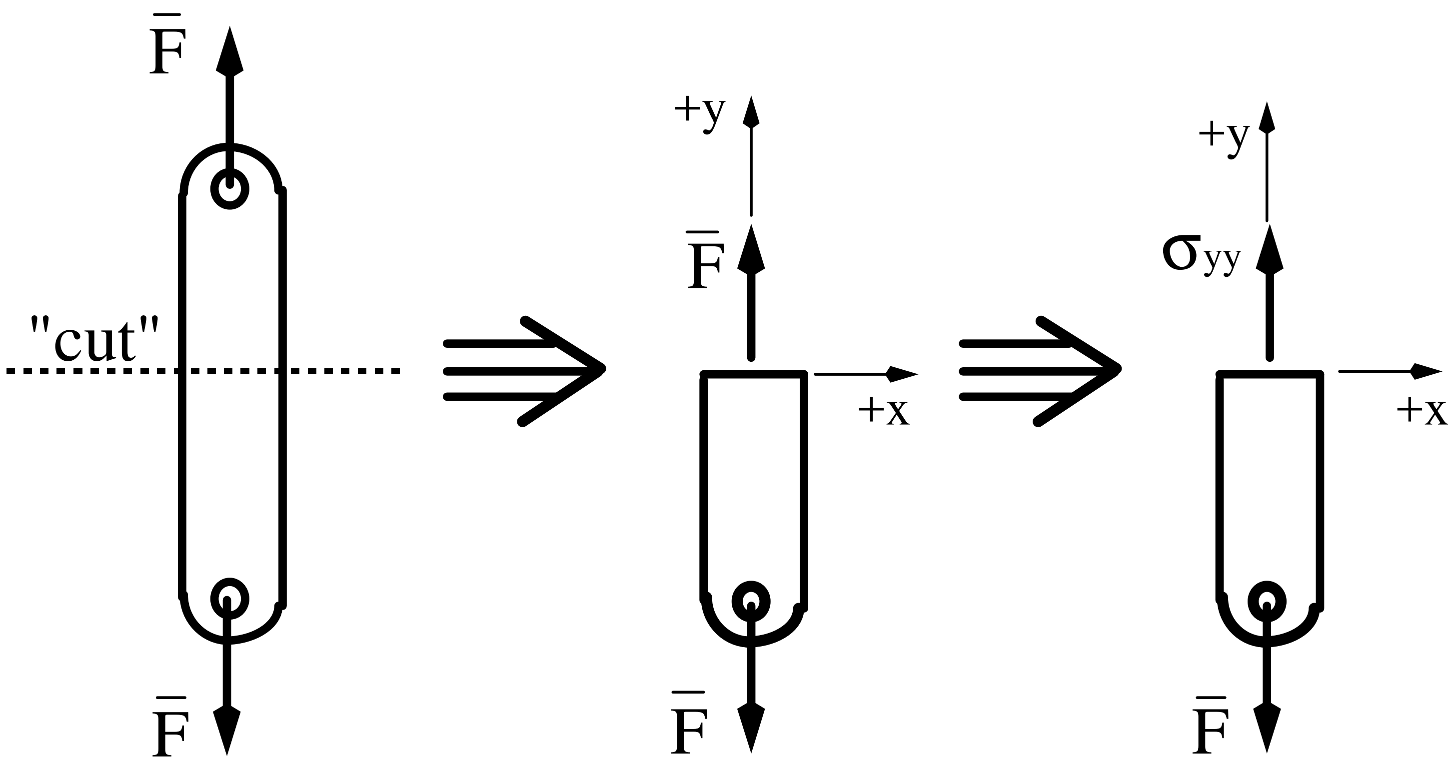


Uniaxial Stress

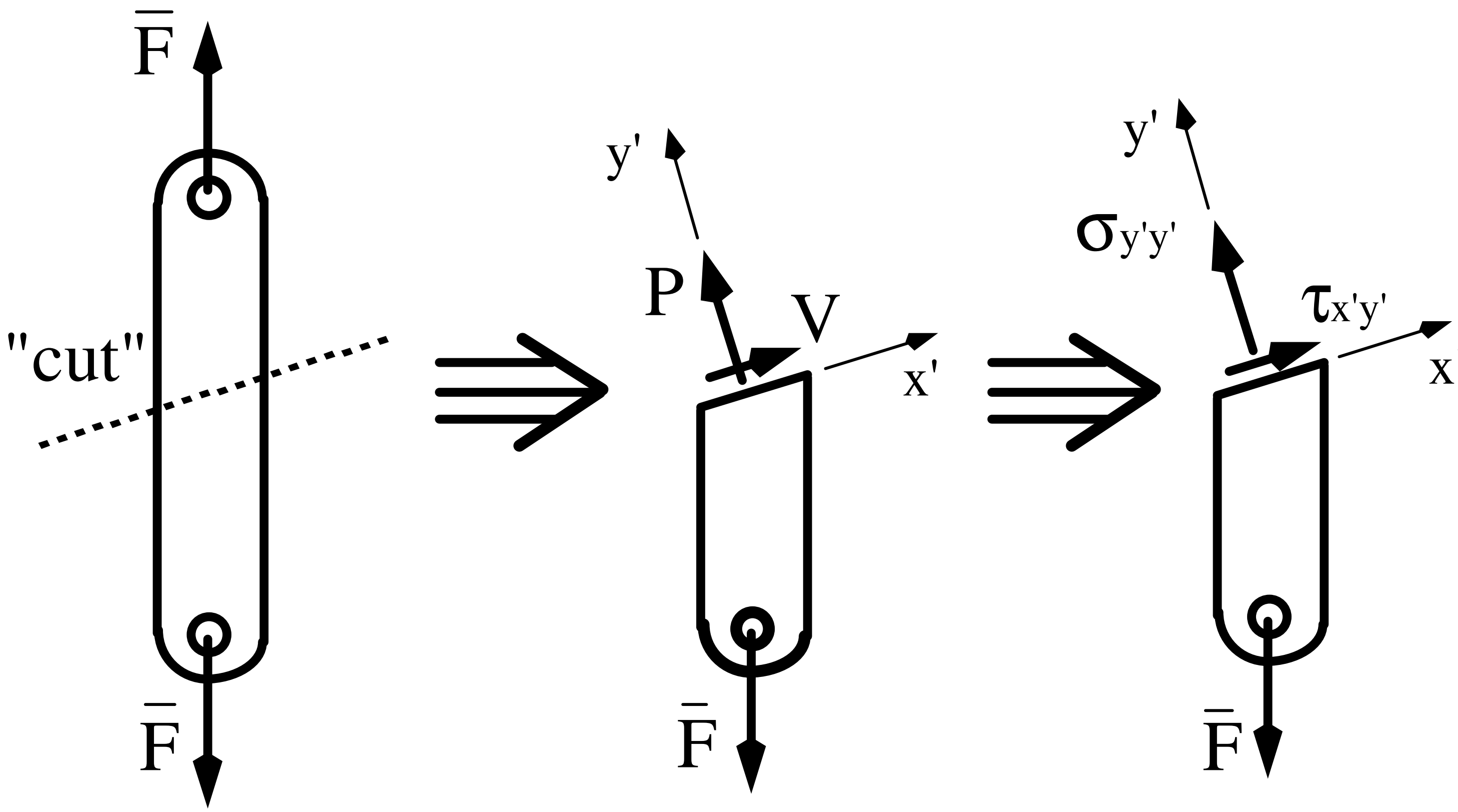
- If only one normal stress exists (if $\sigma_{xx} = \sigma_{zz} = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0$), the state of stress is called a "uniaxial stress"



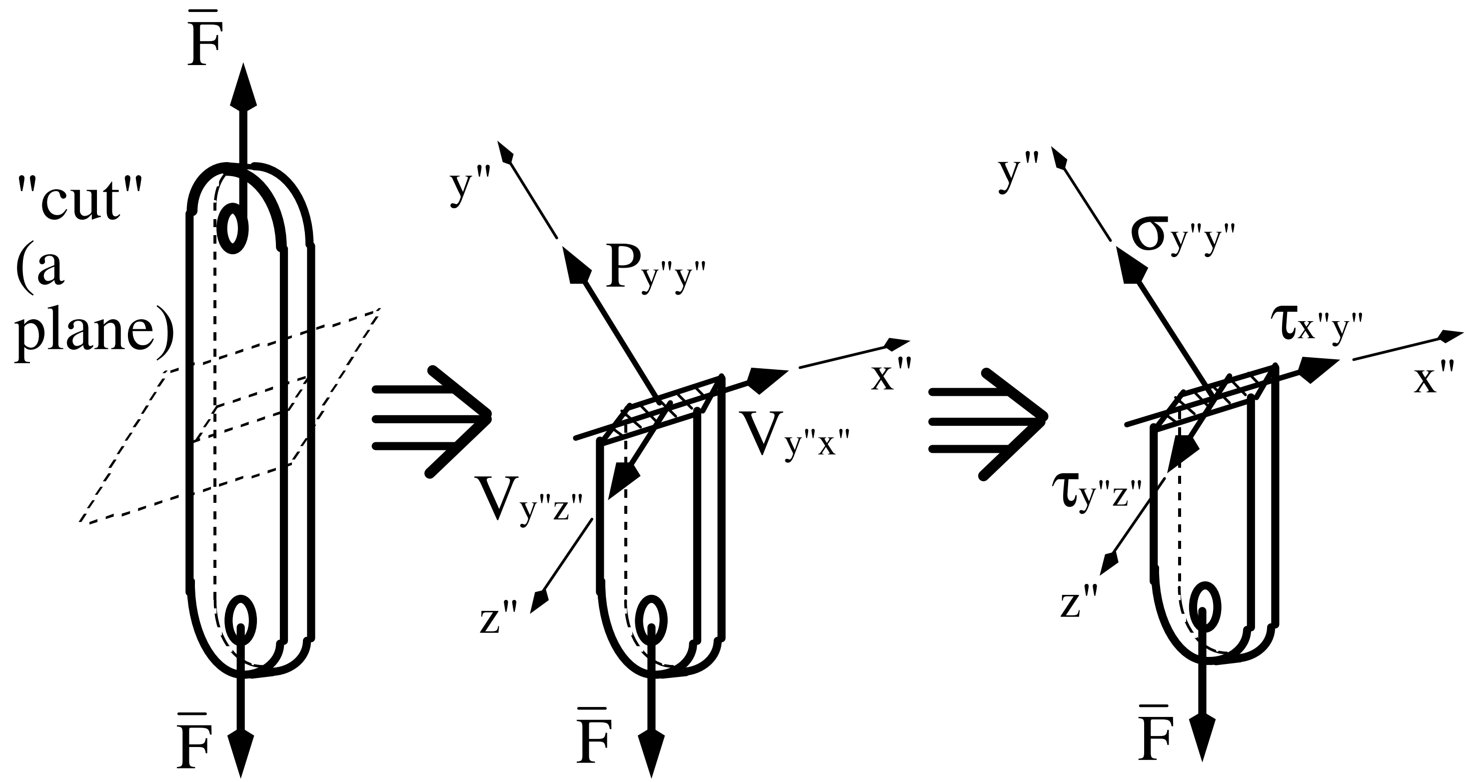
Free Body Diagram *Defines* the Coordinate System



Free Body Diagram *Defines* the Coordinate System

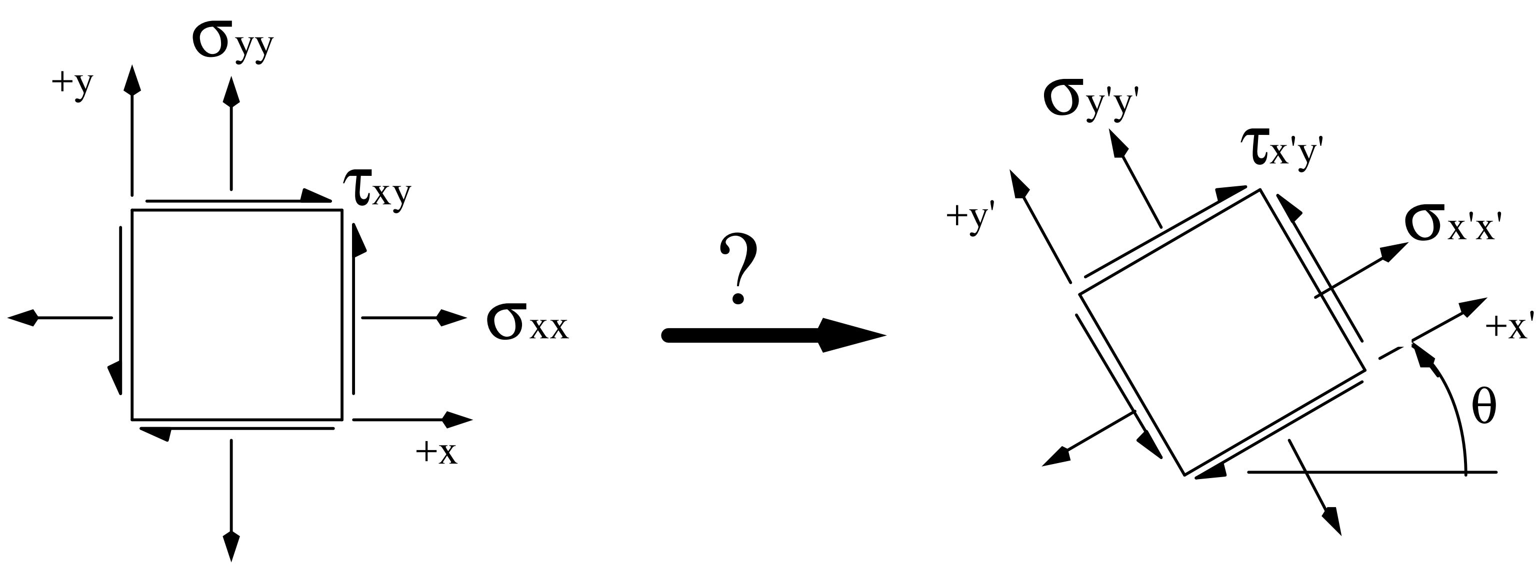


Free Body Diagram *Defines* the Coordinate System



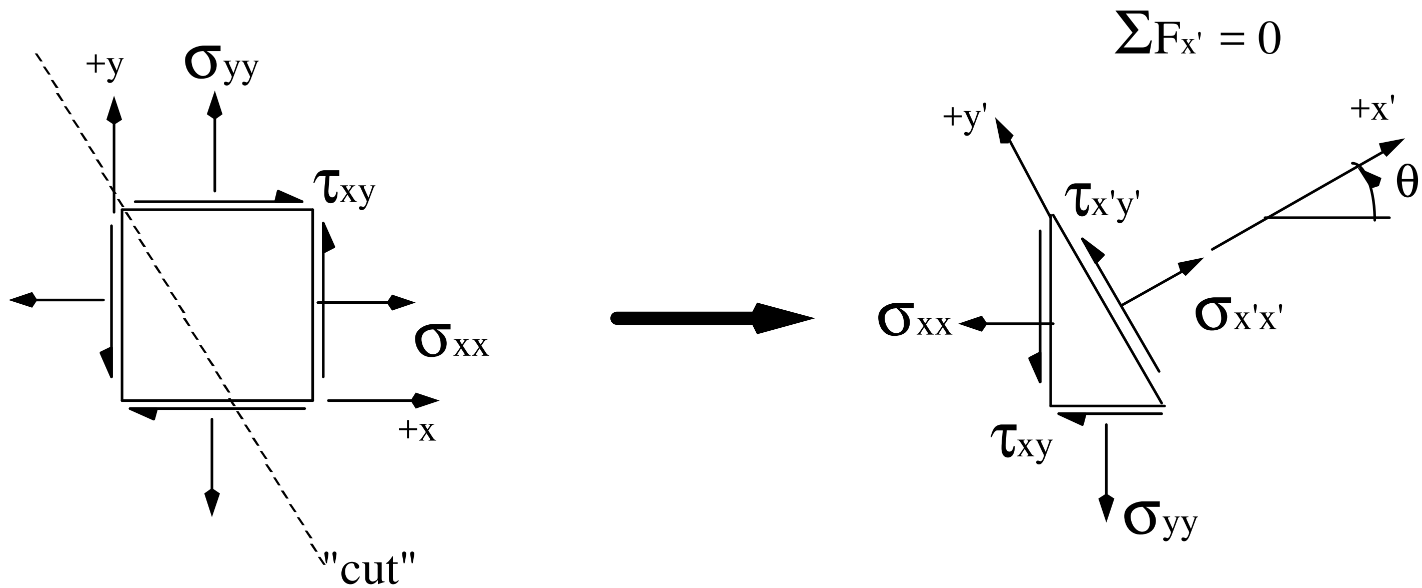
Stress Transformations

- Given stress components in the x-y coordinate system (σ_{xx}, σ_{yy}, τ_{xy}), what are the corresponding stress components in the x'-y' coordinate system?



Stress Transformations

- Stress components in the x'-y' coordinate system may be related to stresses in the x-y coordinate system using a free body diagram and enforcing $\sum \bar{F} = 0$



Stress Transformation Equations

- By enforcing $\Sigma F_{x'} = 0$, $\Sigma F_{y'} = 0$, it can be shown:

$$\sigma_{x'x'} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\cos\theta\sin\theta$$

$$\sigma_{y'y'} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\cos\theta\sin\theta$$

$$\tau_{x'y'} = \left(\sigma_{yy} - \sigma_{xx}\right)\cos\theta\sin\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$

<u>Important note</u>: angle θ is positive *from* the +x-direction *towards* the +y'-direction (for the elements drawn in this presentation, a positive angle θ is counter-clockwise)

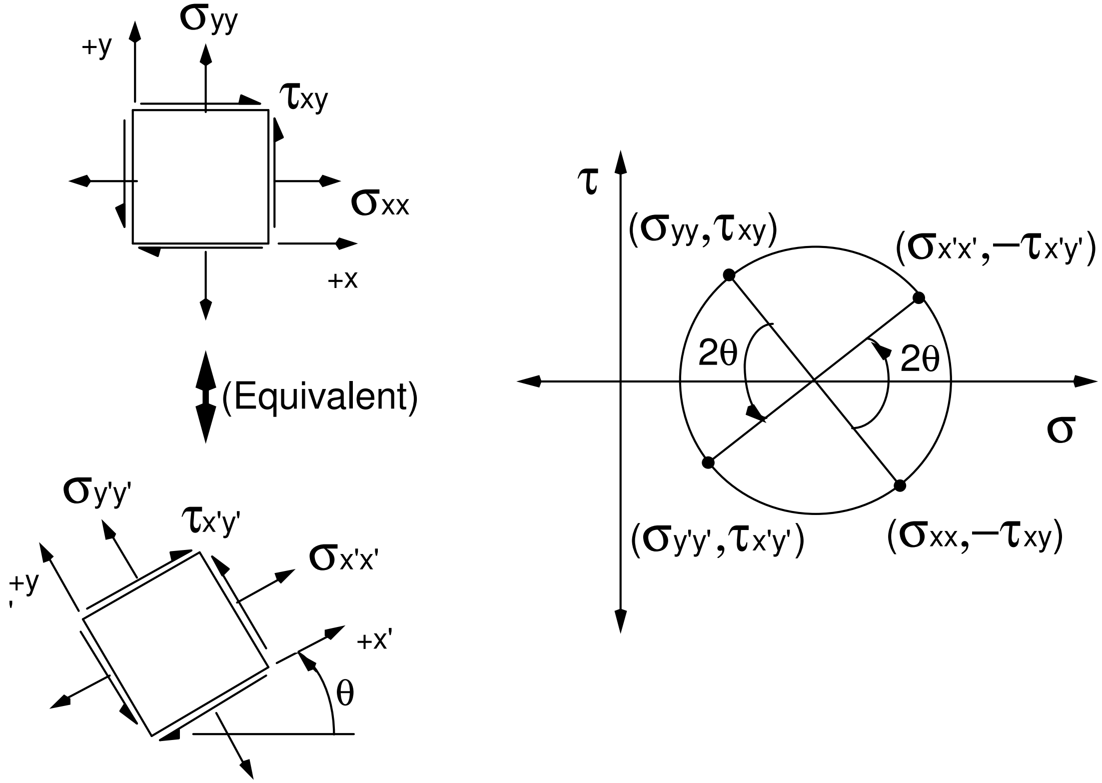
Stress Transformation Equations

- Using matrix notation, these can also be written:

$$\begin{Bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \tau_{x'y'} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & (\cos^2\theta - \sin^2\theta) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}$$

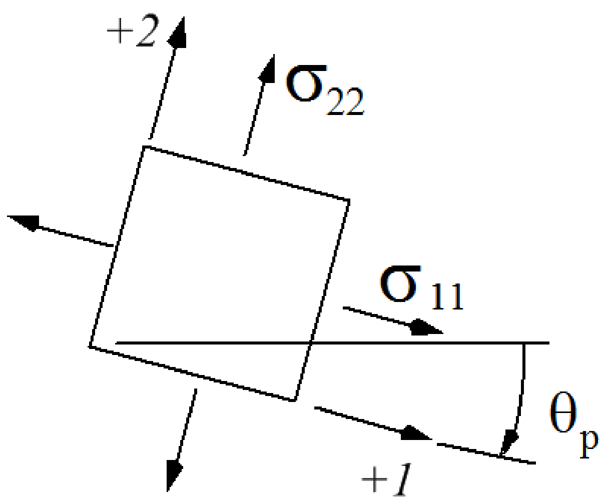
Stress Transformation Equations

- These transformation equations can also be visualized using Mohr's circle of stress:



Principal Stresses

- In the principal stress coordinate system the shear stress is zero, and the normal stresses are max/min...



