A Review of Stress, Strain, Stress/Strain Transformations, and Hooke's Law

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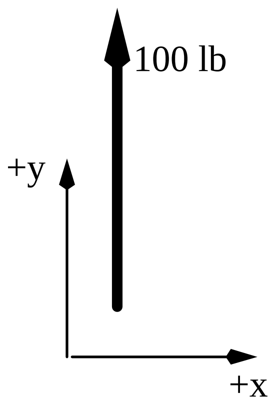
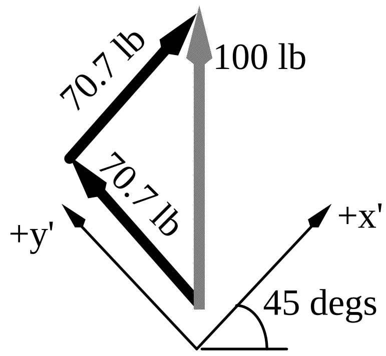
Force Vectors (Tensors)

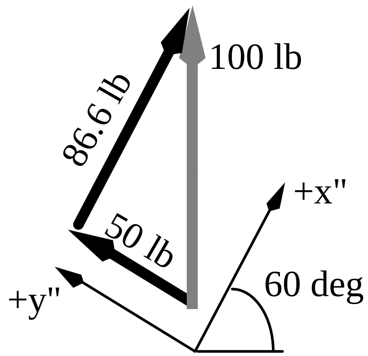
- A force, $\bar{F}$, is a vector (also called a "1st-order tensor")
- The description of any vector (or any tensor) depends on the coordinate system used to describe the vector:



$\bar{F}$ = 100 lb (y-direction)



$\bar{F}$ = 70.7 lb (x'-direction) + 70.7 lb (y'-direction)

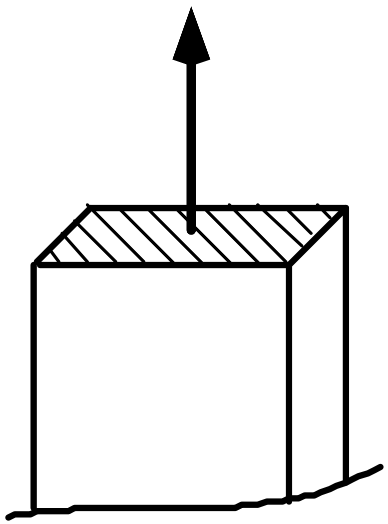


$\bar{F}$ = 86.6 lb (x"-direction) + 50 lb (y"-direction)

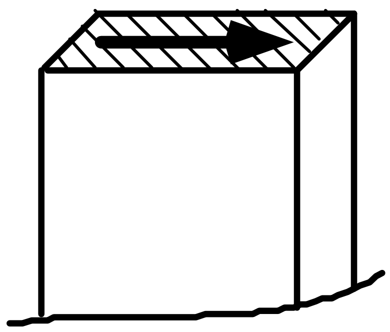
Normal and Shear Forces

- A "normal" force acts perpendicular to a surface
- A "shear" force acts tangent to a surface