$$\sigma_{11},\sigma_{22} = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

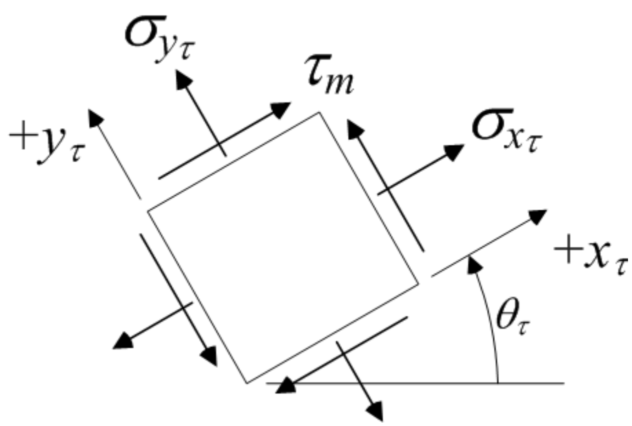
$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_{xx}-\sigma_{yy}}\right)$$

*** OR, EQUIVALENTLY ***

$$\theta_p = \tan^{-1}\left(\frac{\sigma_{11}-\sigma_{xx}}{\tau_{xy}}\right)$$

Maximum Shear Stress

- In the maximum shear stress coordinate system the in-plane shear stress is at a maximum and in-plane normal stresses are equal

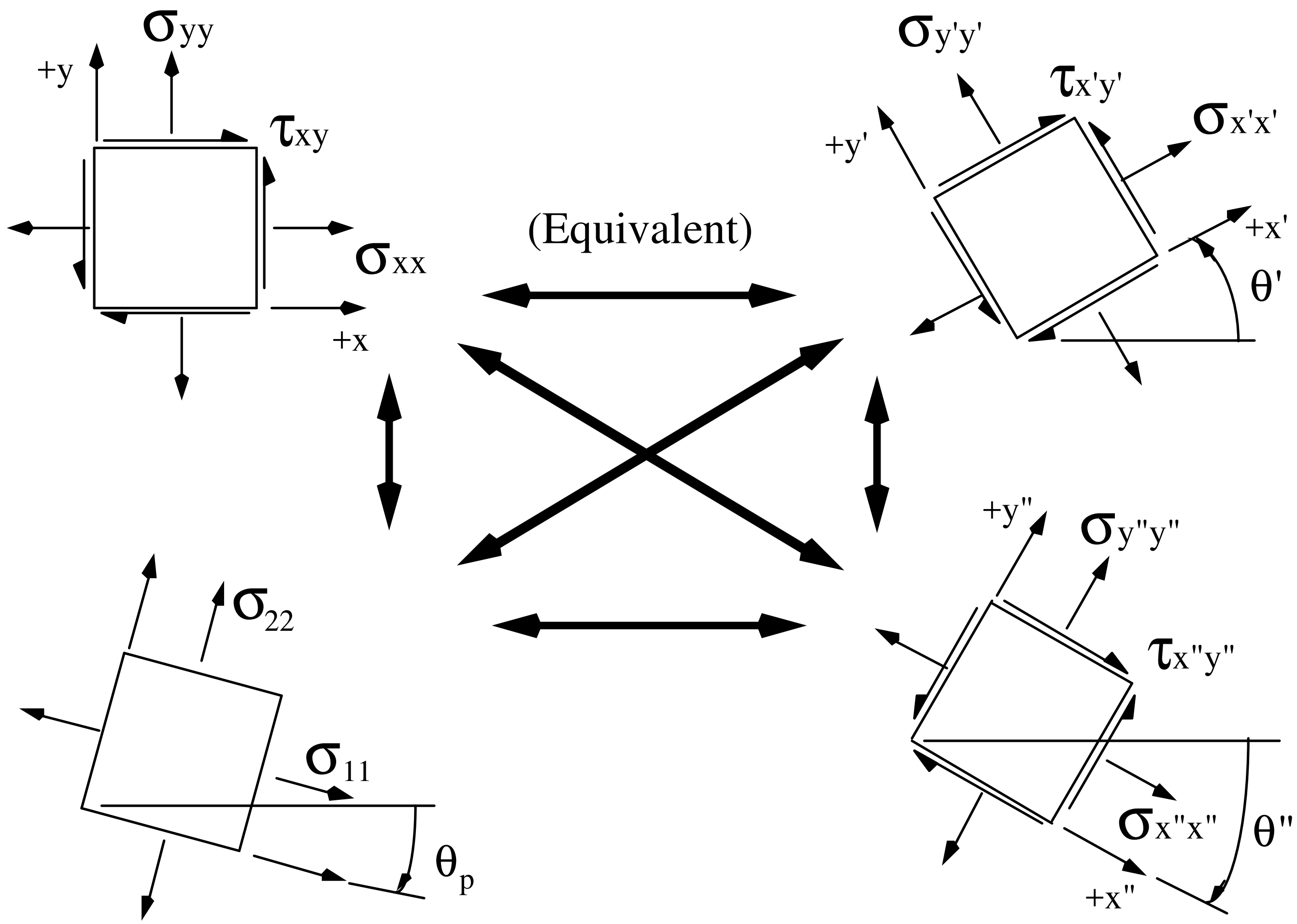


$$\sigma_{x_\tau} = \sigma_{y_\tau} = \frac{\sigma_{xx} + \sigma_{yy}}{2} = \frac{\sigma_{11} + \sigma_{22}}{2}$$

$$\tau_m = \pm\sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

$$\theta_t = \frac{1}{2}\tan^{-1}\left(\frac{\sigma_{yy} - \sigma_{xx}}{2\tau_{xy}}\right)$$

"Transformation" of Stress



Sample Problem

- Given the following stress components (where the x-axis is horizontal and positive to the right, and the y-axis is vertical and positive upwards):

  σ_{xx} = 3500 psi σ_{yy} = -30000 psi τ_{xy} = 20000 psi

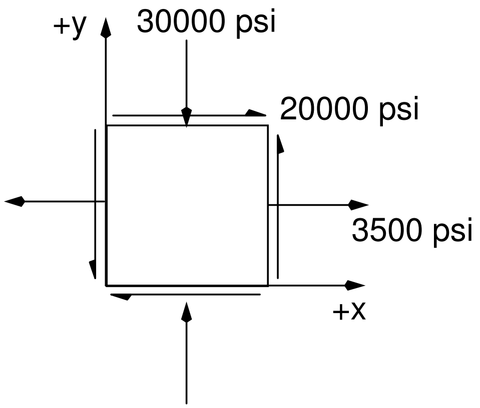
  (a) Sketch the stress element in the x-y coordinate system

  (b) Sketch the stress element in the x'-y' coordinate system, oriented 50°CCW

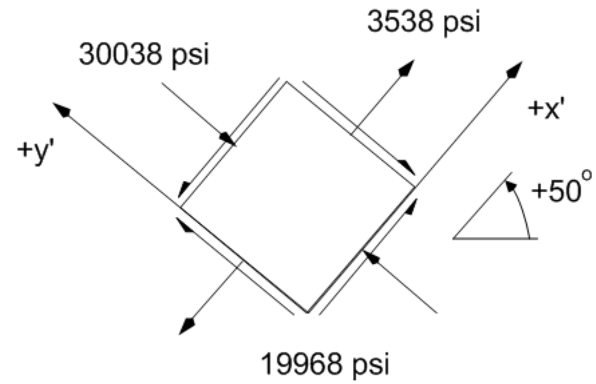
  (c) Sketch the stress element in the principal stress coordinate system

  (d) Sketch the stress element in the maximum shear stress coordinate system

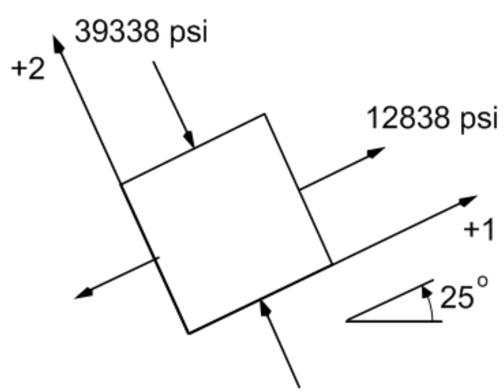
Sample Problem (answers)



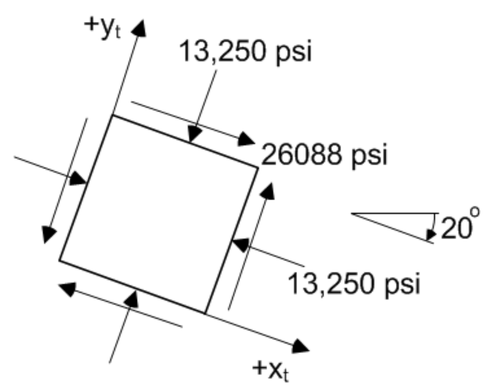
Part (a)



Part (b)



Part (c)



Part (d)

Sample Problem

Use of Stress Transformation Equations (example)

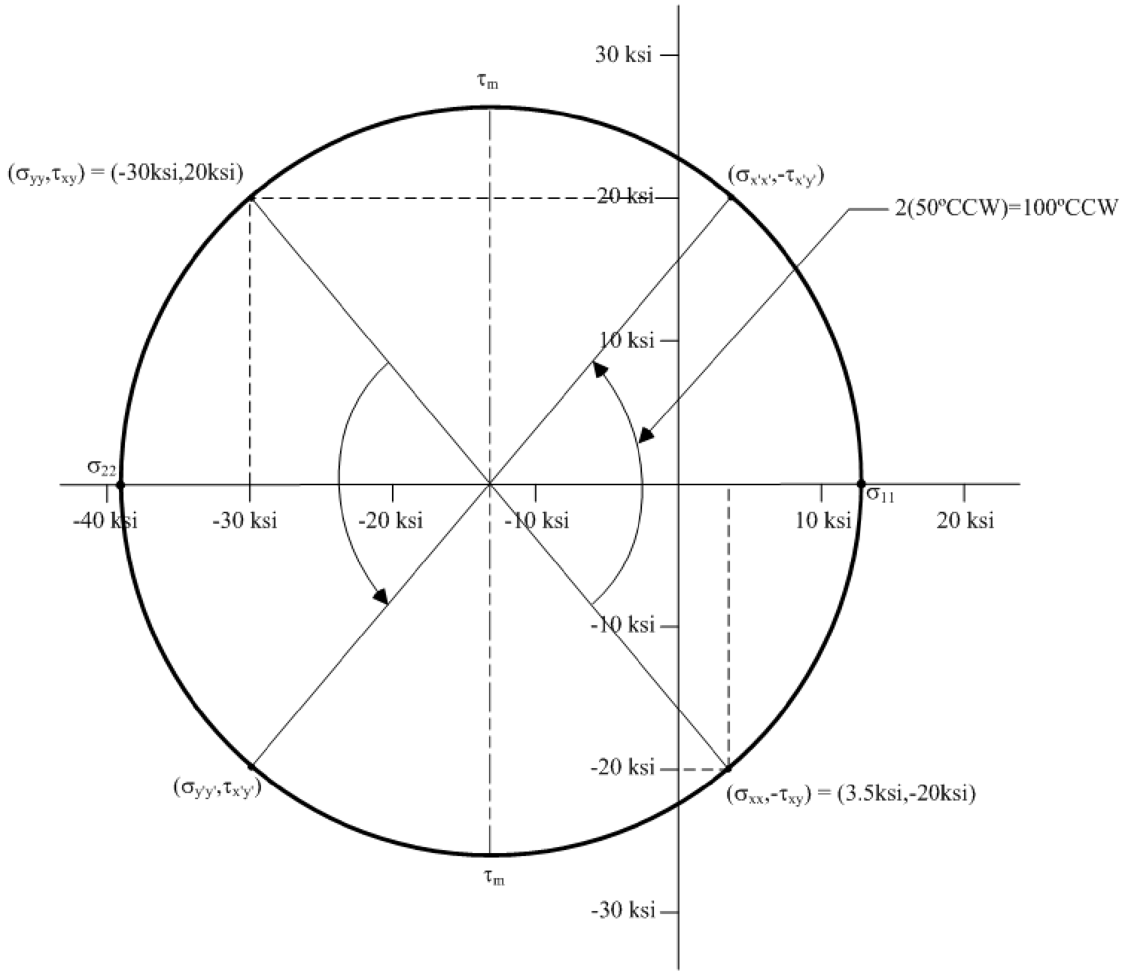
$$\begin{Bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \tau_{x'y'} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & (\cos^2\theta - \sin^2\theta) \end{bmatrix} \begin{Bmatrix} 3.5ksi \\ -30ksi \\ 20ksi \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \tau_{x'y'} \end{Bmatrix} = \begin{bmatrix} \cos^2 50 & \sin^2 50 & 2\cos 50\sin 50 \\ \sin^2 50 & \cos^2 50 & -2\cos 50\sin 50 \\ -\cos 50\sin 50 & \cos 50\sin 50 & (\cos^2 50 - \sin^2 50) \end{bmatrix} \begin{Bmatrix} 3.5ksi \\ -30ksi \\ 20ksi \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \tau_{x'y'} \end{Bmatrix} = \begin{Bmatrix} 3.538ksi \\ -30.038ksi \\ -19.968ksi \end{Bmatrix}$$

Sample Problem

Use of Mohr's Circle



"Stress": Summary of Key Points

- Normal and shear stresses are both defined as a (force/area)
- Six components of stress must be known to specify the "state of stress" at a point
- Stress is a tensorial quantity; values of individual stress components depend on the coordinate system used
- Stress is defined strictly on the basis of static equilibrium; definition is independent of:
  - material properties
  - strain
  - temperature

Strain

Fundamental Definitions

- "Strain" is a measure of the deformation of a solid body
- There are two "types" of strain; normal strain (ε) and shear strain (γ)

$$\varepsilon = \frac{\text{(change in length)}}{\text{(original length)}} \qquad \text{units} = \text{in/in, m/m, etc}$$

$$\gamma = \text{(change in angle)} \qquad \text{units} = \text{radians}$$

The Strain Tensor

- Strain is a 2nd-order tensor, and in the most general case, six components of strain exist "at a point":

$$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}$$

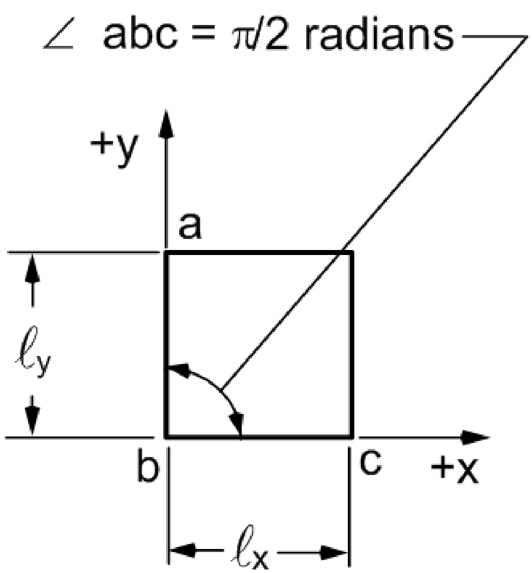
- Since strain is a tensorial quantity, the values of the individual strain components which define the "state of strain" depend on the coordinate system used…

- This review will primarily involve strains which exist within a single plane

Strain Within a Plane

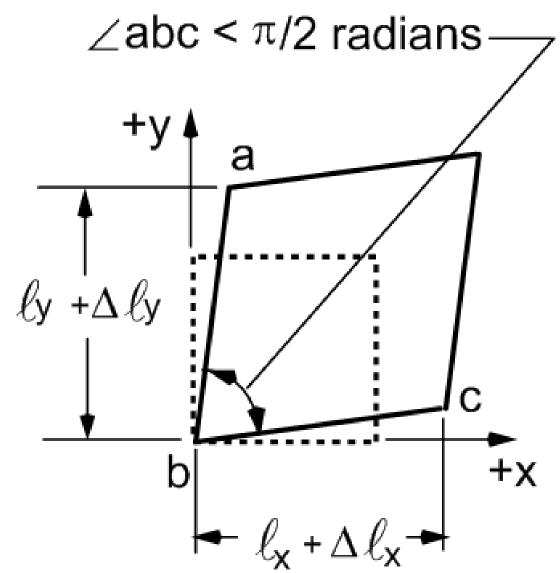
- We often encounter two distinct conditions that result in problems involving "strains within a plane":
  - Plane Stress: All non-zero *stress* components lie within a single plane (e.g., σ_{xx}, σ_{yy}, $\tau_{xy} \neq 0$, $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$). If the material is isotropic, the plane stress condition induces four non-zero strain components: ε_{xx}, ε_{yy}, ε_{zz}, and γ_{xy}
  - Plane Strain: All non-zero *strain* components lie within a single plane (e.g., ε_{xx}, ε_{yy}, $\gamma_{xy} \neq 0$, $\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$). By definition, the plane strain condition involves three non-zero strain components: ε_{xx}, ε_{yy}, and γ_{xy}

Strain Within a Plane



Original Shape

$\Rightarrow$



Deformed Shape

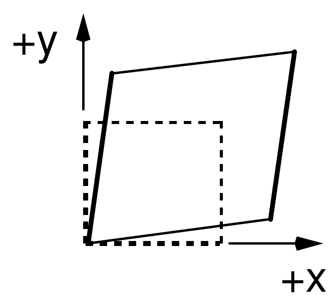
$$\varepsilon_{xx} = \lim_{\ell_x \to 0} (\Delta \ell_x / \ell_x)$$

$$\varepsilon_{yy} = \lim_{\ell_y \to 0} (\Delta \ell_y / \ell_y)$$

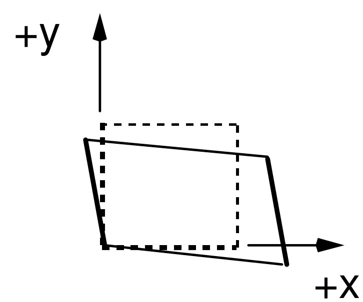
$$\gamma_{xy} = \lim_{\ell_x, \ell_y \to 0} (\Delta \angle abc)$$

Strain Sign Convention

- A positive (tensile) normal strain is associated with an increase in length
- A shear strain is positive if the angle between two positive faces (or two negative faces) decreases



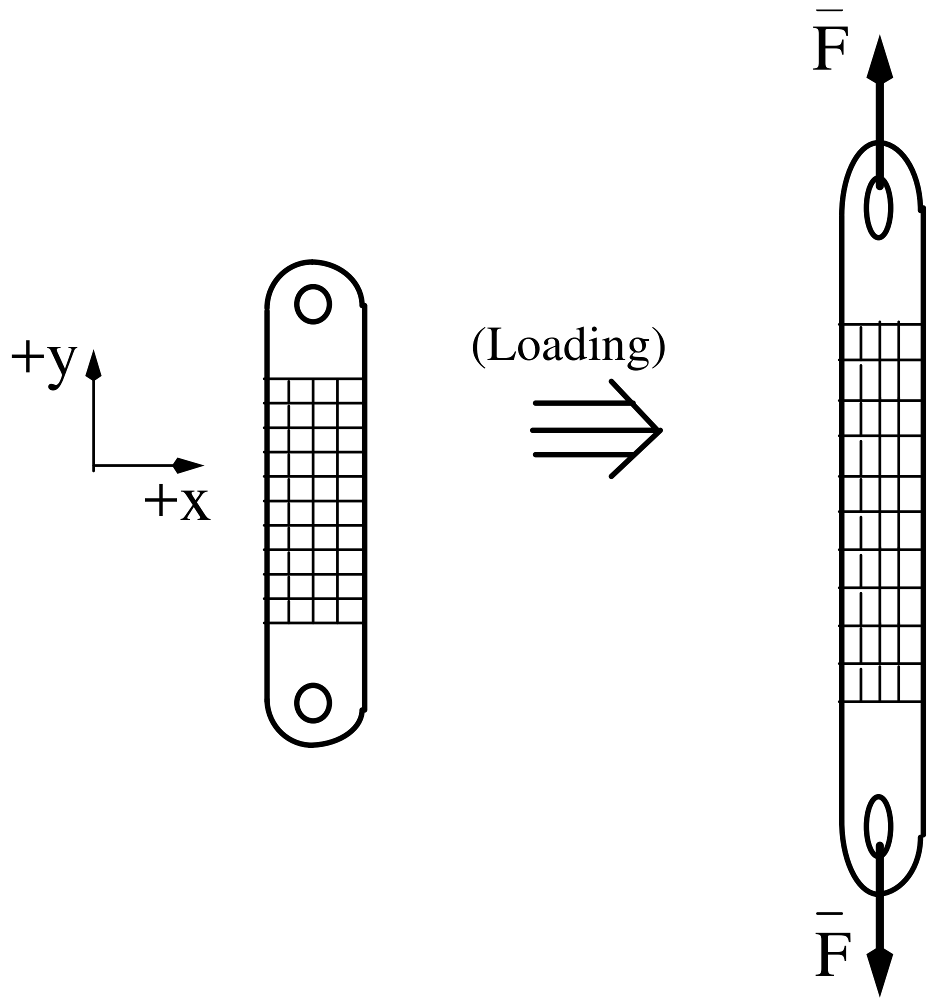
All Strains Positive



ε_{xx} Positive
ε_{yy} and γ_{xy} Negative

Visualization of Strain

(Assuming link made of an isotropic material)