P = Normal Force

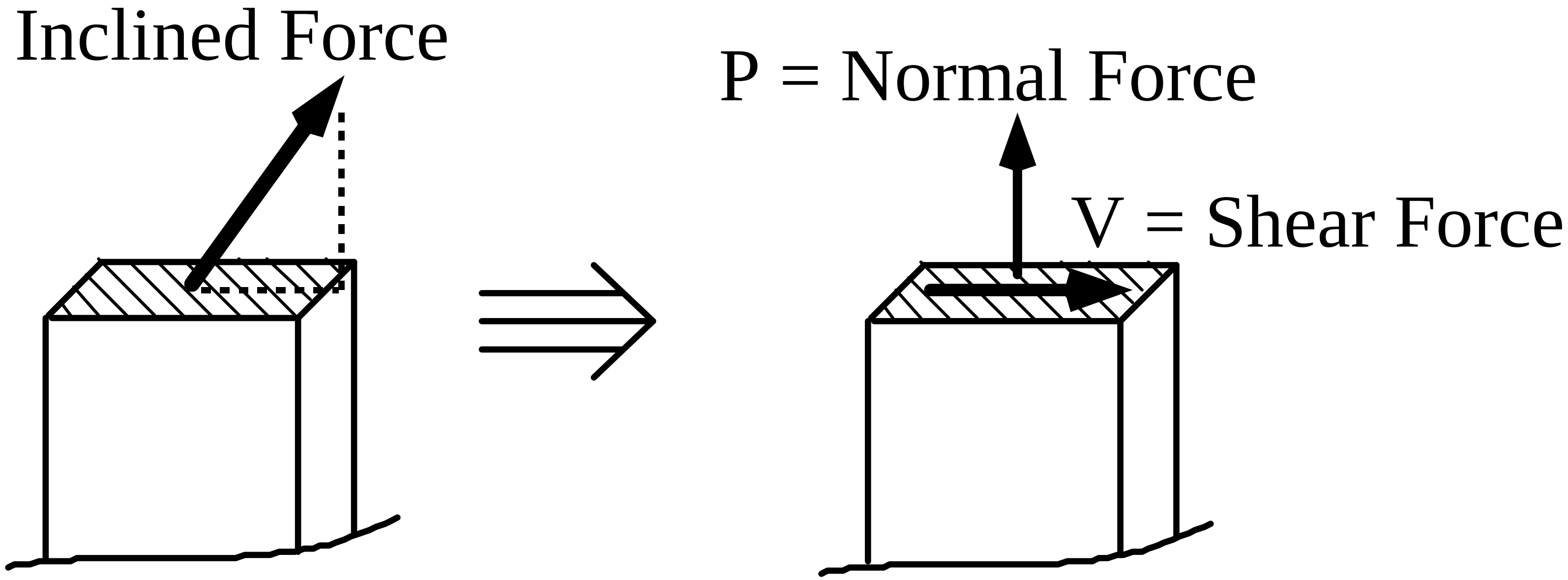


V = Shear Force



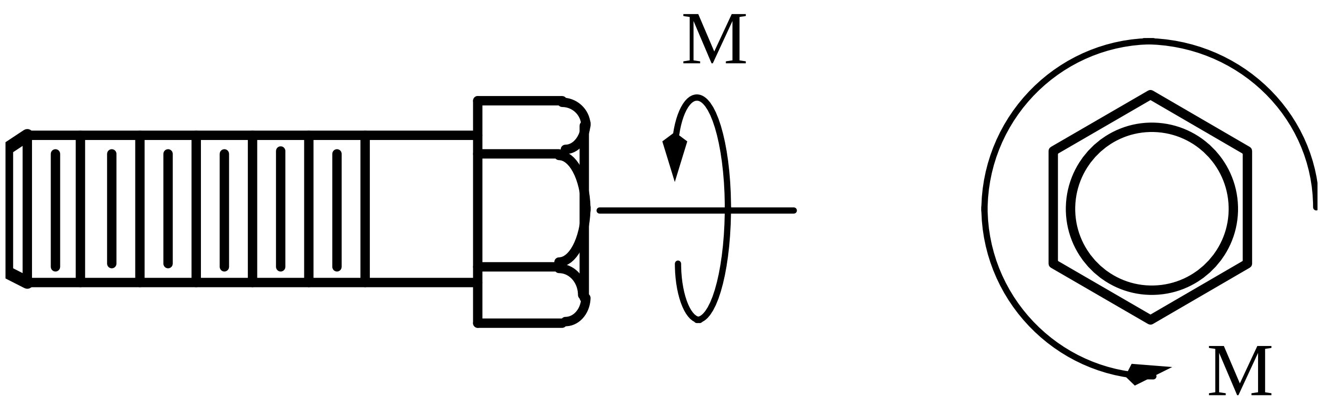
Forces Inclined to a Plane

- A force inclined to a plane can always be described as a combination of normal and shear forces