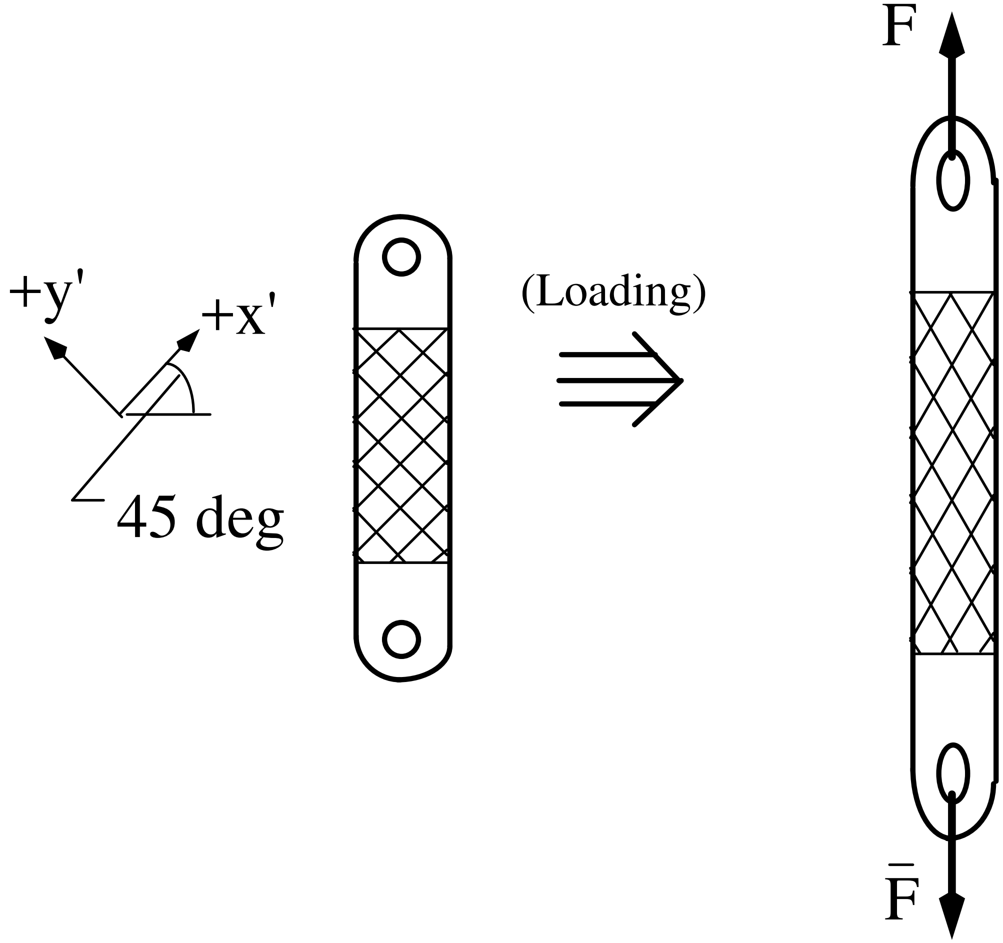
$\varepsilon_{yy} > 0$

$\varepsilon_{xx} < 0$

$\gamma_{xy} = 0$

Visualization of Strain

(Assuming link is made of an isotropic material)

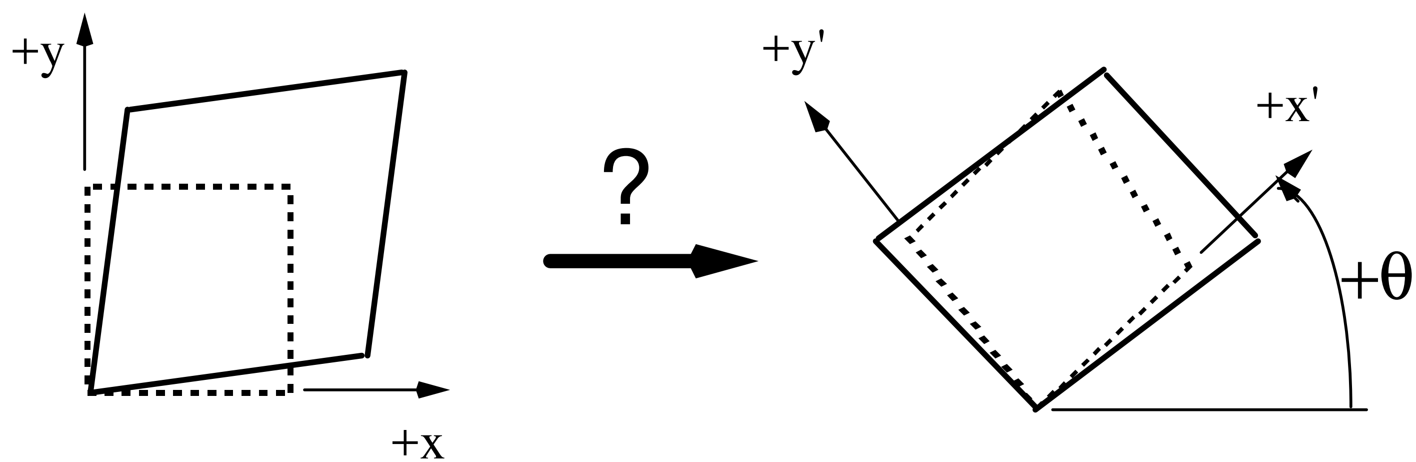


$\varepsilon_{x'x'} = \varepsilon_{y'y'} > 0$

$\gamma_{x'y'} > 0$

Strain Transformations

- Given strain components in the x-y coordinate system, what are the corresponding strain components in the x'-y' coordinate system?



Given: ε_{xx}, ε_{yy}, and γ_{xy}

Find: $\varepsilon_{x'x'}$, $\varepsilon_{y'y'}$, and $\gamma_{x'y'}$

Strain Transformation Equations

- Based strictly on geometry, it can be shown:

$$\varepsilon_{x'x'} = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \left(\frac{\gamma_{xy}}{2}\right)2\cos\theta\sin\theta$$

$$\varepsilon_{y'y'} = \varepsilon_{xx}\sin^2\theta + \varepsilon_{yy}\cos^2\theta - \left(\frac{\gamma_{xy}}{2}\right)2\cos\theta\sin\theta$$

$$\frac{\gamma_{x'y'}}{2} = (\varepsilon_{yy} - \varepsilon_{xx})\cos\theta\sin\theta + \left(\frac{\gamma_{xy}}{2}\right)(\cos^2\theta - \sin^2\theta)$$

Important note: angle θ is positive *from* the +x-direction *towards* the +y-direction (counterclockwise as drawn in this presentation)

Well, I'll Be Darned!!!

- The stress transformation equations are based strictly on the equations of static equilibrium
- The strain transformation equations are based strictly on geometry
- Nevertheless, the stress and strain transformation equations are nearly identical!! (…because both stress and strain are 2^{nd}-order tensors…)

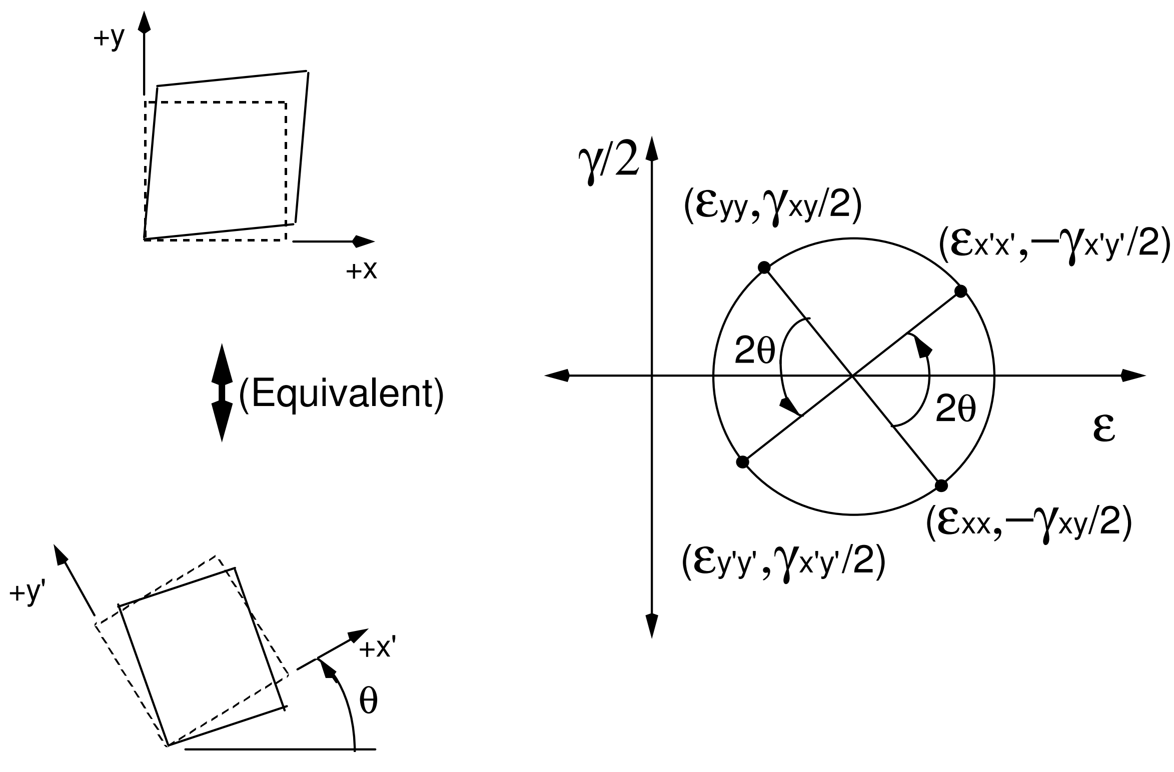
Strain Transformation Equations

- Using matrix notation, these can also be written

$$\begin{Bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ \gamma_{x'y'}/2 \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & (\cos^2\theta-\sin^2\theta) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy}/2 \end{Bmatrix}$$

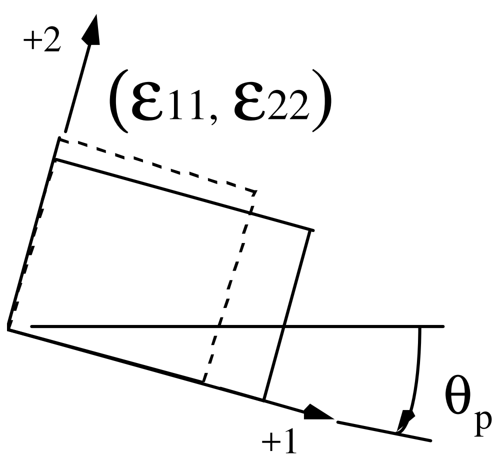
Strain Transformation Equations

- These transformation equations can also be visualized using Mohr's circle of strain:



Principal Strains

- In the principal strain coordinate system the shear strain is zero and the normal strains are max/min...



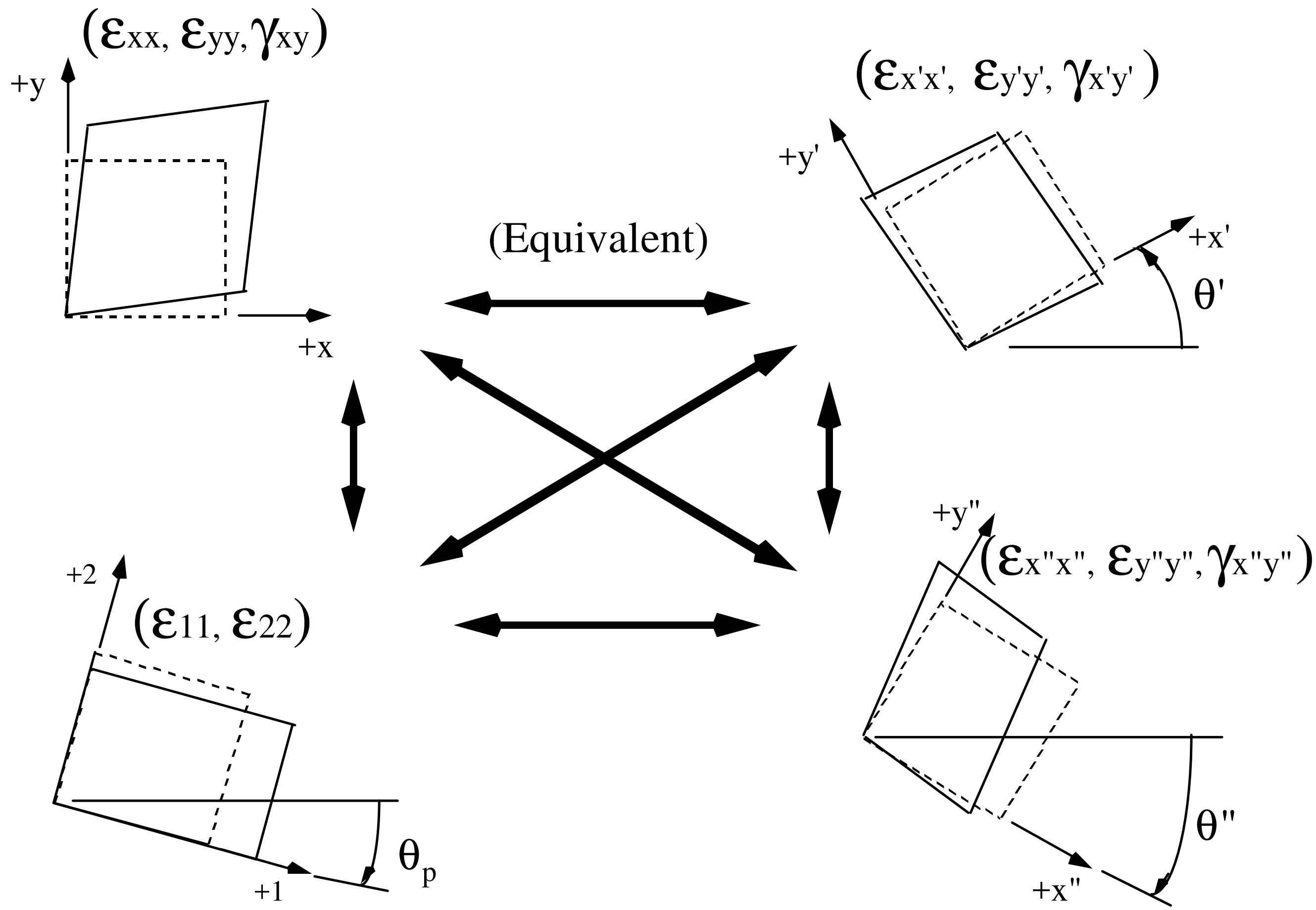
$$\varepsilon_{11},\varepsilon_{22} = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{\gamma_{xy}}{\varepsilon_{xx}-\varepsilon_{yy}}\right)$$

$$***\text{OR, EQUIVALENTLY}***$$

$$\theta_p = \tan^{-1}\left(\frac{2(\varepsilon_{11}-\varepsilon_{xx})}{\gamma_{xy}}\right)$$

“Transformation” of Strain



"Strain": Summary of Key Points

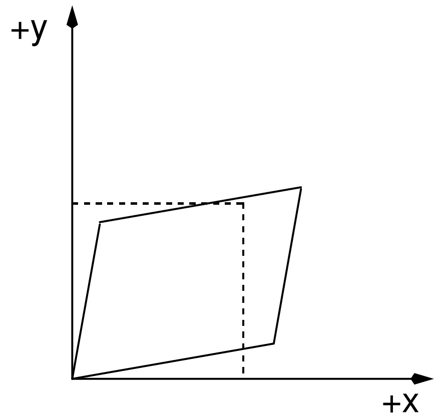
- ε = (Δ length)/(original length) γ = (Δ angle)
- Six components of strain specify the "state of strain"
- Strain is a tensorial quantity; numerical values of individual strain components depend on the coordinate system used
- Strain is defined strictly on the basis of a change in shape; definition is independent of:
  - material properties
  - stress
  - temperature
- "Suprisingly," the stress and strain transformation equations are nearly identical

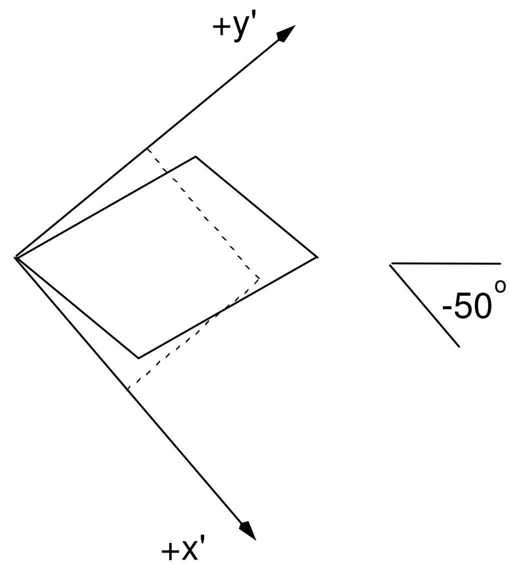
Sample Problem

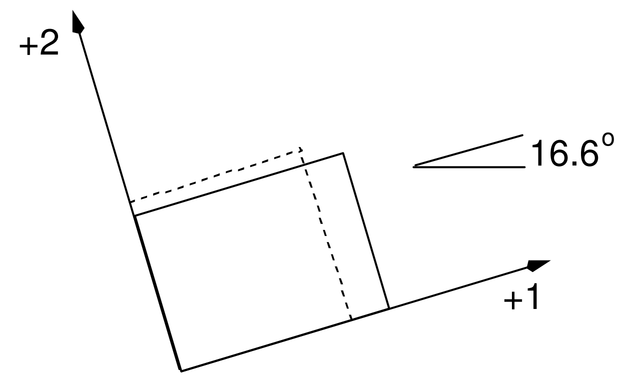
- Given the following strain components (where the x-axis is horizontal and positive to the right, and the y-axis is vertical and positive upwards):
  ε_{xx} = 2000 μin/in ε_{yy} = -1350 μin/in γ_{xy} = 2200 μrad

- (a) Sketch (not to scale) the strain element in the x-y coordinate system
  (b) Sketch (not to scale) the strain element in the x'-y' coordinate system, oriented 50°CW
  (c) Sketch (not to scale) the strain element in the principal strain coordinate system

Sample Problem (answers)







Part (a):
ε_{xx} = 2000 μin/in
ε_{yy} = -1350 μin/in
γ_{xy} = 2200 μrad

Part (b):
$\varepsilon_{x'x'}$ = -1049 μin/in
$\varepsilon_{y'y'}$ = 1699 μin/in
$\gamma_{x'y'}$ = 2917 μrad

Part (c):
ε_{11} = 2329 μin/in
ε_{22} = -1679 μin/in
θ_p = 16.6°

Hooke’s Law

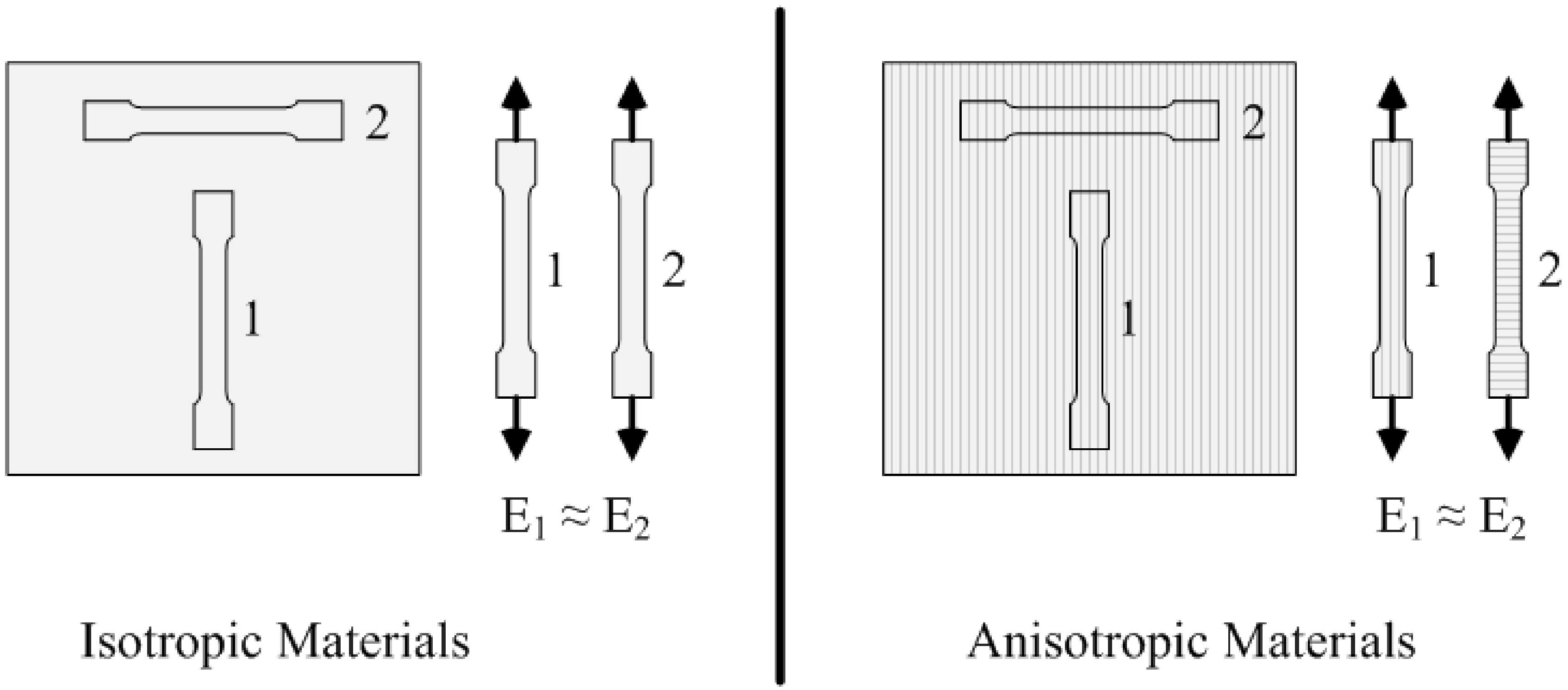
- The structural engineer is typically interested in the measuring the state of stress induced in a structure during service
- The state of stress cannot be measured directly….
- The state of strain can be measured directly….
- Hence, we must develop a relationship between the stress tensor and the strain tensor…this relationship is called a “constitutive model”, and the most common is “Hooke’s Law”