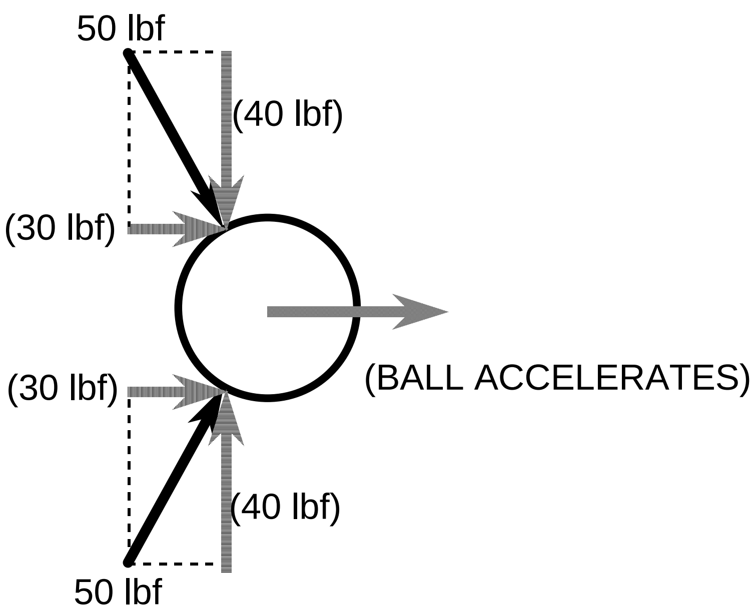
Moments

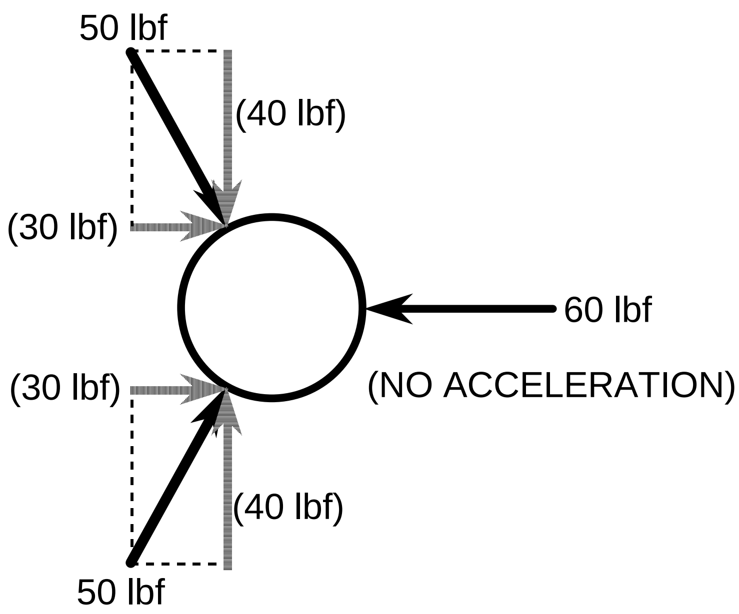
- A moment (also called a "torque" or a "couple") is a force which tend to cause rotation of a rigid body
- A moment is also vectoral quantity...



Static Equilibrium

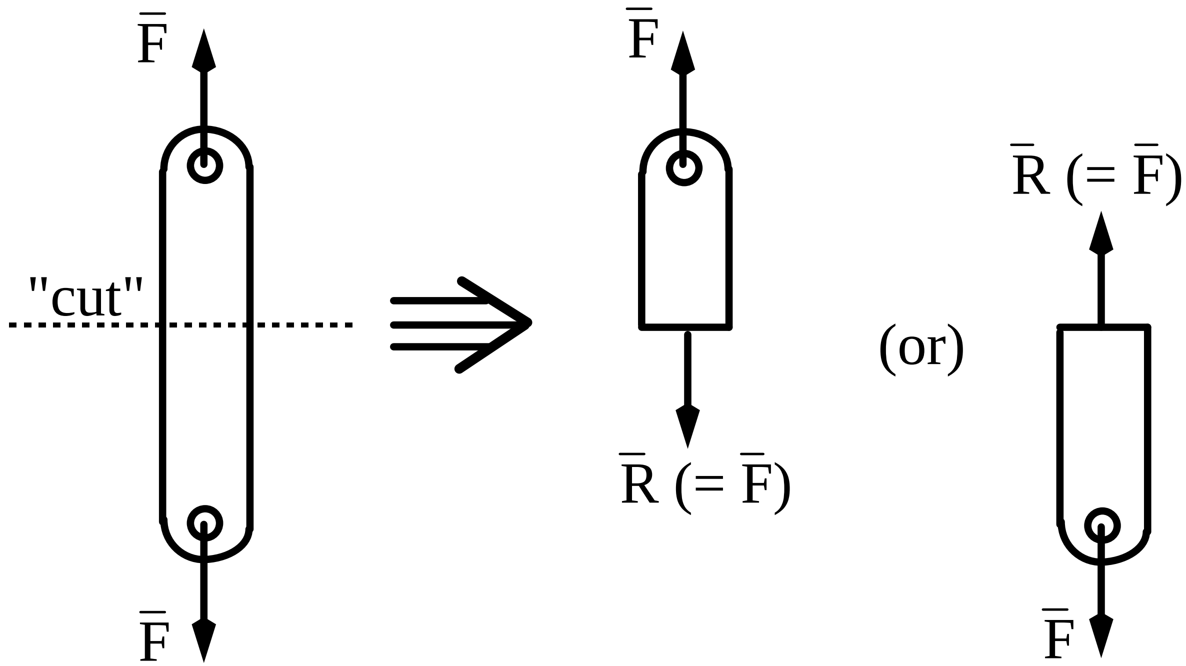
- A rigid solid body is in "static equilibrium" if it is:
  - at rest, or
  - moves with a constant velocity
- Static equilibrium exists if: $\sum \overline{F} = 0$ and $\sum \overline{M} = 0$





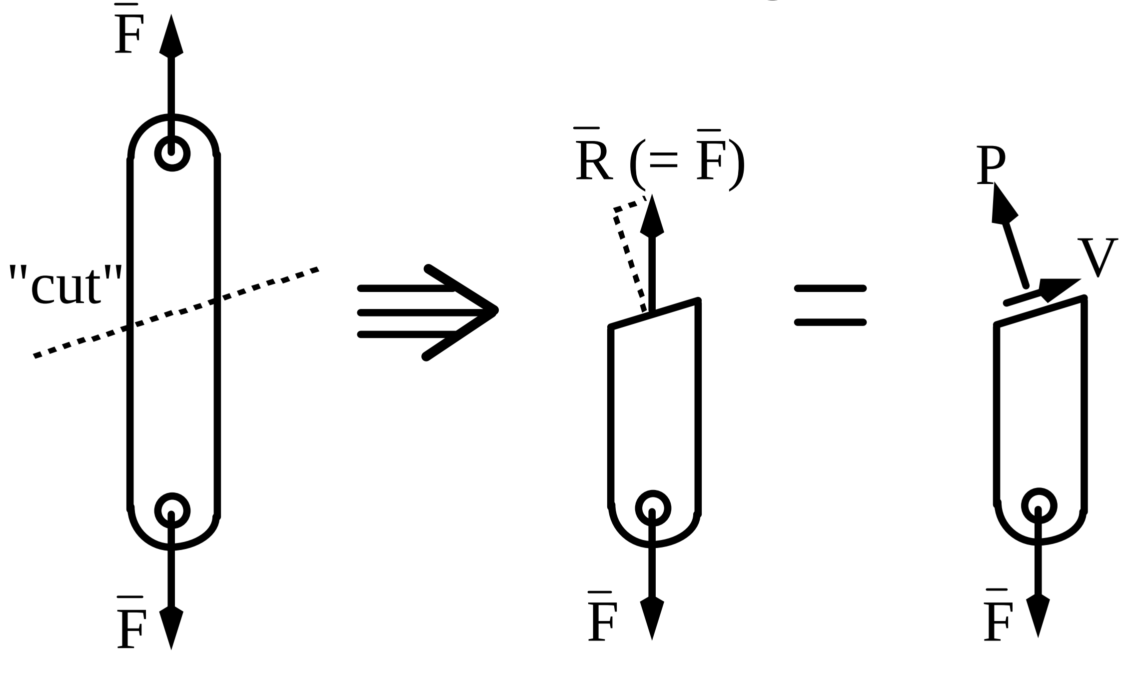
Free Body Diagrams and "Internal Forces"

- An imaginary "cut" is made at plane of interest
- Apply $\sum \overline{F} = 0$ and $\sum \overline{M} = 0$ to either half to determine internal forces, $\overline{R}$



Free Body Diagrams and "Internal Forces"

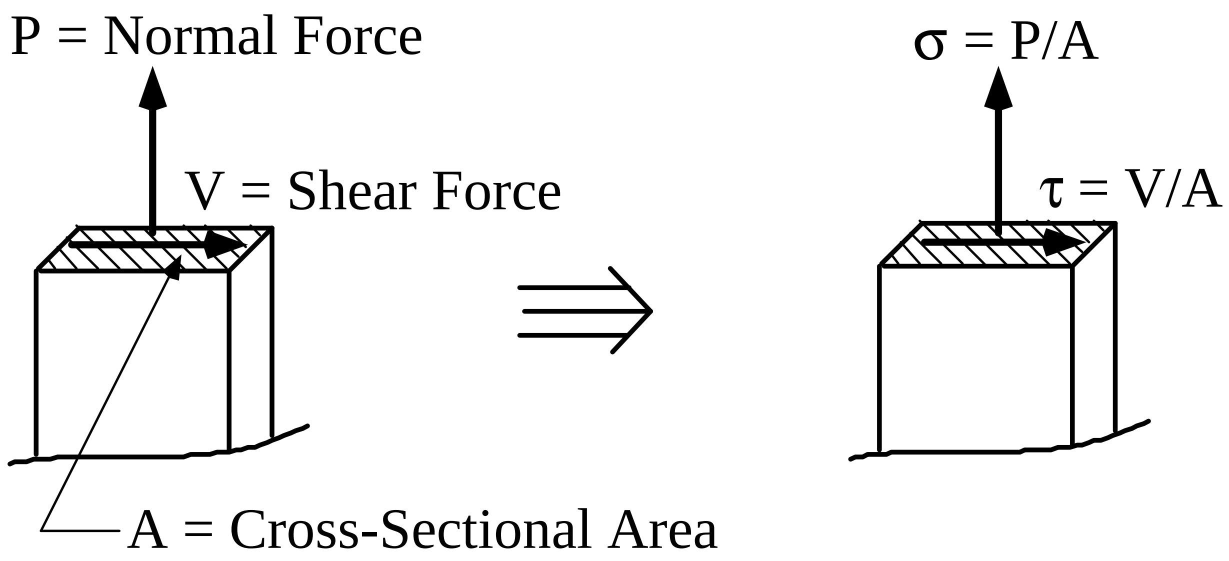
- The imaginary cut can be made along an arbitrary plane
- Internal force $\overline{R}$ can be decomposed to determine the normal and shear forces acting on the arbitrary plane