Hooke's Law (Cont'd)

- The form of Hooke's depends on whether the material is isotropic or anisotropic:
  - Isotropic materials: same properties in all directions
  - Anisotropic materials: properties vary with direction
- More than one "type" of anisotropic behavior. Three will be mentioned in this review:
  - Transversely isotropic
  - Orthotropic
  - Generally anisotropic

Isotropic vs Anisotropic Materials

Anisotropy occurs because of some type of order in the microstructure



Hooke's Law

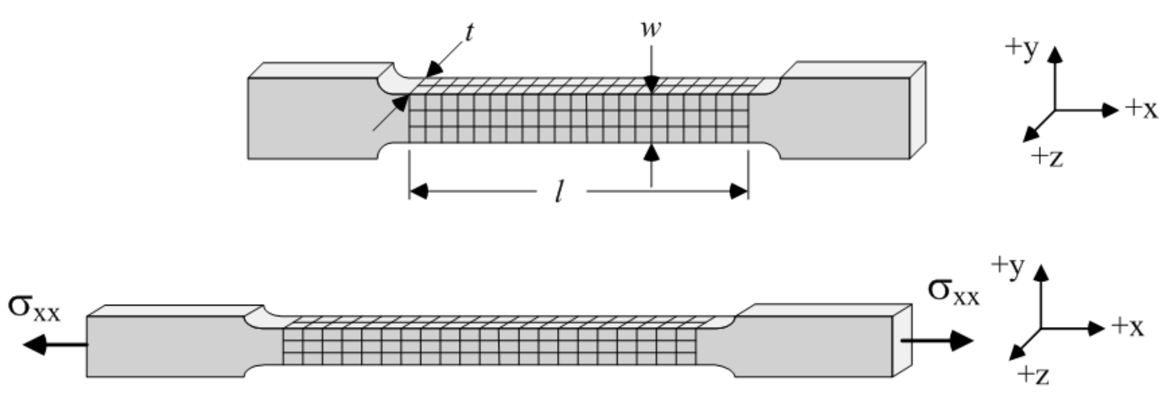
- Hooke's Law will be reviewed/discussed in the following order:
  - Isotropic materials
  - Anisotropic
    - Transversely isotropic
    - Orthotropic
    - Generally anisotropic

Isotropic Material Properties

The Uniaxial Tensile Test

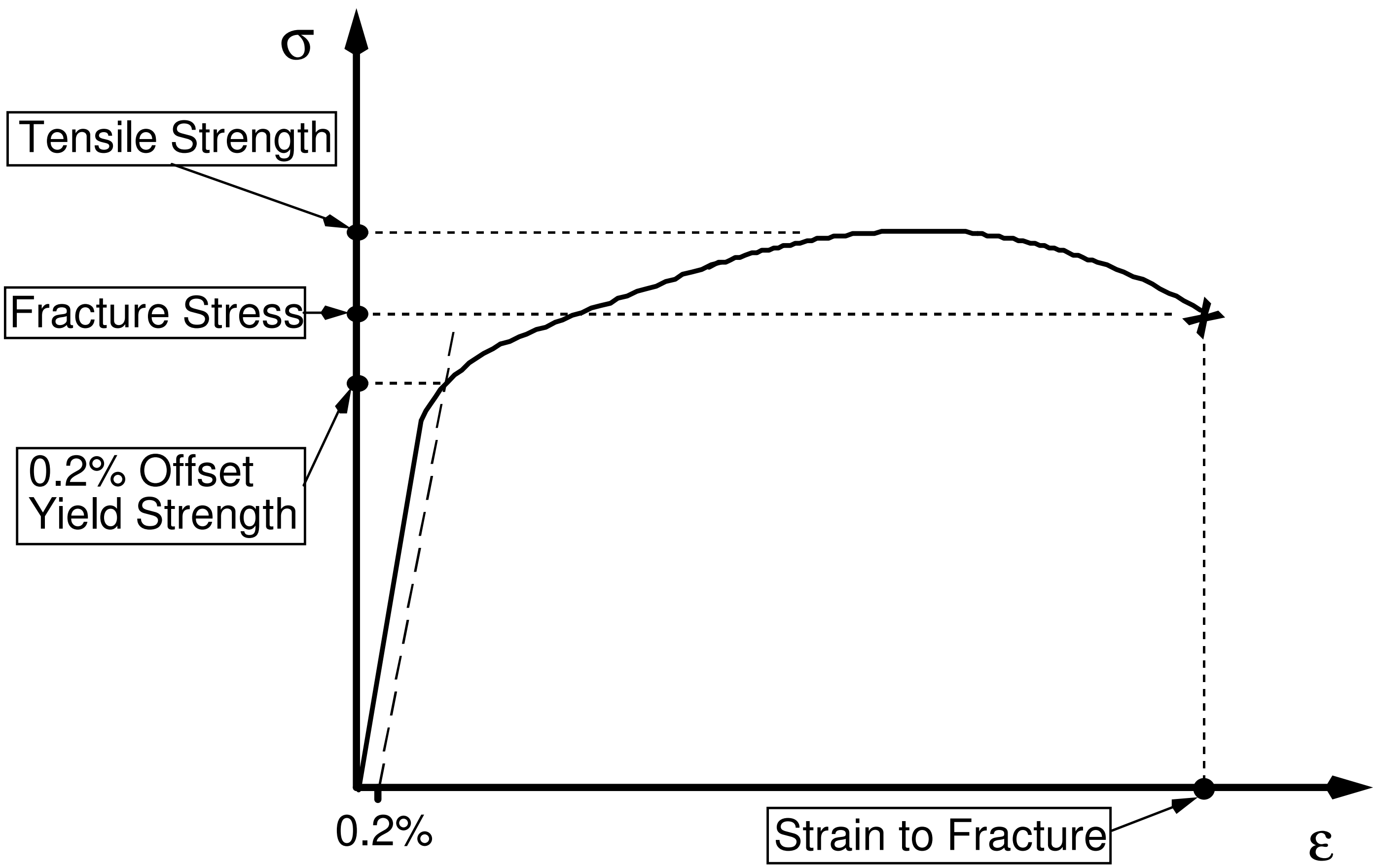
- Specimen is subjected to axial tensile force, inducing a *uniaxial* state of stress in the "gage" region
- Stress is increased until fracture occurs; corresponding axial and transverse strains are measured throughout the test (all shear strains = 0)

$$\varepsilon_{xx} = \Delta l/l \qquad \varepsilon_{yy} = \varepsilon_{zz} = \Delta w/w = \Delta t/t \qquad \gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0$$

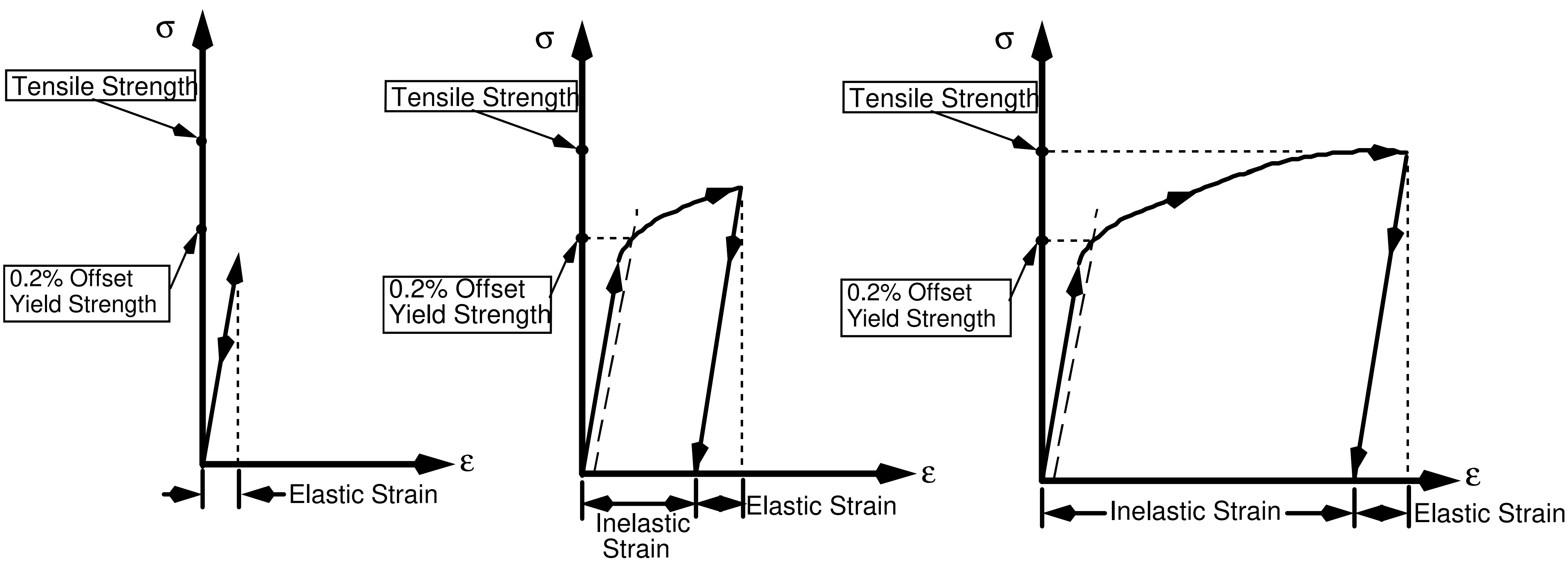


The Tensile Stress-Strain Curve

A plot of axial stress vs axial strain



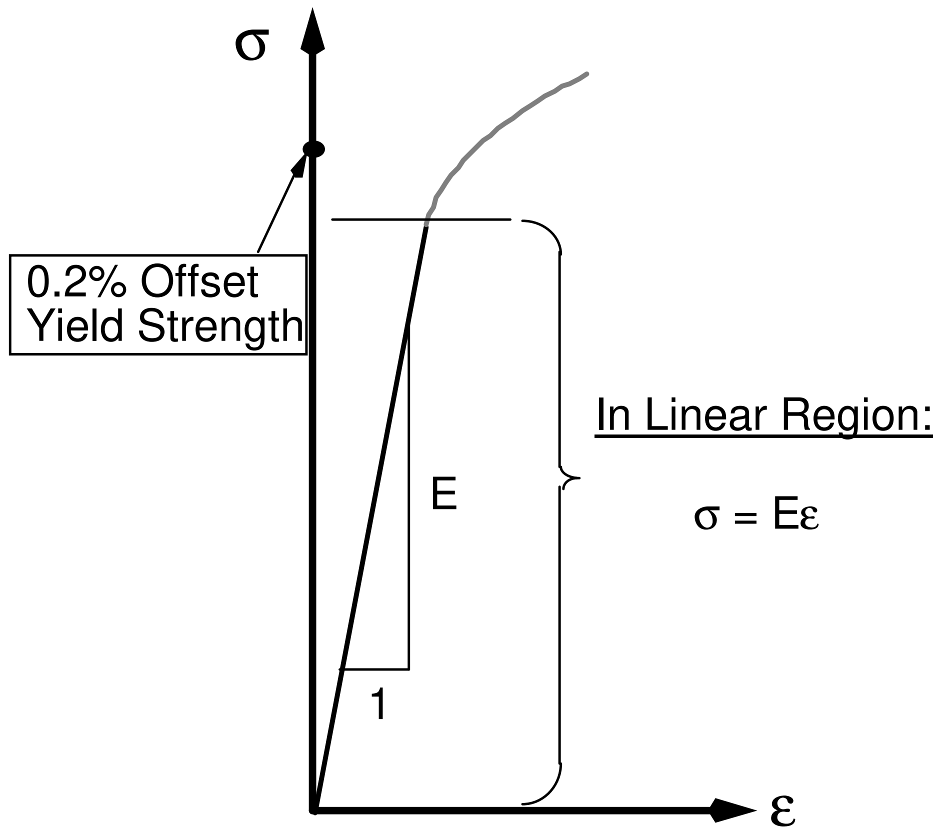
Load-Unload Cycles



Material Property:
Young's Modulus

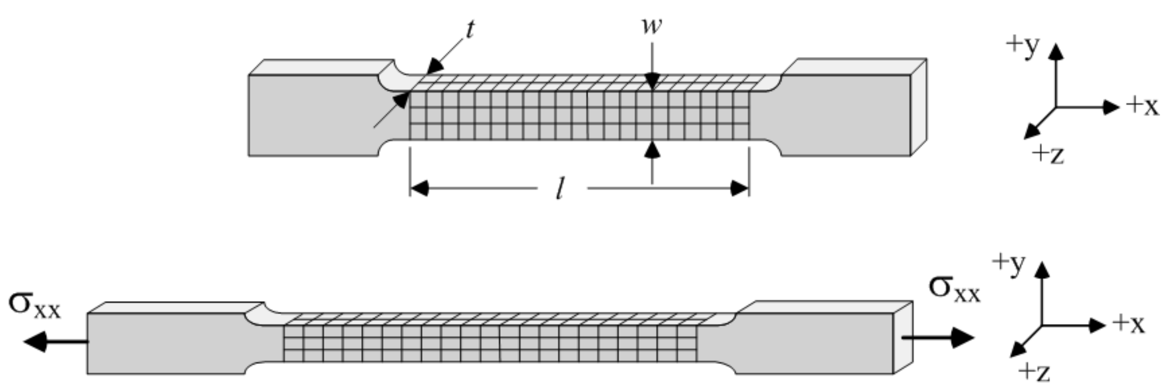
- At stress levels below the yield stress the response is called "linear elastic"
- The slope of the linear region is called "Young's modulus" or the "modulus of elasticity", E
- In the linear region and *for a uniaxial stress-state* (<u>only</u>!!!):

$\sigma = E\varepsilon$ (or) $\varepsilon = \sigma/E$



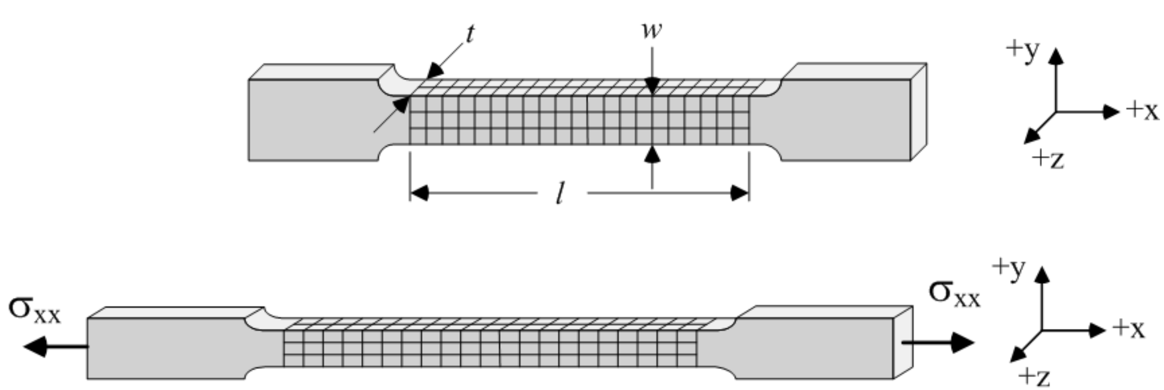
Material Property:
Poisson's Ratio

- Poisson's ratio is based on the ratio of two normal strains cause by a *uniaxial* stress: $\nu = -(\varepsilon_t/\varepsilon_a)$
- Poisson's ratio is a measure of the *coupling* between σ_{xx} and ε_{yy}, ε_{zz}
- In this case: $\nu = (-\varepsilon_{yy}/\varepsilon_{xx}) = (-\varepsilon_{zz}/\varepsilon_{xx})$



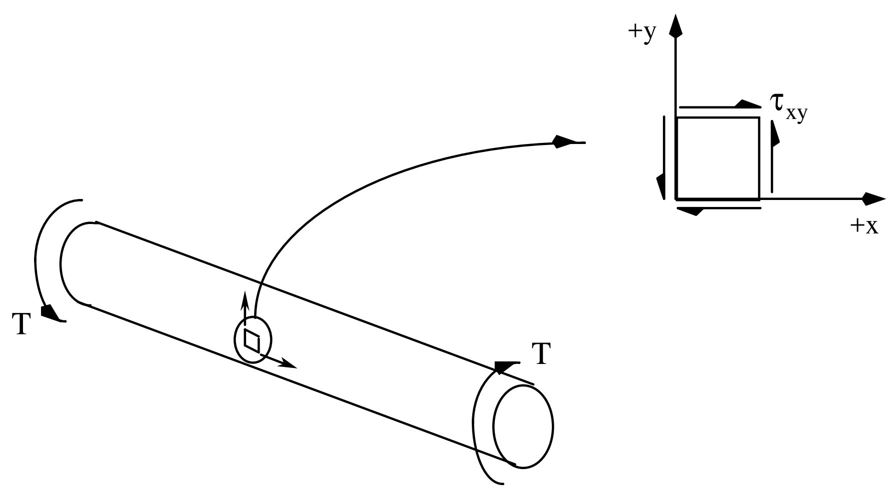
Material Property:
Poisson's Ratio

- Poisson's ratio is based on the ratio of two normal strains cause by a *uniaxial* stress: $\nu = -(\varepsilon_t/\varepsilon_a)$
- Poisson's ratio is a measure of the *coupling* between σ_{xx} and ε_{yy}, ε_{zz}
- In this case: $\nu = (-\varepsilon_{yy}/\varepsilon_{xx}) = (-\varepsilon_{zz}/\varepsilon_{xx})$



Material Properties
The Torsion Test

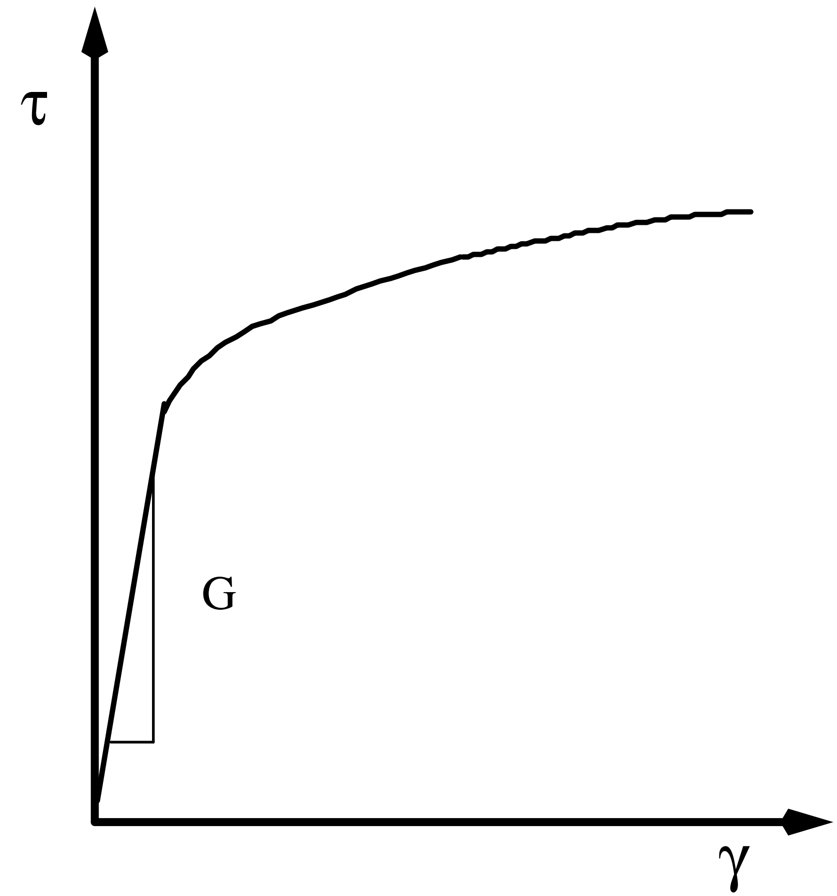
- Thin-walled cylindrical specimen subjected to a torque, inducing a uniform shear stress τ_{xy} in the gage region of the specimen
- Shear stress (i.e., torque) increased until fracture occurs; shear strain measured throughout test



The Shear Stress- Shear Strain Curve

- At linear levels, the slope of the shear stress -shear strain curve is called the "shear modulus"

- In the linear region (only!!) $\tau_{xy} = G\gamma_{xy}$ (or) $\gamma_{xy} = \tau_{xy}/G$



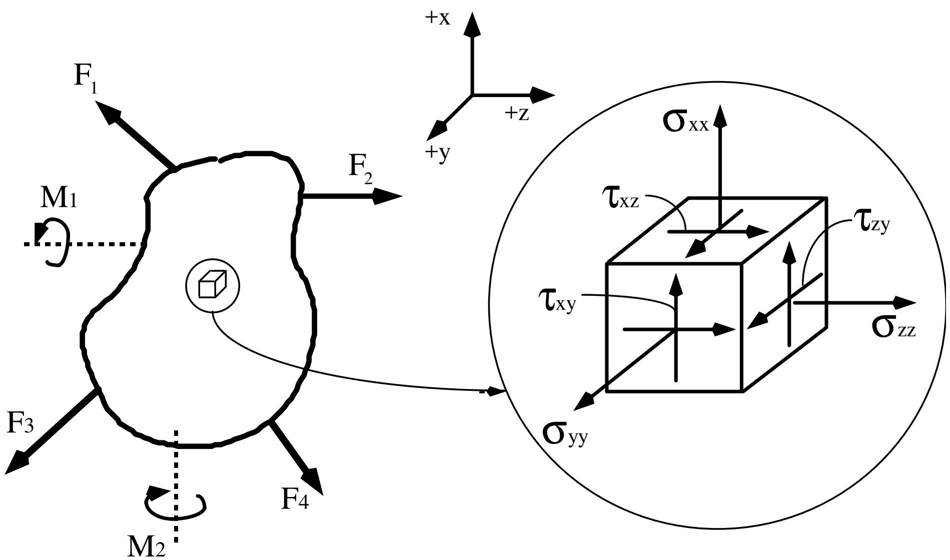
Number of Independent Material Properties

- Three material properties have been defined; E, ν, and G
- For an isotropic material, only two of these three properties are independent…it can be shown:

$$G = \frac{E}{2(1+\nu)}$$

Derivation of Hooke's Law

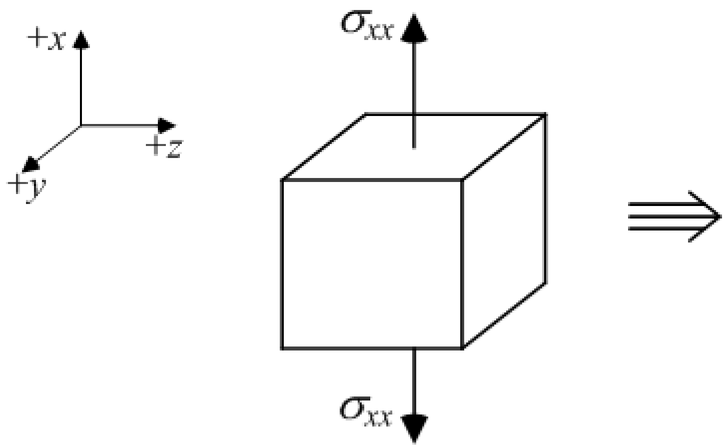
For an isotropic material subjected to general 3D streses



- We assume the strain tensor is linearly related to the stress tensor….(when is this a bad assumption?)
- Assuming the linear assumption is appropriate, the principle of superposition can be used to develop a Hooke's law:

Hooke's Law (cont'd)

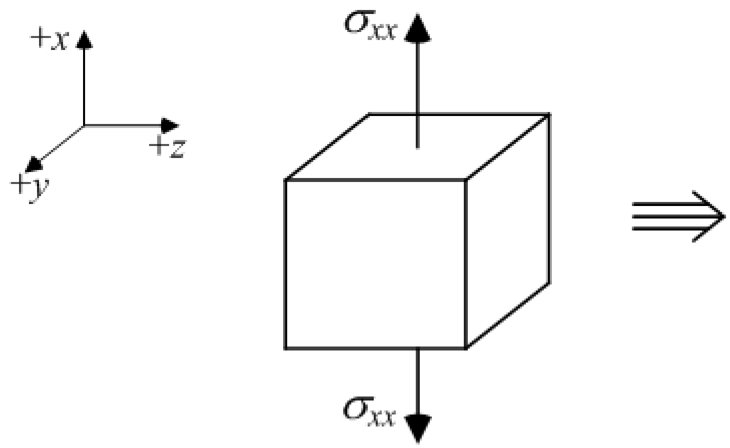
Strains caused by σ_{xx} only:



(What strains are induced by σ_{xx} only?)

Hooke's Law (cont'd)

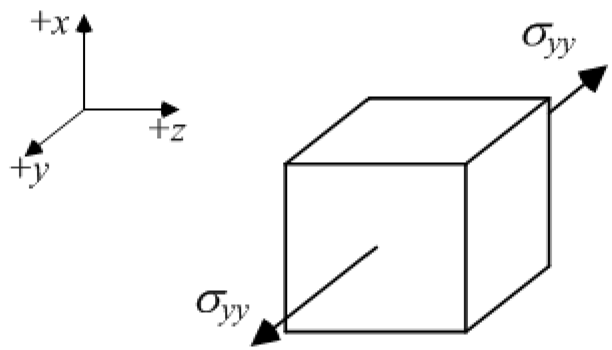
Strains caused by σ_{xx} only:



$$\varepsilon_{xx} = k_{11}\,\sigma_{xx} = (1/E)\,\sigma_{xx}$$
$$\varepsilon_{yy} = k_{21}\,\sigma_{xx} = (-\nu/E)\,\sigma_{xx}$$
$$\varepsilon_{zz} = k_{31}\,\sigma_{xx} = (-\nu/E)\,\sigma_{xx}$$
$$\gamma_{xy} = k_{41}\,\sigma_{xx} = 0$$
$$\gamma_{yz} = k_{51}\,\sigma_{xx} = 0$$
$$\gamma_{zx} = k_{61}\,\sigma_{xx} = 0$$

Hooke's Law (cont'd)

Strains caused by σ_{yy} only:

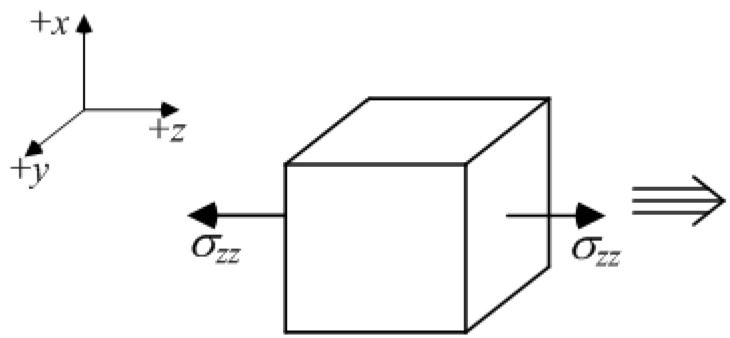


$$\Longrightarrow$$

$$\varepsilon_{xx} = k_{12}\,\sigma_{yy} = (-\nu/E)\,\sigma_{yy}$$
$$\varepsilon_{yy} = k_{22}\,\sigma_{yy} = (1/E)\,\sigma_{yy}$$
$$\varepsilon_{zz} = k_{32}\,\sigma_{yy} = (-\nu/E)\,\sigma_{yy}$$
$$\gamma_{xy} = k_{42}\,\sigma_{yy} = 0$$
$$\gamma_{yz} = k_{52}\,\sigma_{yy} = 0$$
$$\gamma_{zx} = k_{62}\,\sigma_{yy} = 0$$

Hooke's Law (cont'd)

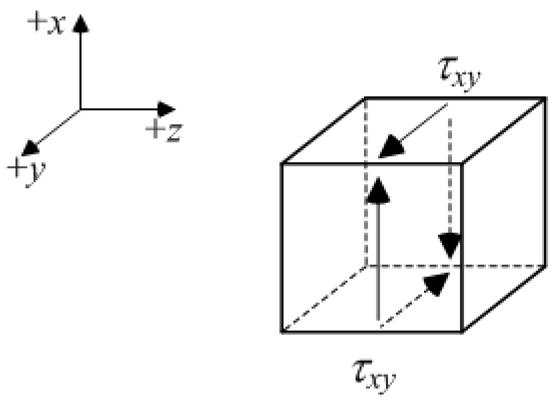
Strains caused by σ_{zz} only:



$$\varepsilon_{xx} = k_{13}\,\sigma_{zz} = (-\nu/E)\,\sigma_{zz}$$
$$\varepsilon_{yy} = k_{23}\,\sigma_{zz} = (-\nu/E)\,\sigma_{zz}$$
$$\varepsilon_{zz} = k_{33}\,\sigma_{zz} = (1/E)\,\sigma_{zz}$$
$$\gamma_{xy} = k_{43}\,\sigma_{zz} = 0$$
$$\gamma_{yz} = k_{53}\,\sigma_{zz} = 0$$
$$\gamma_{zx} = k_{63}\,\sigma_{zz} = 0$$

Hooke's Law (cont'd)

Strains caused by τ_{xy} only:

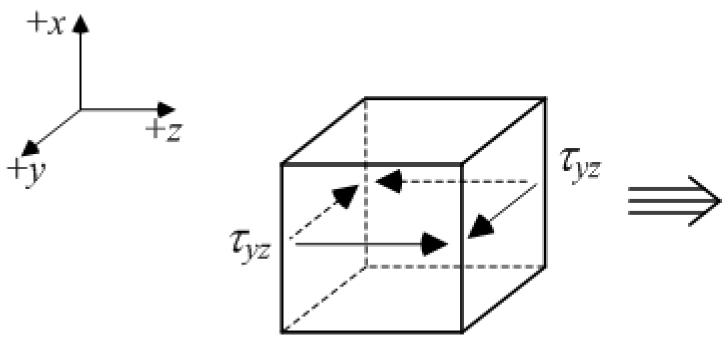


$\Longrightarrow$

$$\varepsilon_{xx} = k_{14}\ \tau_{xy} = 0$$
$$\varepsilon_{yy} = k_{24}\ \tau_{xy} = 0$$
$$\varepsilon_{zz} = k_{34}\ \tau_{xy} = 0$$
$$\gamma_{xy} = k_{44}\ \tau_{xy} = (1/G)\tau_{xy} = [2(1+\nu)/E]\ \tau_{xy}$$
$$\gamma_{yz} = k_{54}\ \tau_{xy} = 0$$
$$\gamma_{zx} = k_{64}\ \tau_{xy} = 0$$

Hooke's Law (cont'd)

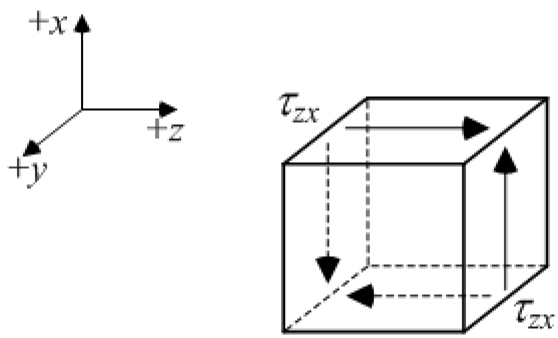
Strains caused by τ_{yz} only:



$$\varepsilon_{xx} = k_{15}\ \tau_{yz} = 0$$
$$\varepsilon_{yy} = k_{25}\ \tau_{yz} = 0$$
$$\varepsilon_{zz} = k_{35}\ \tau_{yz} = 0$$
$$\gamma_{xy} = k_{45}\ \tau_{yz} = 0$$
$$\gamma_{yz} = k_{55}\ \tau_{yz} = (1/G)\tau_{yz} = [2(1+\nu)/E]\ \tau_{yz}$$
$$\gamma_{zx} = k_{65}\ \tau_{yz} = 0$$

Hooke's Law (cont'd)

Strains caused by τ_{zx} only:

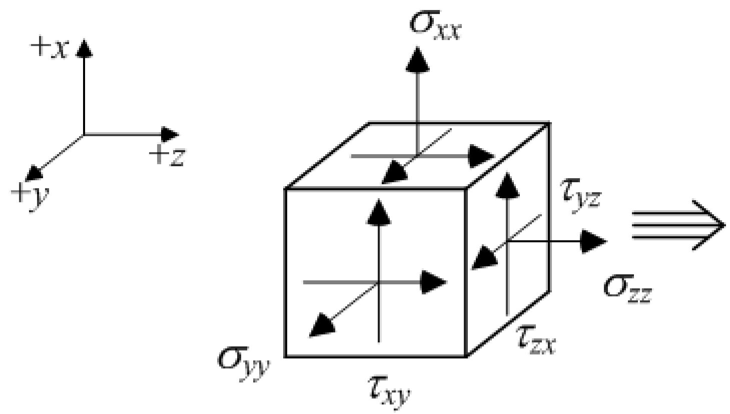


$$\Rightarrow$$

$$\varepsilon_{xx} = k_{16}\,\tau_{zx} = 0$$
$$\varepsilon_{yy} = k_{26}\,\tau_{zx} = 0$$
$$\varepsilon_{zz} = k_{36}\,\tau_{zx} = 0$$
$$\gamma_{xy} = k_{46}\,\tau_{zx} = 0$$
$$\gamma_{yz} = k_{56}\,\tau_{zx} = 0$$
$$\gamma_{zx} = k_{66}\,\tau_{zx} = (1/G)\tau_{zx} = [2(1+\nu)/E]\,\tau_{zx}$$

Hooke's Law (cont'd)

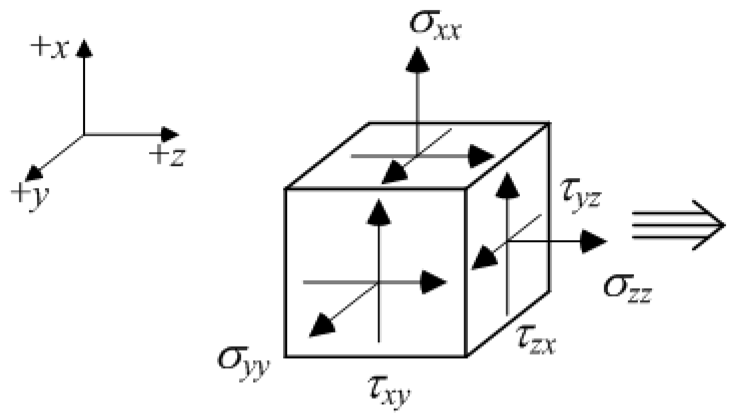
Strain ε_{xx} caused by all stress components acting simultaneously:



$$\varepsilon_{xx} = ?$$

Hooke's Law (cont'd)

Strain ε_{xx} caused by all stress components acting simultaneously:

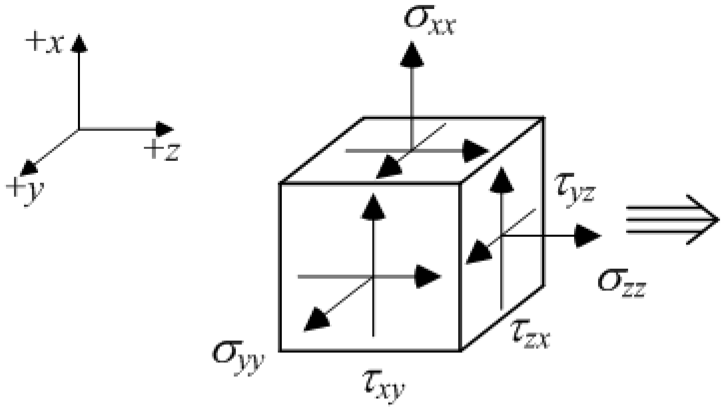


$\varepsilon_{xx} = ?$

(since strain-stress relation assumed linear, we can apply the principle of superposition and simply add up contribution of each stress component):

Hooke's Law (cont'd)

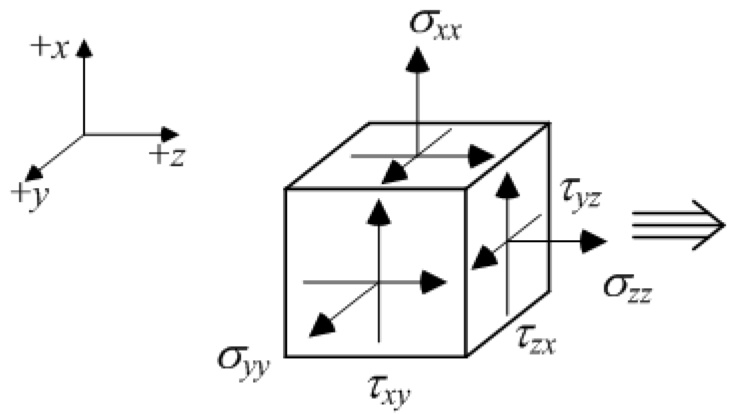
Strain ε_{xx} caused by all stress components acting simultaneously:



$$\varepsilon_{xx} = k_{11}\,\sigma_{xx} + k_{12}\,\sigma_{yy} + k_{13}\,\sigma_{zz} + k_{14}\,\tau_{xy} + k_{15}\,\tau_{yz} + k_{16}\,\tau_{zx}$$

Hooke's Law (cont'd)

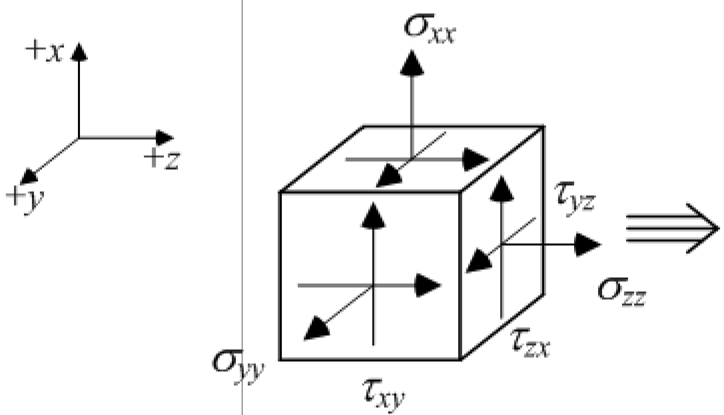
Strain ε_{xx} caused by all stress components acting simultaneously:



$$\begin{aligned}\varepsilon_{xx} = (1/E)\,\sigma_{xx} &\\ + (-\nu/E)\sigma_{yy} &\\ + (-\nu/E)\,\sigma_{zz} &\\ + (0)\,\tau_{xy} &\\ + (0)\,\tau_{yz} &\\ + (0)\,\tau_{zx} &\end{aligned}$$

Hooke's Law (cont'd)

Rearranging:



$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right]$$

Hooke's Law

Repeating process for all six strain components

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \qquad \gamma_{xy} = \frac{2(1+\nu)\tau_{xy}}{E}$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] \qquad \gamma_{xz} = \frac{2(1+\nu)\tau_{xz}}{E}$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] \qquad \gamma_{yz} = \frac{2(1+\nu)\tau_{yz}}{E}$$

Hooke's Law

Matrix Notation

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$$

Hooke's Law

Inverting the six equations leads to a more convenient form for experimental analysis…

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu(\varepsilon_{yy} + \varepsilon_{zz})\right] \qquad \tau_{xy} = \frac{E\gamma_{xy}}{2(1+\nu)}$$

$$\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{yy} + \nu(\varepsilon_{xx} + \varepsilon_{zz})\right] \qquad \tau_{xz} = \frac{E\gamma_{xz}}{2(1+\nu)}$$

$$\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{zz} + \nu(\varepsilon_{xx} + \varepsilon_{yy})\right] \qquad \tau_{yz} = \frac{E\gamma_{yz}}{2(1+\nu)}$$

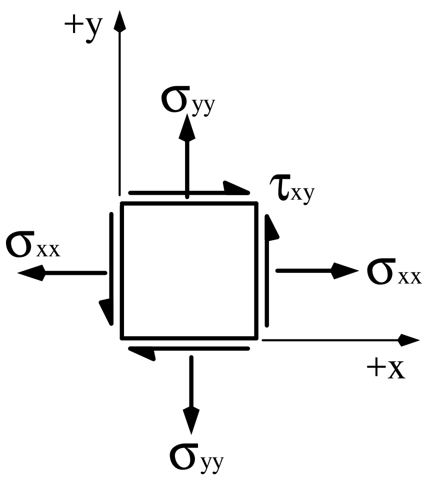
Hooke's Law

Matrix Notation

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ \nu & (1-\nu) & \nu & 0 & 0 & 0 \\ \nu & \nu & (1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

Hooke's Law For Plane Stress

assume $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$



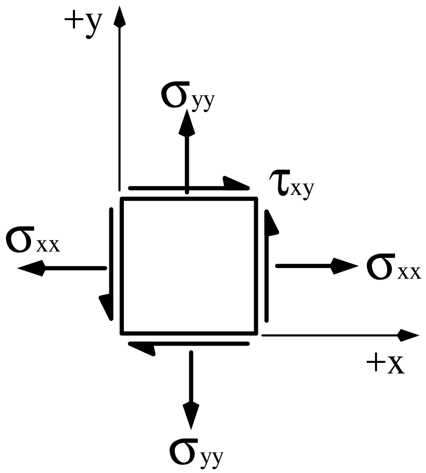
$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \qquad \gamma_{xy} = \frac{2(1+\nu)\tau_{xy}}{E}$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] \qquad \gamma_{xz} = \frac{2(1+\nu)\tau_{xz}}{E}$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] \qquad \gamma_{yz} = \frac{2(1+\nu)\tau_{yz}}{E}$$

Hooke's Law For Plane Stress

assume $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$



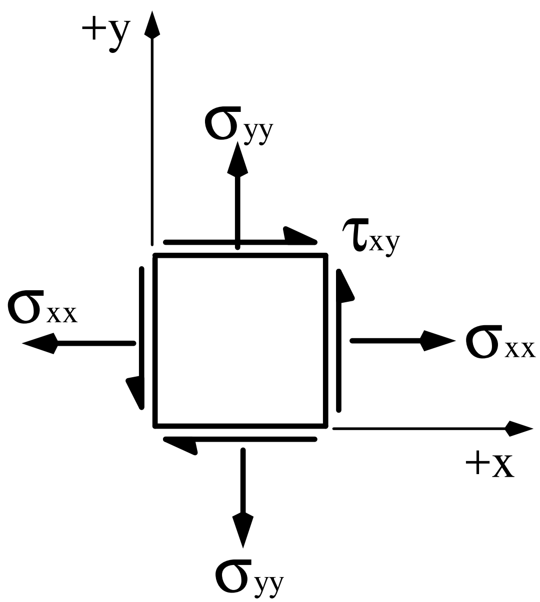
$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \cancelto{0}{\sigma_{zz}})\right] \qquad \gamma_{xy} = \frac{2(1+\nu)\tau_{xy}}{E}$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \cancelto{0}{\sigma_{zz}})\right] \qquad \gamma_{xz} = \frac{2(1+\nu)\cancelto{0}{\tau_{xz}}}{E}$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\cancelto{0}{\sigma_{zz}} - \nu(\sigma_{xx} + \sigma_{yy})\right] \qquad \gamma_{yz} = \frac{2(1+\nu)\cancelto{0}{\tau_{yz}}}{E}$$