Stress

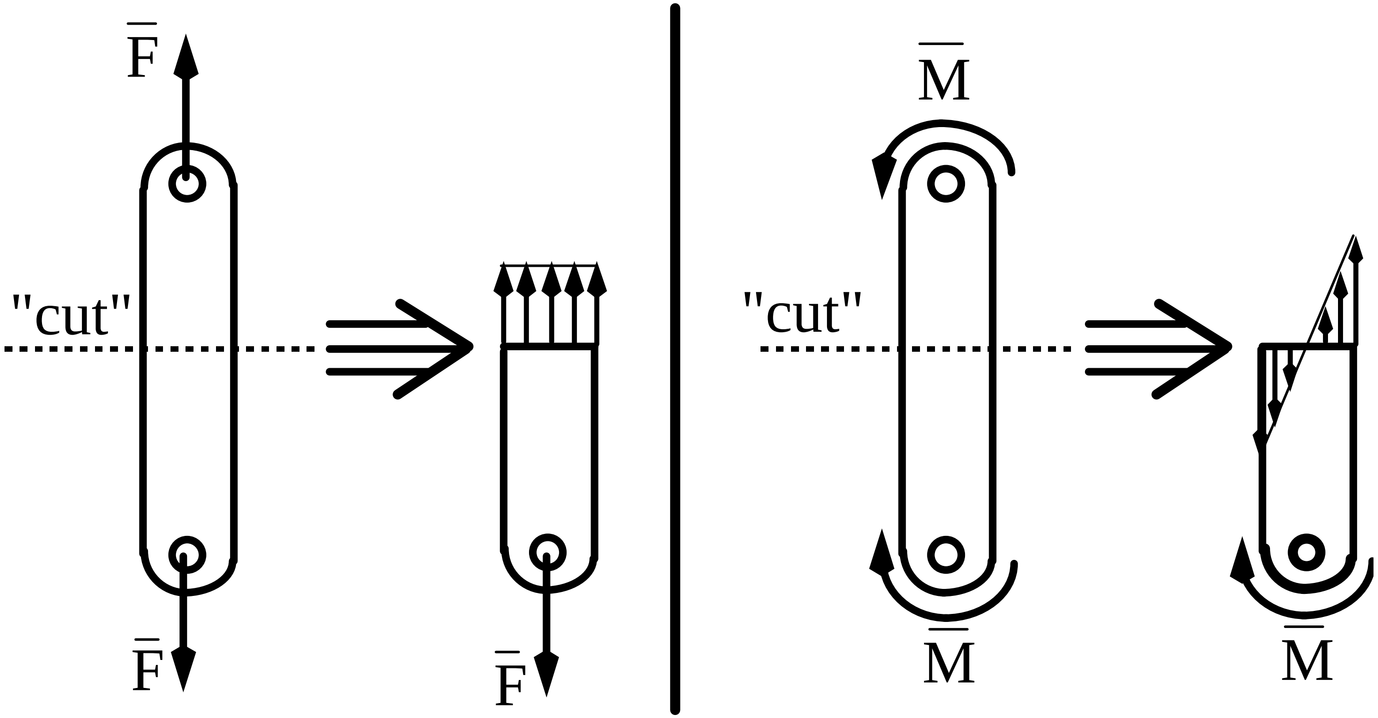
Fundamental Definitions

- Two "types" of stress:
  - normal stress = σ = P/A
  - shear stress = τ = V/A
  - where P and V must be *uniformly distributed* over A