Hooke's Law For Plane Stress

assume $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$



$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right)$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx}\right)$$

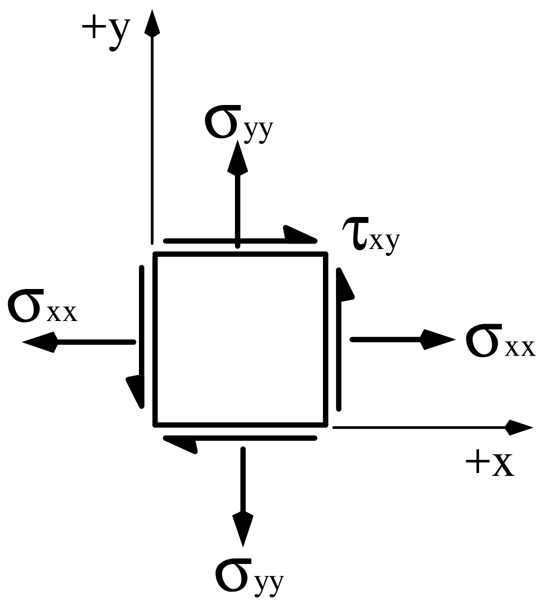
$$\varepsilon_{zz} = \frac{-\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right)$$

$$\gamma_{xy} = \frac{2(1+\nu)\tau_{xy}}{E}$$

$$\gamma_{xz} = \gamma_{yz} = 0$$

Hooke's Law For Plane Stress

assume $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$



$$\sigma_{xx} = \frac{E}{(1-\nu^2)}\left[\varepsilon_{xx} + \nu\varepsilon_{yy}\right]$$

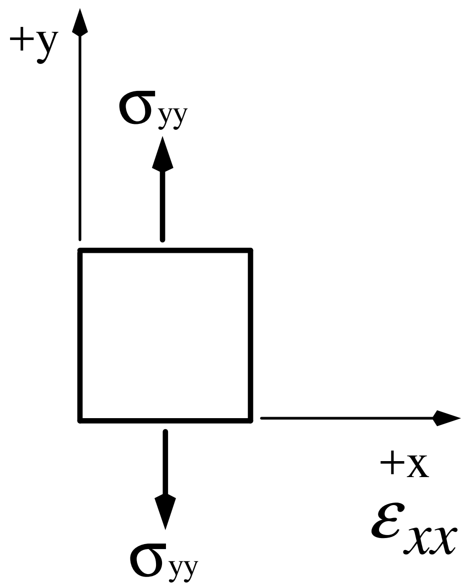
$$\sigma_{yy} = \frac{E}{(1-\nu^2)}\left[\varepsilon_{yy} + \nu\varepsilon_{xx}\right]$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

$$\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$$

Hooke's Law for Uniaxial Stress

$$\textit{If } \sigma_{xx} = \sigma_{zz} = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0$$



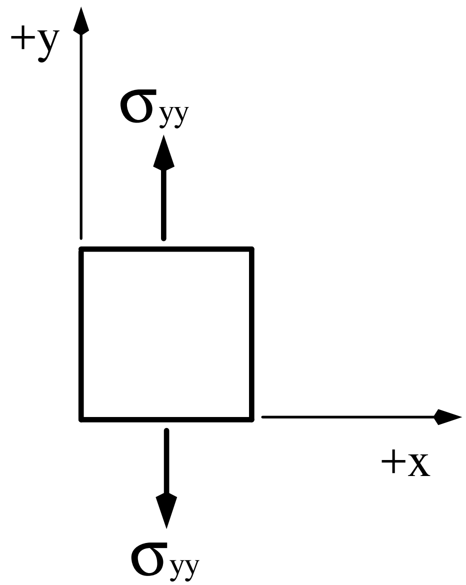
$$\varepsilon_{xx} = \frac{1}{E}\left[\cancelto{0}{\sigma_{xx}} - \nu(\sigma_{yy} + \cancelto{0}{\sigma_{zz}})\right] \qquad \gamma_{xy} = \cancelto{0}{\frac{2(1+\nu)\tau_{xy}}{E}}$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\cancelto{0}{\sigma_{xx}} + \cancelto{0}{\sigma_{zz}})\right] \qquad \gamma_{xz} = \cancelto{0}{\frac{2(1+\nu)\tau_{xz}}{E}}$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\cancelto{0}{\sigma_{zz}} - \nu(\cancelto{0}{\sigma_{xx}} + \sigma_{yy})\right] \qquad \gamma_{yz} = \cancelto{0}{\frac{2(1+\nu)\tau_{yz}}{E}}$$

Hooke's Law for Uniaxial Stress

$$\text{If } \sigma_{xx} = \sigma_{zz} = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0$$



$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E}$$

$$\varepsilon_{xx} = \varepsilon_{zz} = \frac{-\nu\sigma_{yy}}{E}$$

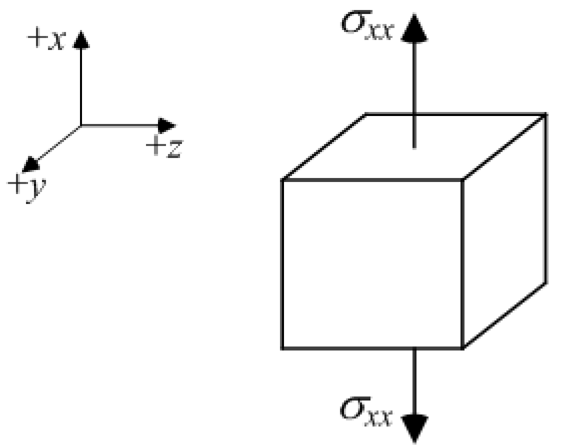
$$\gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0$$

Hooke's Law
Anisotropic materials

- As before, assume stress is linearly related to strain....
- Anisotropic material exhibit two "unusual" features (as compared to isotropic materials):
  - Properties differ with direction (e.g, in general $E_{xx} \neq E_{yy} \neq E_{zz}$)
  - This can lead to unusual "coupling" effects:
    - A *normal* stress may cause *shear* strains
    - A *shear* stress may cause *normal* strains

Hooke's Law – Anisotropic Materials

Strains caused by σ_{xx} only:

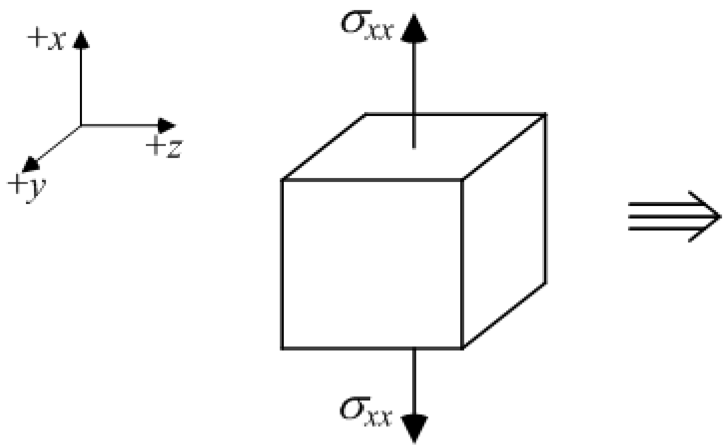


$\Rrightarrow$

(What strains are induced by σ_{xx} only?)

Hooke's Law – Anisotropic Materials

Strains caused by σ_{xx} only:

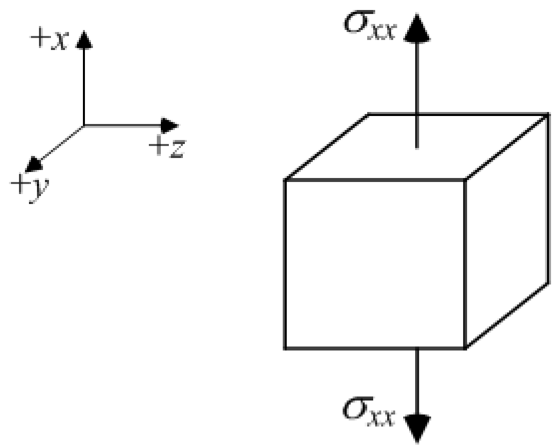


(What strains are induced by σ_{xx} only?)

…for generally anisotropic materials, σ_{xx} will induce *six* components of strain (i.e., σ_{xx} will induce a 3-D strain tensor)

Hooke's Law – Anisotropic Materials

Strains caused by σ_{xx} only:

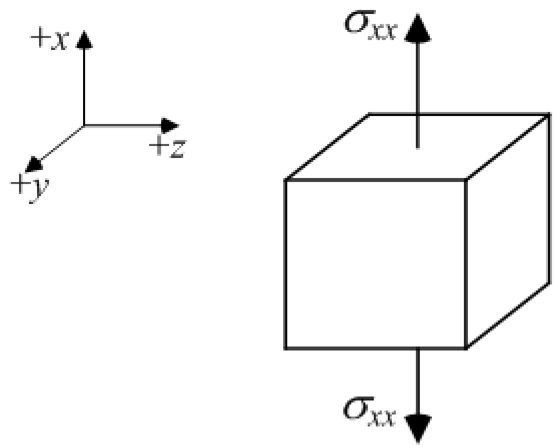


$\Longrightarrow$

$$\varepsilon_{xx} = k_{11}\,\sigma_{xx}$$
$$\varepsilon_{yy} = k_{21}\,\sigma_{xx}$$
$$\varepsilon_{zz} = k_{31}\,\sigma_{xx}$$
$$\gamma_{xy} = k_{41}\,\sigma_{xx}$$
$$\gamma_{yz} = k_{51}\,\sigma_{xx}$$
$$\gamma_{zx} = k_{61}\,\sigma_{xx}$$

Hooke's Law – Anisotropic Materials

Strains caused by σ_{xx} only:



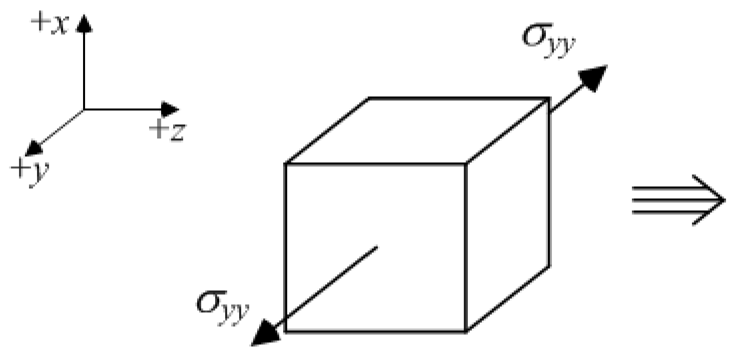
$$\Longrightarrow$$

$$\varepsilon_{xx} = k_{11}\,\sigma_{xx}$$
$$\varepsilon_{yy} = k_{21}\,\sigma_{xx}$$
$$\varepsilon_{zz} = k_{31}\,\sigma_{xx}$$
$$\gamma_{xy} = k_{41}\,\sigma_{xx}$$
$$\gamma_{yz} = k_{51}\,\sigma_{xx}$$
$$\gamma_{zx} = k_{61}\,\sigma_{xx}$$

In general: $k_{11} \neq k_{21} \neq k_{31} \neq k_{41} \neq k_{51} \neq k_{61} \neq 0$

Hooke's Law – Anisotropic Materials

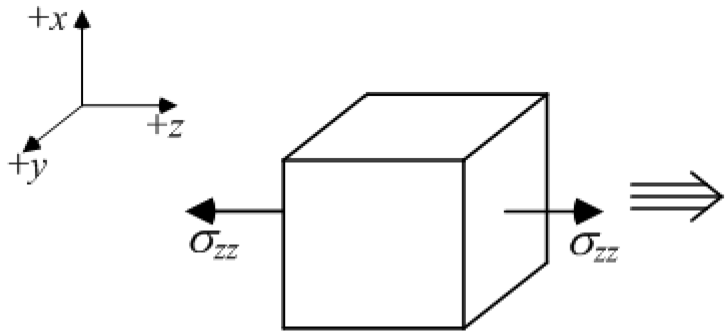
Strains caused by σ_{yy} only:



$$\varepsilon_{xx} = k_{12}\,\sigma_{yy}$$
$$\varepsilon_{yy} = k_{22}\,\sigma_{yy}$$
$$\varepsilon_{zz} = k_{32}\,\sigma_{yy}$$
$$\gamma_{xy} = k_{42}\,\sigma_{yy}$$
$$\gamma_{yz} = k_{52}\,\sigma_{yy}$$
$$\gamma_{zx} = k_{62}\,\sigma_{yy}$$

Hooke's Law – Anisotropic Materials

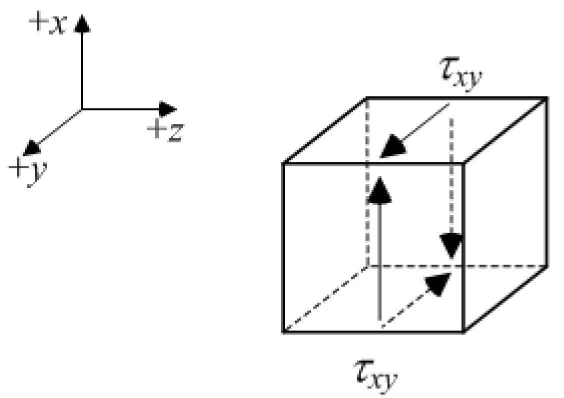
Strains caused by σ_{zz} only:



$$\varepsilon_{xx} = k_{13}\,\sigma_{zz}$$
$$\varepsilon_{yy} = k_{23}\,\sigma_{zz}$$
$$\varepsilon_{zz} = k_{33}\,\sigma_{zz}$$
$$\gamma_{xy} = k_{43}\,\sigma_{zz}$$
$$\gamma_{yz} = k_{53}\,\sigma_{zz}$$
$$\gamma_{zx} = k_{63}\,\sigma_{zz}$$

Hooke's Law – Anisotropic Materials

Strains caused by τ_{xy} only:

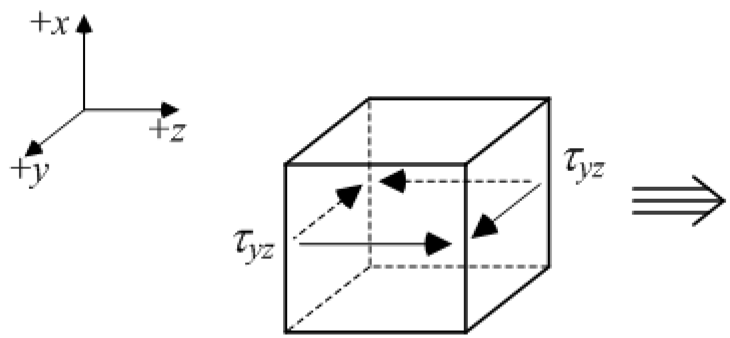


$$\Longrightarrow$$

$$\varepsilon_{xx} = k_{14}\,\tau_{xy}$$
$$\varepsilon_{yy} = k_{24}\,\tau_{xy}$$
$$\varepsilon_{zz} = k_{34}\,\tau_{xy}$$
$$\gamma_{xy} = k_{44}\,\tau_{xy}$$
$$\gamma_{yz} = k_{54}\,\tau_{xy}$$
$$\gamma_{zx} = k_{64}\,\tau_{xy}$$

Hooke's Law – Anisotropic Materials

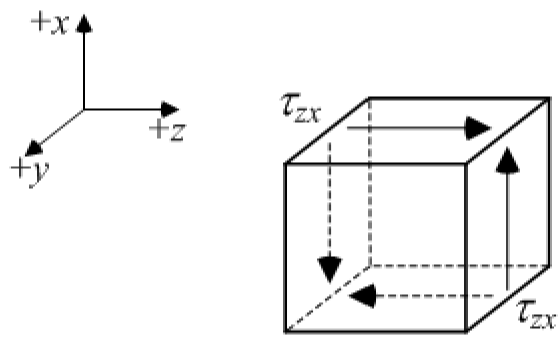
Strains caused by τ_{yz} only:



$$\varepsilon_{xx} = k_{15}\,\tau_{yz}$$
$$\varepsilon_{yy} = k_{25}\,\tau_{yz}$$
$$\varepsilon_{zz} = k_{35}\,\tau_{yz}$$
$$\gamma_{xy} = k_{45}\,\tau_{yz}$$
$$\gamma_{yz} = k_{55}\,\tau_{yz}$$
$$\gamma_{zx} = k_{65}\,\tau_{yz}$$

Hooke's Law – Anisotropic Materials

Strains caused by τ_{zx} only:

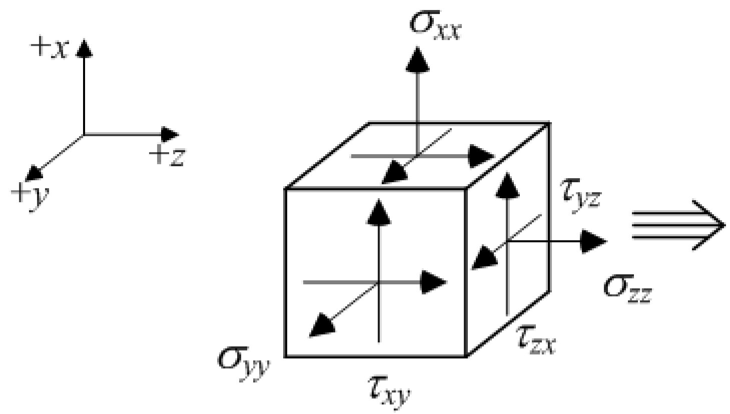


$\Longrightarrow$

$$\varepsilon_{xx} = k_{16}\,\tau_{zx}$$
$$\varepsilon_{yy} = k_{26}\,\tau_{zx}$$
$$\varepsilon_{zz} = k_{36}\,\tau_{zx}$$
$$\gamma_{xy} = k_{46}\,\tau_{zx}$$
$$\gamma_{yz} = k_{56}\,\tau_{zx}$$
$$\gamma_{zx} = k_{66}\,\tau_{zx}$$

Hooke's Law – Anisotropic Materials

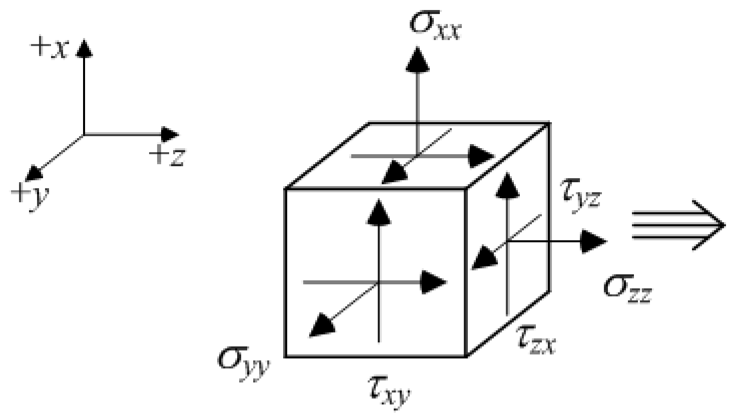
Strain ε_{xx} caused by all stress components acting simultaneously:



$\varepsilon_{xx} = ?$

Hooke's Law – Anisotropic Materials

Strain ε_{xx} *caused by all stress components acting simultaneously:*

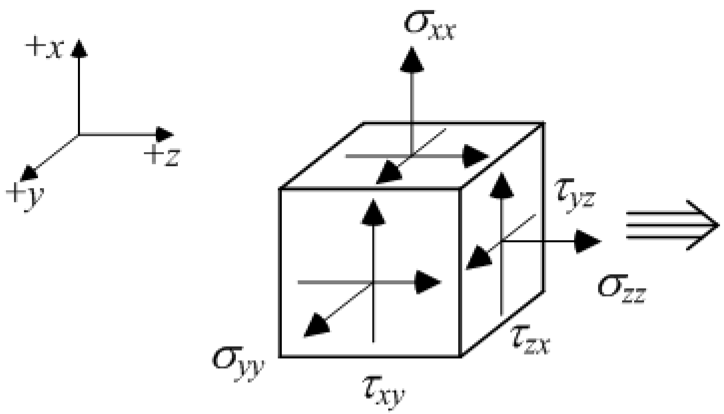


$\varepsilon_{xx} = ?$

(since strain-stress relation assumed linear, we can apply the principle of superposition and simply add up contribution of each stress component):

Hooke's Law – Anisotropic Materials

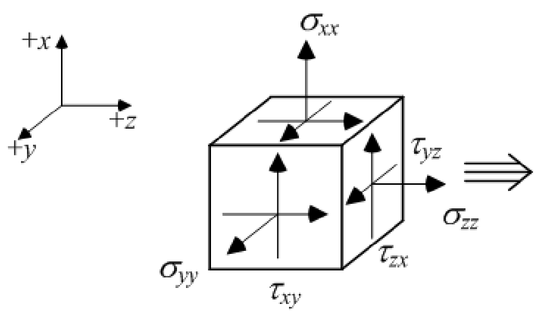
Strain ε_{xx} caused by all stress components acting simultaneously:



$$\varepsilon_{xx} = k_{11}\,\sigma_{xx} + k_{12}\,\sigma_{yy} + k_{13}\,\sigma_{zz} + k_{14}\,\tau_{xy} + k_{15}\,\tau_{yz} + k_{16}\,\tau_{zx}$$