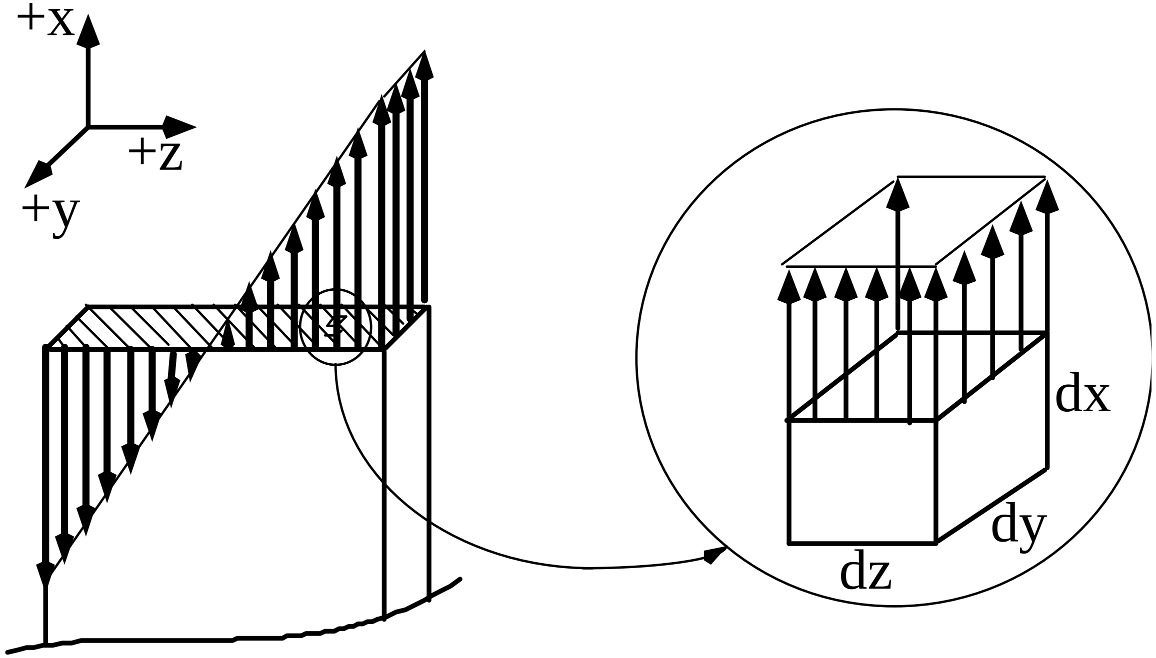
Distribution of Internal Forces

- Forces are distributed over the internal plane... they may or may not be uniformly distributed



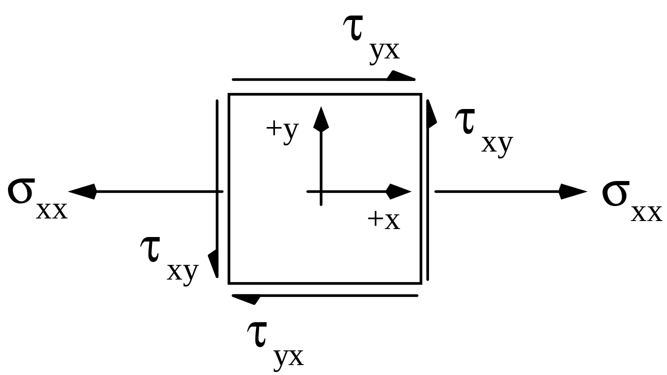
Infinitesimal Elements

- A free-body diagram of an "infinitesimal element" is used to define "stress at a point"
- Forces can be considered "uniform" over the infinitesimally small elemental surfaces

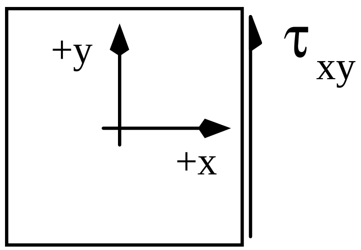


Labeling Stress Components

- Two subscripts are used to identify a stress component, e.g., "σ_{xx}" or "τ_{xy}" (note: for convenience we sometimes write $\sigma_x = \sigma_{xx}$ or $\sigma_{xy} = \tau_{xy}$)

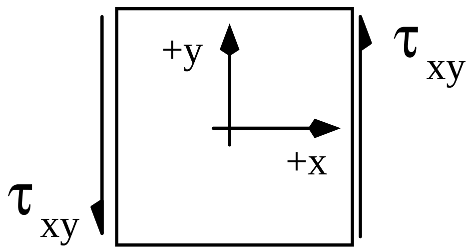


Admissable Pure Shear Stress States



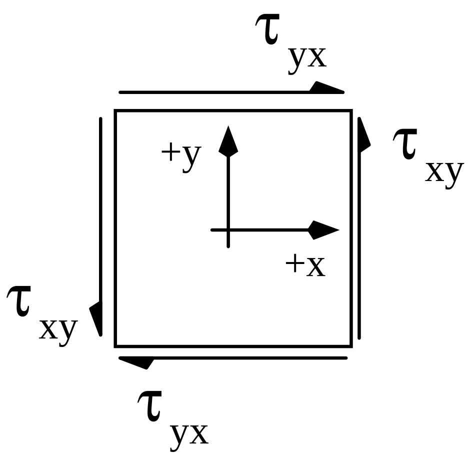
$$\sum \overline{F} \neq 0$$
$$\sum \overline{M} \neq 0$$