Hooke's Law – Anisotropic Materials

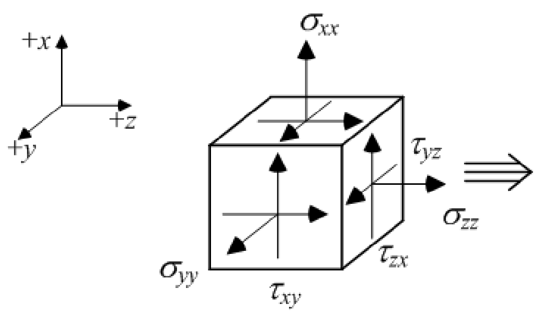
Repeating this process for each strain component results in six simultaneous equations



$$\varepsilon_{xx} = k_{11}\,\sigma_{xx} + k_{12}\,\sigma_{yy} + k_{13}\,\sigma_{zz} + k_{14}\,\tau_{xy} + k_{15}\,\tau_{yz} + k_{16}\,\tau_{zx}$$
$$\varepsilon_{yy} = k_{21}\,\sigma_{xx} + k_{22}\,\sigma_{yy} + k_{23}\,\sigma_{zz} + k_{24}\,\tau_{xy} + k_{25}\,\tau_{yz} + k_{26}\,\tau_{zx}$$
$$\varepsilon_{zz} = k_{31}\,\sigma_{xx} + k_{32}\,\sigma_{yy} + k_{33}\,\sigma_{zz} + k_{34}\,\tau_{xy} + k_{35}\,\tau_{yz} + k_{36}\,\tau_{zx}$$
$$\gamma_{xy} = k_{41}\,\sigma_{xx} + k_{42}\,\sigma_{yy} + k_{43}\,\sigma_{zz} + k_{44}\,\tau_{xy} + k_{45}\,\tau_{yz} + k_{46}\,\tau_{zx}$$
$$\gamma_{yz} = k_{51}\,\sigma_{xx} + k_{52}\,\sigma_{yy} + k_{53}\,\sigma_{zz} + k_{54}\,\tau_{xy} + k_{55}\,\tau_{yz} + k_{56}\,\tau_{zx}$$
$$\gamma_{zx} = k_{61}\,\sigma_{xx} + k_{62}\,\sigma_{yy} + k_{63}\,\sigma_{zz} + k_{64}\,\tau_{xy} + k_{65}\,\tau_{yz} + k_{66}\,\tau_{zx}$$

Hooke's Law – Anisotropic Materials

…the six eq's can be expressed using matrix notation



$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

Hooke's Law – Anisotropic Materials

Inverting the six equations:

$$
\begin{aligned}
\sigma_{xx} &= K_{11}\,\varepsilon_{xx} + K_{12}\,\varepsilon_{yy} + K_{13}\,\varepsilon_{zz} + K_{14}\,\gamma_{xy} + K_{15}\,\gamma_{yz} + K_{16}\,\gamma_{zx} \\
\sigma_{yy} &= K_{21}\,\varepsilon_{xx} + K_{22}\,\varepsilon_{yy} + K_{23}\,\varepsilon_{zz} + K_{24}\,\gamma_{xy} + K_{25}\,\gamma_{yz} + K_{26}\,\gamma_{zx} \\
\sigma_{zz} &= K_{31}\,\varepsilon_{xx} + K_{32}\,\varepsilon_{yy} + K_{33}\,\varepsilon_{zz} + K_{34}\,\gamma_{xy} + K_{35}\,\gamma_{yz} + K_{36}\,\gamma_{zx} \\
\tau_{xy} &= K_{41}\,\varepsilon_{xx} + K_{42}\,\varepsilon_{yy} + K_{43}\,\varepsilon_{zz} + K_{44}\,\gamma_{xy} + K_{45}\,\gamma_{yz} + K_{46}\,\gamma_{zx} \\
\tau_{yz} &= K_{51}\,\varepsilon_{xx} + K_{52}\,\varepsilon_{yy} + K_{53}\,\varepsilon_{zz} + K_{54}\,\gamma_{xy} + K_{55}\,\gamma_{yz} + K_{56}\,\gamma_{zx} \\
\tau_{zx} &= K_{61}\,\varepsilon_{xx} + K_{62}\,\varepsilon_{yy} + K_{63}\,\varepsilon_{zz} + K_{64}\,\gamma_{xy} + K_{65}\,\gamma_{yz} + K_{66}\,\gamma_{zx}
\end{aligned}
\tag{2.11}
$$

Hooke's Law – Anisotropic Materials

Using matrix notation:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

- K_{ij} = $[K_{ij}]$ = "coefficients of elasticity"

Hooke's Law – Anisotropic Materials

Inverting the six equations:

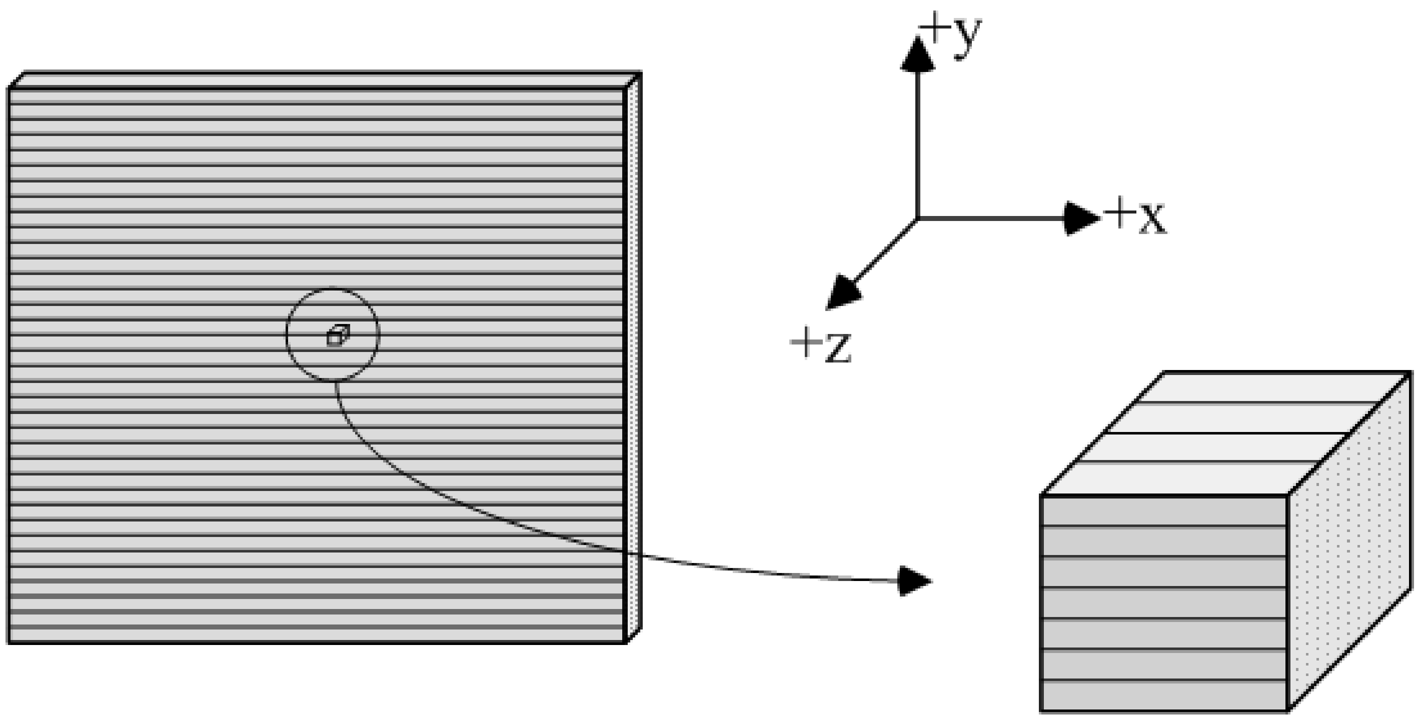
- Apparently, there are 36 coefficients of elasticity, however
- Strain energy considerations show that the $[K_{ij}]$ matrix *must be symmetric*….number of independent coefficients reduces from 36 to 21:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{12} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{13} & K_{23} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{14} & K_{24} & K_{34} & K_{44} & K_{45} & K_{46} \\ K_{15} & K_{25} & K_{35} & K_{45} & K_{55} & K_{56} \\ K_{16} & K_{26} & K_{36} & K_{46} & K_{56} & K_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

Hooke's Law – Anisotropic Materials

Anisotropy originates due to microstructure

• Unidirectional composites possess an inherent "principal material coordinate system", defined by the fiber orientation



Hooke's Law – Anisotropic Materials

Anisotropy originates due to microstructure

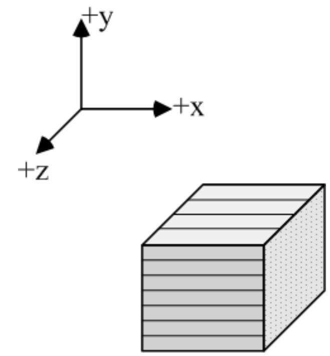
• If the stress and strain tensor are described relative to the principal material coordinate system, then there is no coupling between normal stress and shear strain, and no coupling between shear stress and normal strain:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 \\ K_{13} & K_{23} & K_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{44} & 0 & 0 \\ 0 & 0 & 0 & & K_{55} & 0 \\ & & & & & K_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

Hooke's Law – Anisotropic Materials

Anisotropy originates due to microstructure

- If fiber distribution *differs* in y- and z-directions, then:



$E_{xx} > E_{yy} \neq E_{zz}$

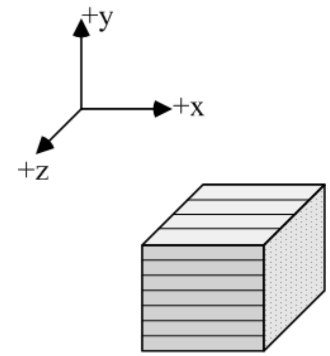
__*Orthotropic Material*__

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 \\ K_{13} & K_{23} & K_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{44} & 0 & 0 \\ 0 & 0 & 0 & & K_{55} & 0 \\ & & & & & K_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{2.21}$$

Hooke's Law – Anisotropic Materials

Anisotropy originates due to microstructure

- If fiber distribution in y- and z-directions is identical, then:



$E_{xx} > E_{yy} = E_{zz}$

__Transversely Isotropic Material__

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{12} & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 \\ K_{12} & K_{23} & K_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{K_{22}-K_{23}}{2} & 0 & 0 \\ 0 & 0 & 0 & & K_{66} & 0 \\ & & & & & K_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

(2.32)

Hooke's Law
Summary of Key Points

- Hooke's Law is valid under linear-elastic conditions only
- The mathematical form of Hooke's Law depends on the problem involved:
  - Isotropic vs anisotropic materials
  - 3-D stress/strains
  - Plane stress states
  - Plane strain states
  - Uniaxial stress

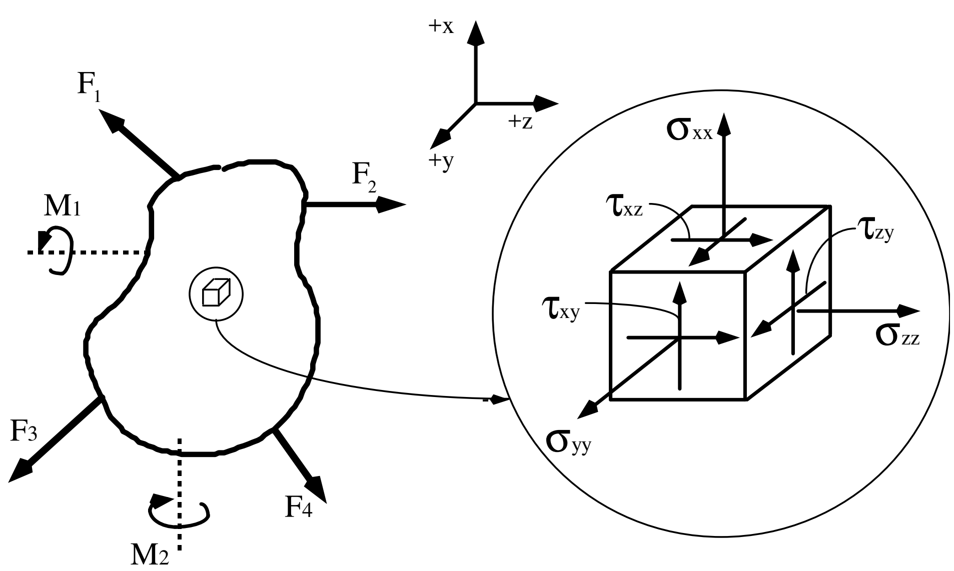
Failure Predictions

- Isotropic metals: methods to predict failure are reasonably well-developed and are typically based on:
  - Yield criterion (i.e., predict the stress or strain tensor necessary to cause nonlinear behavior), and/or
  - Fracture mechanics (i.e., predict the stress or strain tensor that will cause either stable or unstable crack growth)
- Anisotropic materials (composites): methods to predict failure are not as well-developed, and often vary from company-to-company or industry-to-industry…predicting failure of anisotropic materials is a active research topic and will not be discussed in this review

Failure Predictions for Isotropic Metals

Yield criterion

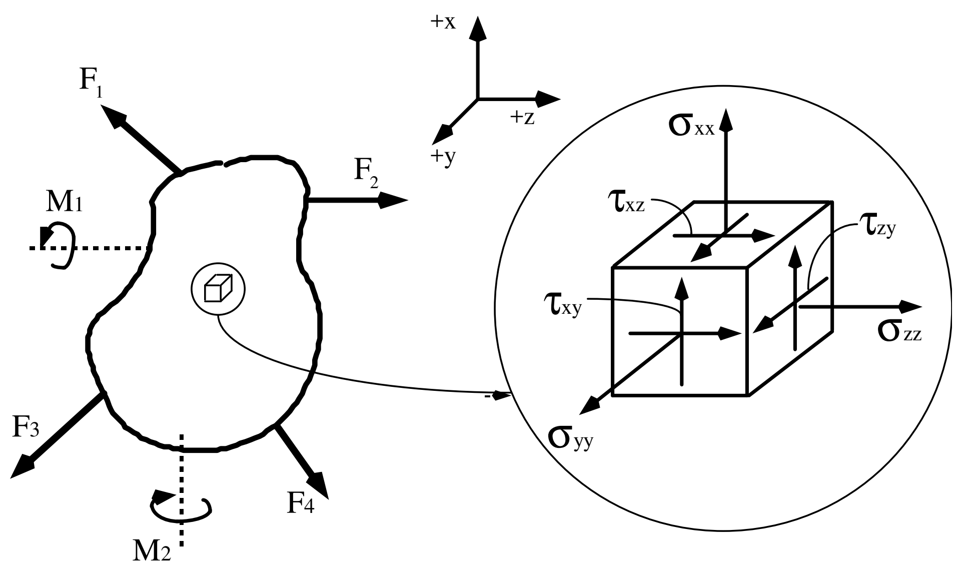
- Context: a structure is subjected to external loads, causing a 3-D state of stress. What load level will cause nonlinear behavior (yielding)?



Failure Predictions for Isotropic Metals

Yield criterion

- The von Mises criterion is most commonly used to predict yielding of isotropic metals such as steel or aluminum alloys (aka the Maxwell-Huber-Hencky-von Mises criterion)
- Other common yield criterion include the Tresca, Max Normal Stress, or Mohr's criterion…these criterion will not be reviewed here



Failure Predictions for Isotropic Metals

The von Mises criterion: 3D stress states

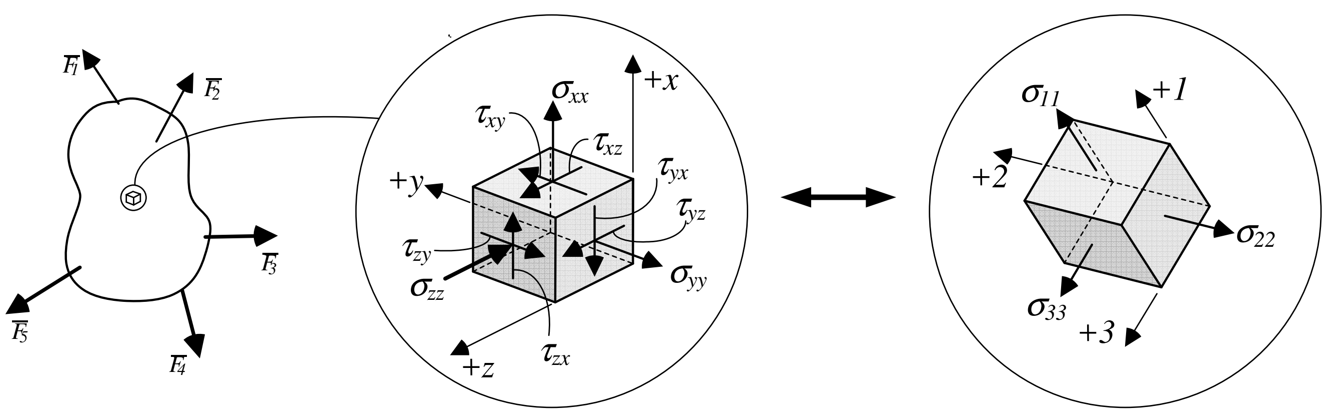
- Yielding occurs if:

$$\frac{1}{2}\left[(\sigma_{xx}-\sigma_{yy})^2+(\sigma_{yy}-\sigma_{zz})^2+(\sigma_{zz}-\sigma_{xx})^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{xz}^2)\right]\geq\sigma_Y^2$$

or, equivalently, if:

$$\frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2\right]\geq\sigma_Y^2$$

where σ_Y = 0.2% offset yield strength (usually tensile)



Failure Predictions for Isotropic Metals

The von Mises criterion – Plane stress states

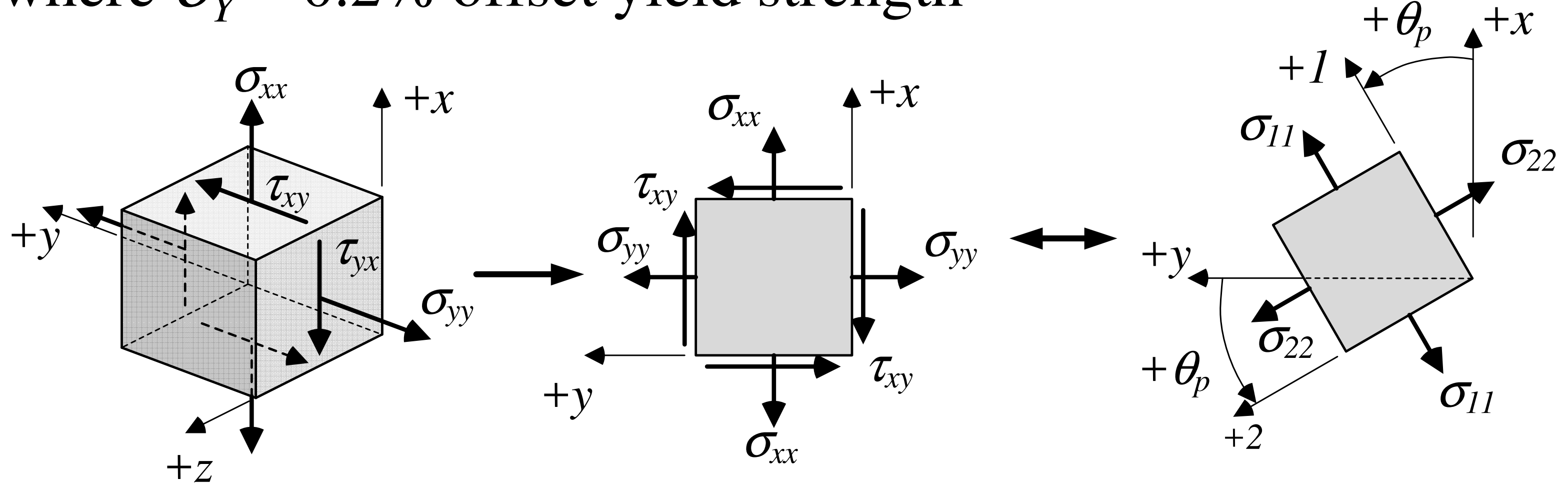
- If a plane stress state exists, yielding occurs if:

$$\left[\sigma_{xx}^2 - \sigma_{xx}\sigma_{yy} + \sigma_{yy}^2 + 3\tau_{xy}^2\right] \geq \sigma_Y^2$$

or, equivalently, if:

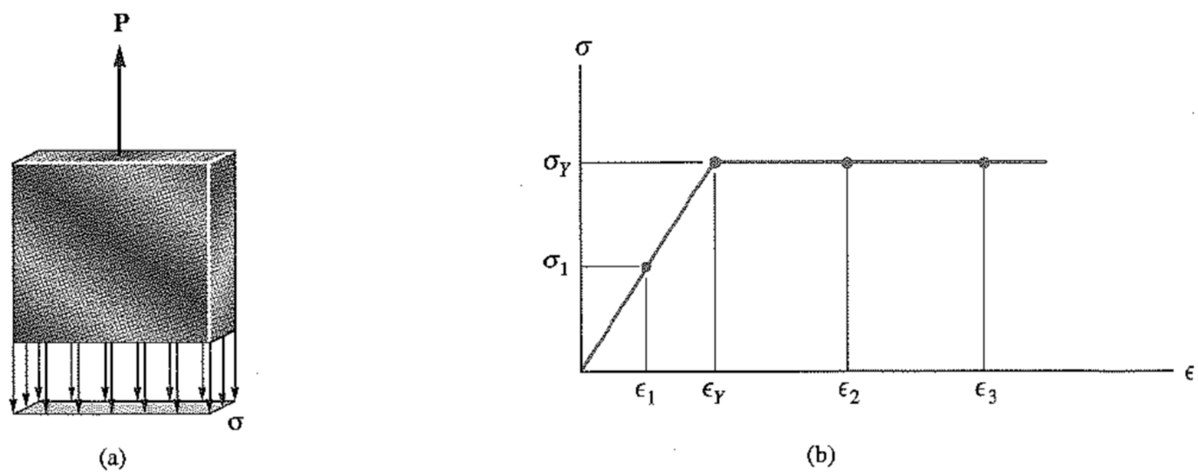
$$\left[\sigma_{11}^2 - \sigma_{11}\sigma_{22} + \sigma_{22}^2\right] \geq \sigma_Y^2$$

where $\sigma_Y = 0.2\%$ offset yield strength



Failure Predictions for Isotropic Metals
Application

- Failure predictions based on yield criterion are typically based on the concept of “fully plastic” loading
  - Elastic-perfectly plastic behavior assumed (aka elastoplastic)
  - Failure predicted when entire cross-section is predicted to have yielded
  - This approach provides a reasonably conservative estimate, since in reality metal alloys strain-harden (i.e., are not elastic-perfectly plastic)