(inadmissable)



$$\sum \overline{F} = 0$$
$$\sum \overline{M} \neq 0$$

(inadmissable)



$$\text{If}: \left|\tau_{yx}\right| = \left|\tau_{xy}\right|$$
$$\text{then}:$$
$$\sum \overline{F} = 0$$
$$\sum \overline{M} = 0$$

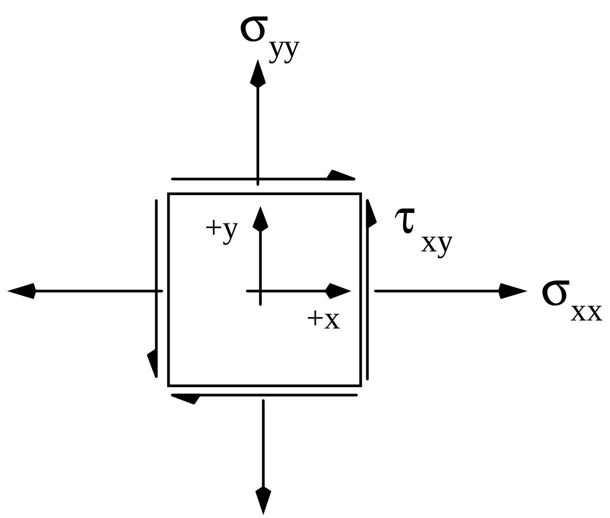
(admissable)

Stress Sign Conventions

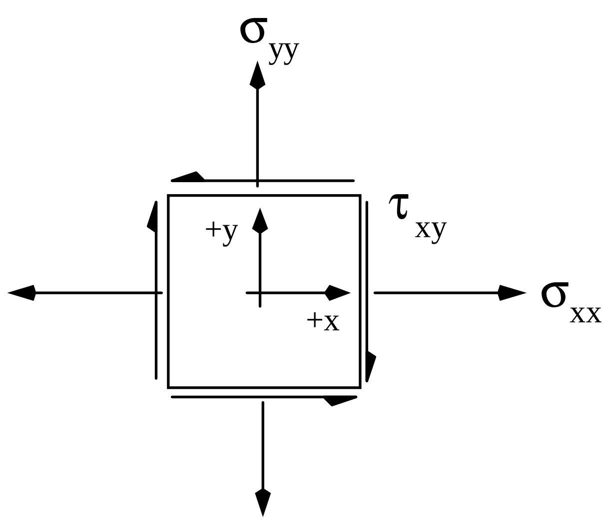
- The algebraic sign of an element face is "positive" of the outward-pointing unit normal to the face "points" in a positive coordinate direction

- A stress component is positive if:
  - stress component acts on a positive face and "points" in a positive coordinate direction, or
  - stress component acts on a negative face and "points" in a negative coordinate direction