Failure Predictions for Isotropic Metals
Application

- Failure predictions based on yield criterion are typically based on the concept of "fully plastic" loading
  - Elastic-perfectly plastic behavior assumed (aka elastoplastic)
  - Failure predicted when entire cross-section is predicted to have yielded
  - This approach provides a reasonably conservative estimate, since in reality metal alloys strain-harden (i.e., are not elastic-perfectly plastic)

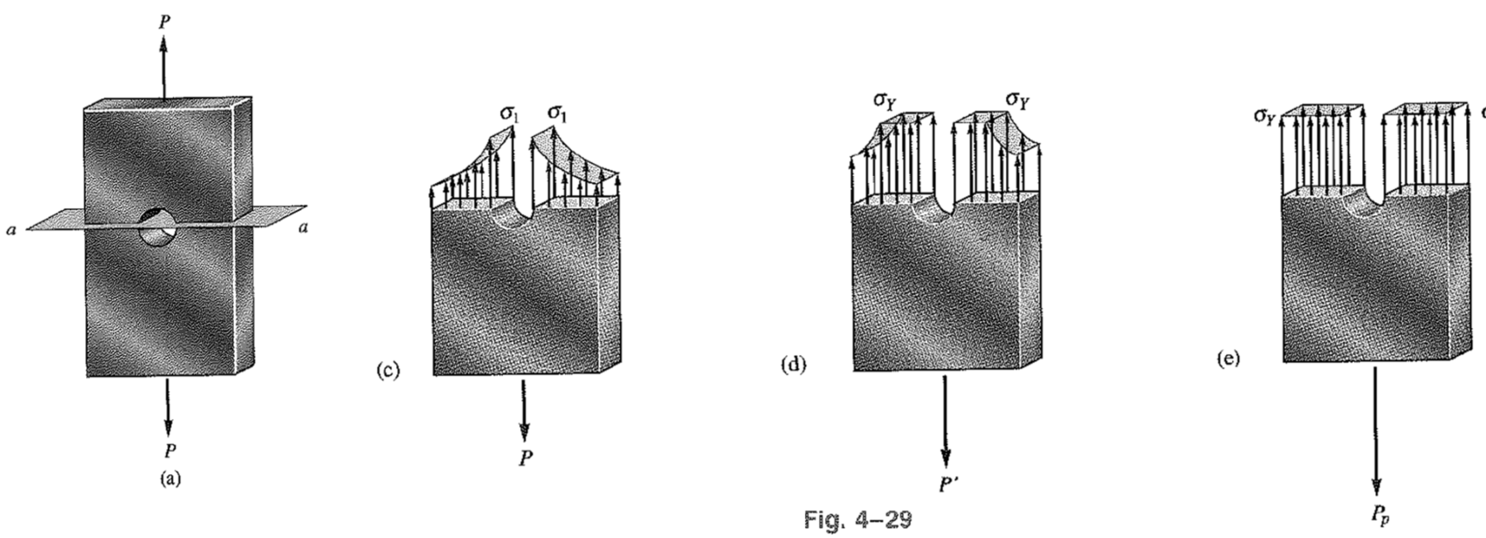
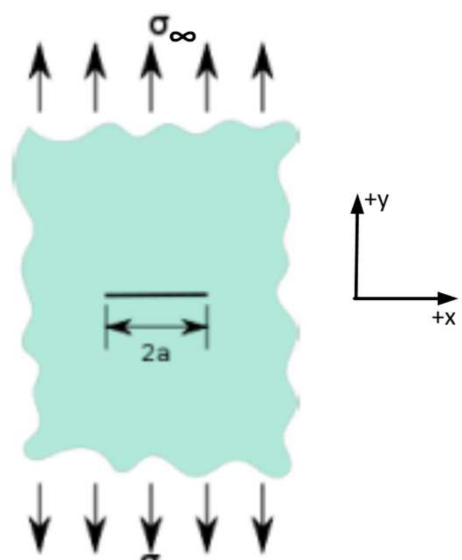


Fig. 4–29

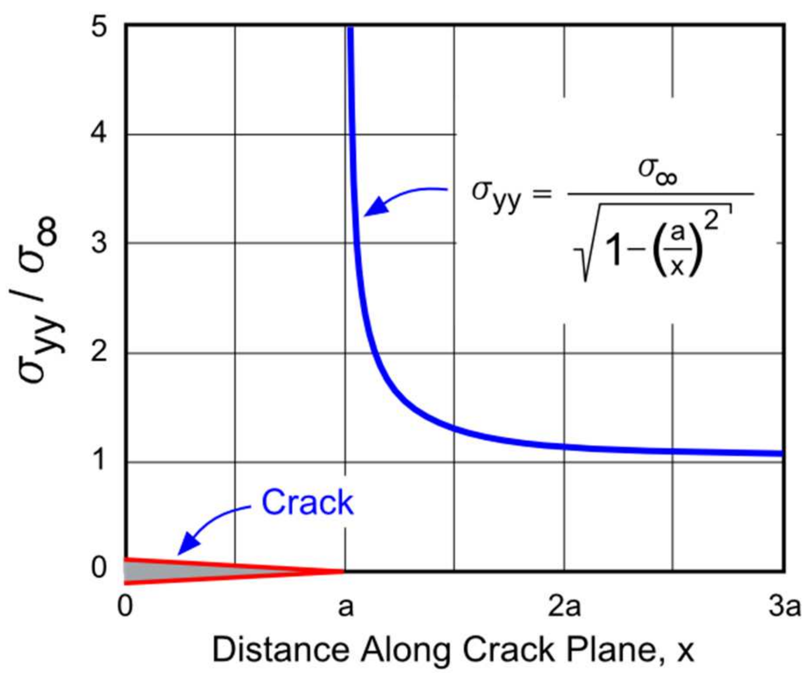
Failure Predictions for Isotropic Metals

Fracture Mechanics

- Inglis (1913) and Westergaard (1939) derived theoretical solutions for stresses near a crack based on the theory of elasticity (which is, in turn, based on Hooke's Law)
- These solutions show that the stresses near a crack tip are "singular"



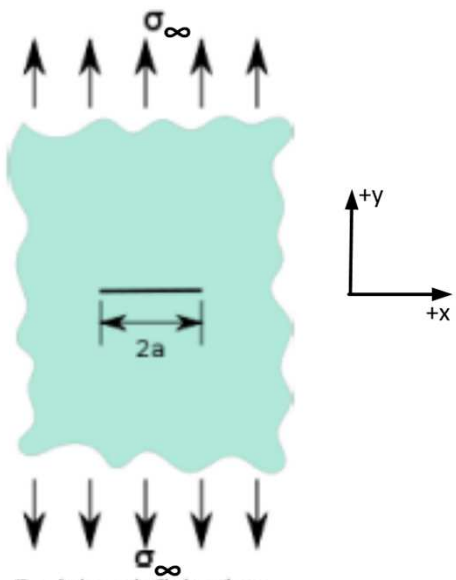
Crack in an infinite plate under mode I loading.



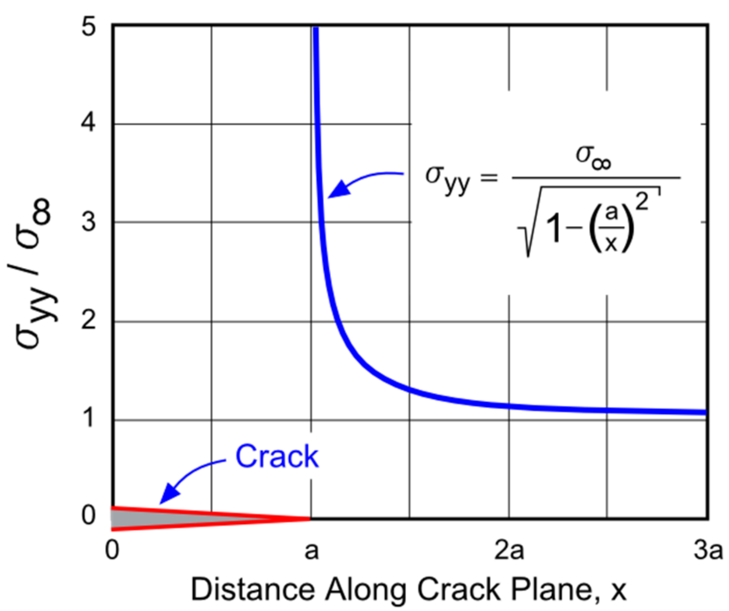
Failure Predictions for Isotropic Metals

Fracture Mechanics

- The Inglis and Westergaard solutions are obviously incorrect near the crack tip…otherwise $\sigma_{yy}\big|_{x \to a} = \infty$ if $\sigma_\infty \neq 0.00000$….
- Due to very high stresses and stress gradients, traditional yield criterion (e.g., the von Mises criterion) cannot be applied to predict failure at/near the crack…must use fracture mechanics instead



Crack in an infinite plate under mode I loading.

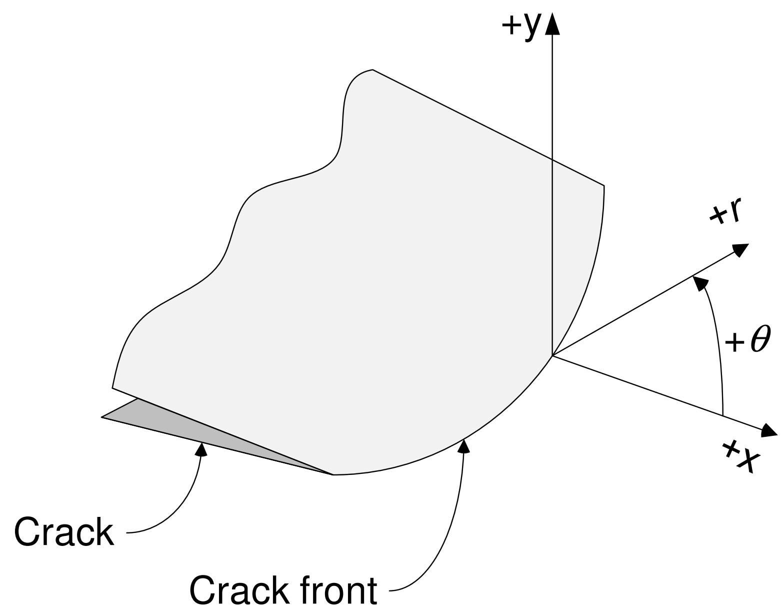


Failure Predictions for Isotropic Metals

Fracture Mechanics

- In 1958 Irwin noted that the general solutions for stresses near a crack tip can be written in polar coordinates (and for small *r*) as:

$$\sigma_{ij}(r,\theta) = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta) + \text{higher order terms}$$



where:

$f_{ij}(\theta)$ = dimensionless function that depends on geometry

K = the "stress intensity factor," whose value depends on geometry, size and location of the crack, and the *magnitude of loading* ; units = stress-length$^{1/2}$

Failure Predictions for Isotropic Metals
Fracture Mechanics

- Three types of loading modes are defined:
  - Mode I: the "opening mode"… characterized by K_I
  - Mode II: the "shearing mode"… characterized by K_{II}
  - Mode III: the "tearing mode"… characterized by K_{III}

- Modes I and II are most commonly encountered in practice…if loads associated with both Mode I and Mode II are present the loading condition is called "mixed mode" loading

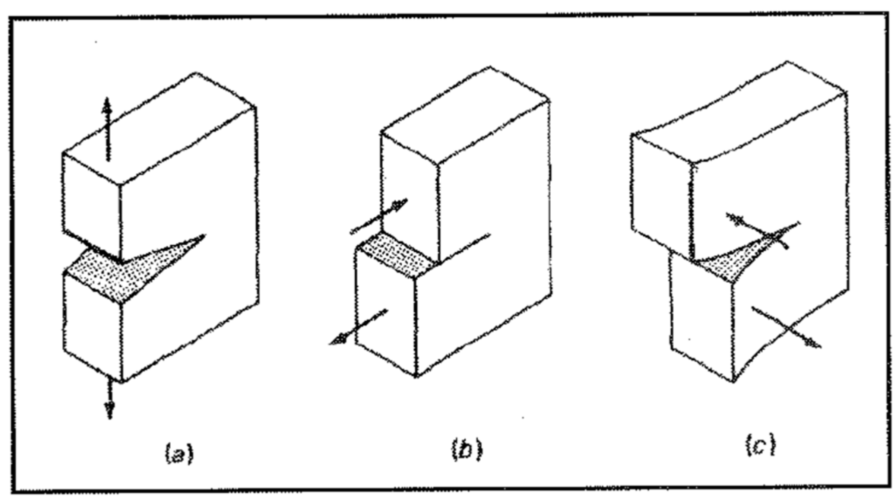


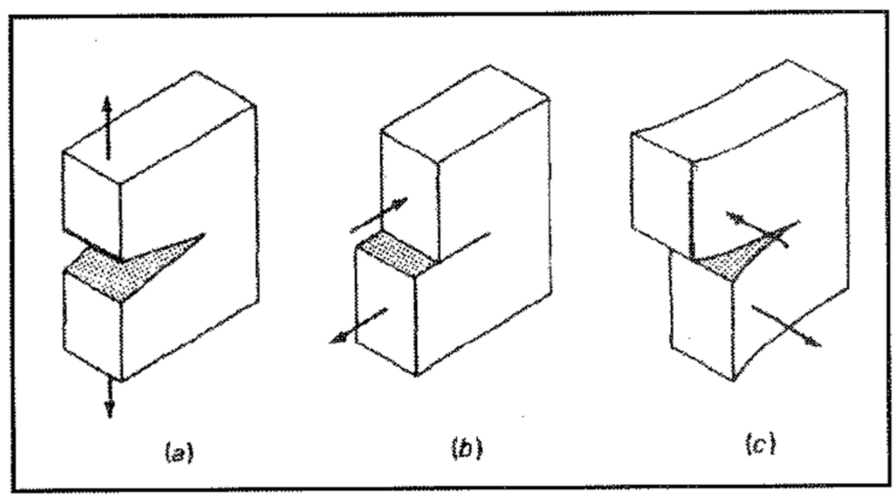
Figure 4.10 Three modes of crack loading: (a) opening; (b) shearing; (c) tearing.

Failure Predictions for Isotropic Metals

Fracture Mechanics

- Since K increases as loading is increased, it was proposed the failure will occur when K is increases to a "critical" level.
- The "critical stress intensity factor" for each mode (K_{Ic}, K_{IIc}, K_{IIIc}) is treated as a material property and tabulated like characteristic properties (e.g., E, ν, or σ_Y)
- Predictions based on K_c are most accurate for brittle materials or materials with very low levels of ductility (…why?)
- Failure of highly ductile materials is better predicted using the critical strain energy release rate (G_c) …not reviewed here

Figure 4.10 Three modes of crack loading: (a) opening; (b) shearing; (c) tearing.



Failure Predictions for Isotropic Metals

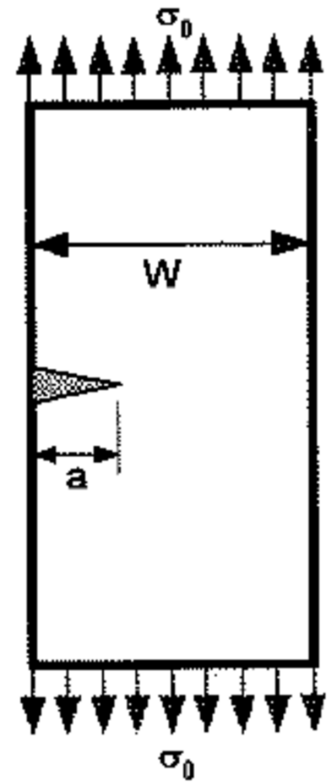
Fracture Mechanics

- To repeat, the value of the stress intensity factor geometry, size and location of the crack, and the magnitude of loading
- K_I has been tabulated for *many* geometries (both standard lab specimens and typical structural geometries)…for example:
  - Murakami, Y (ed), Stress Intensity Factors Handbook, Vols 1&2, Pergamon Press, (1987)
  - https://en.wikipedia.org/wiki/Stress_intensity_factor
  - (A few are listed in Shukla and Dally, sec 4.4)
- Values for K_{II} and K_{III} are also available but for fewer geometries
- The tabulated values of K_I, K_{II}, K_{III} are often curve fits of data obtained from:
  - Experiments, especially photoelasticity
  - Numerical analyses (usually finite-element analyses)

Failure Predictions for Isotropic Metals
Fracture Mechanics

- For example, K_I for an edge crack in a plate under uniaxial stress is (Shukla and Dally, Sec 4.4, combine Eqs 4.22 and 4.23):

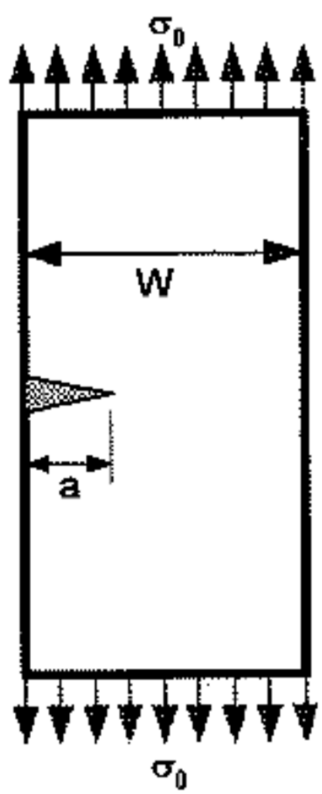
$$K_I = \sigma_0\sqrt{\pi a}\left[1.12 - 0.231\left(\frac{a}{W}\right) + 10.55\left(\frac{a}{W}\right)^2 - 21.71\left(\frac{a}{W}\right)^3 + 30.38\left(\frac{a}{W}\right)^4\right]$$

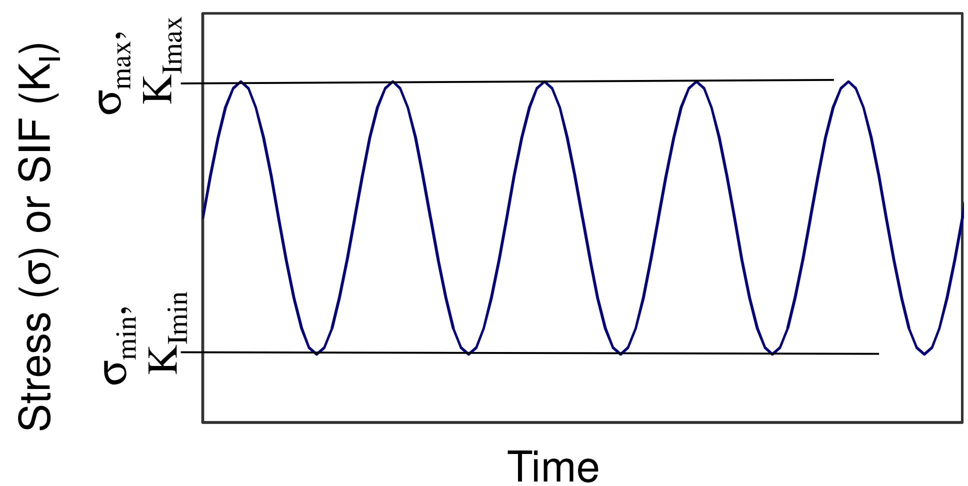


Failure Predictions for Isotropic Metals

Fracture Mechanics applied to fatigue failure

- The variation in the stress intensity factor can be used to predict sub-critical crack growth during fatigue loading:





Failure Predictions for Isotropic Metals

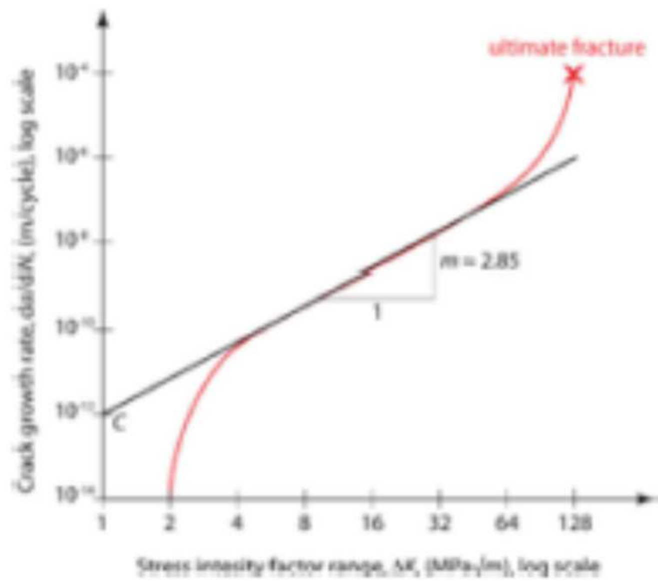
Fracture Mechanics applied to fatigue failure

- The variation in the stress intensity factor can be combined with the Paris Law to predict sub-critical crack growth during fatigue loading:

$$\frac{da}{dN} = C(\Delta K_I)^m$$

where:

N = load cycle
C, m = experimentally-determined constants
$\Delta K_I = K_{Imax} - K_{Imin}$



Idealized plot extracted from:
https://en.wikipedia.org/wiki/Paris%27_law