Stress Sign Conventions



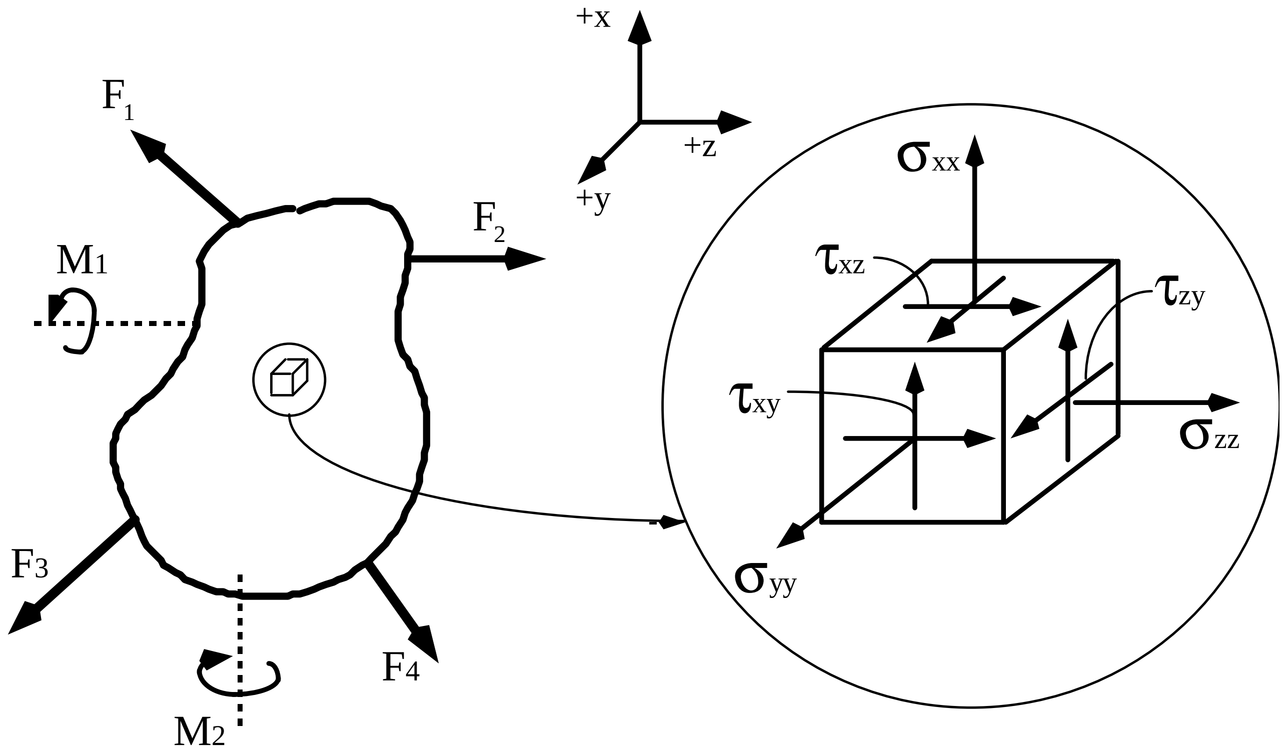
all stresses positive



σ_{xx} and σ_{yy} positive,
τ_{xy} negative

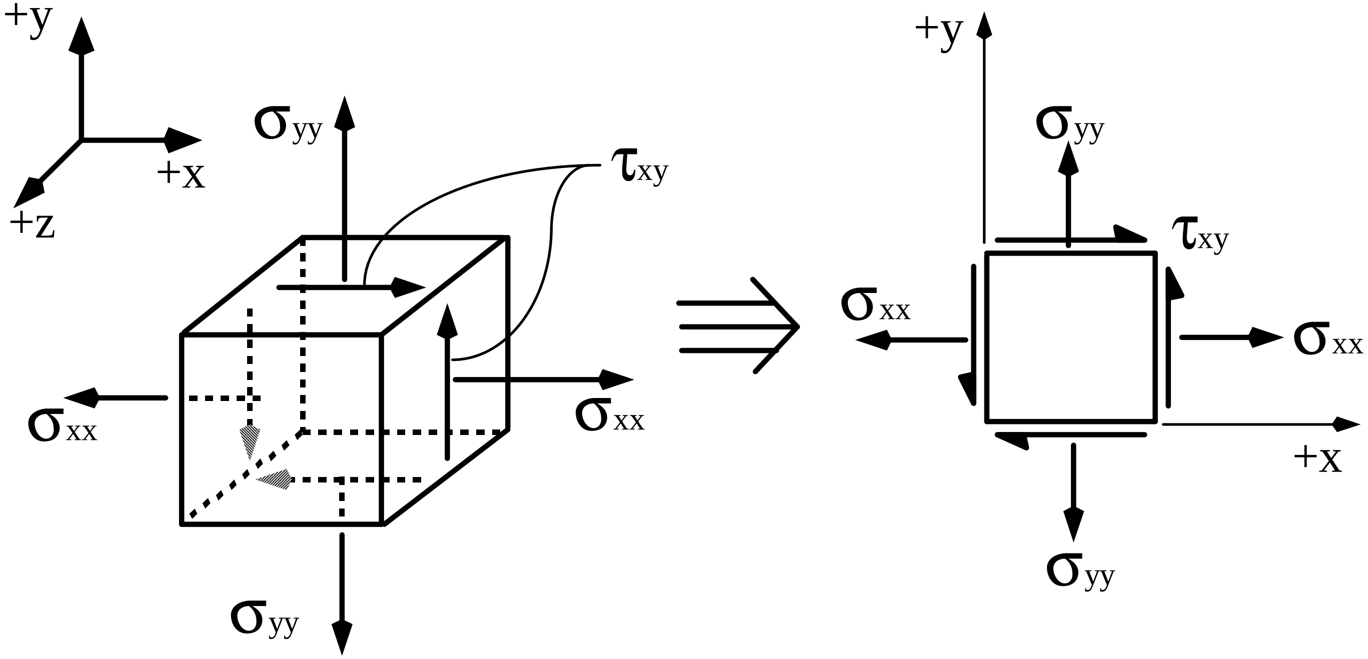
The Stress Tensor

- Stress is a "2nd-order tensor", and in the most general case six components of stress exist "at a point"



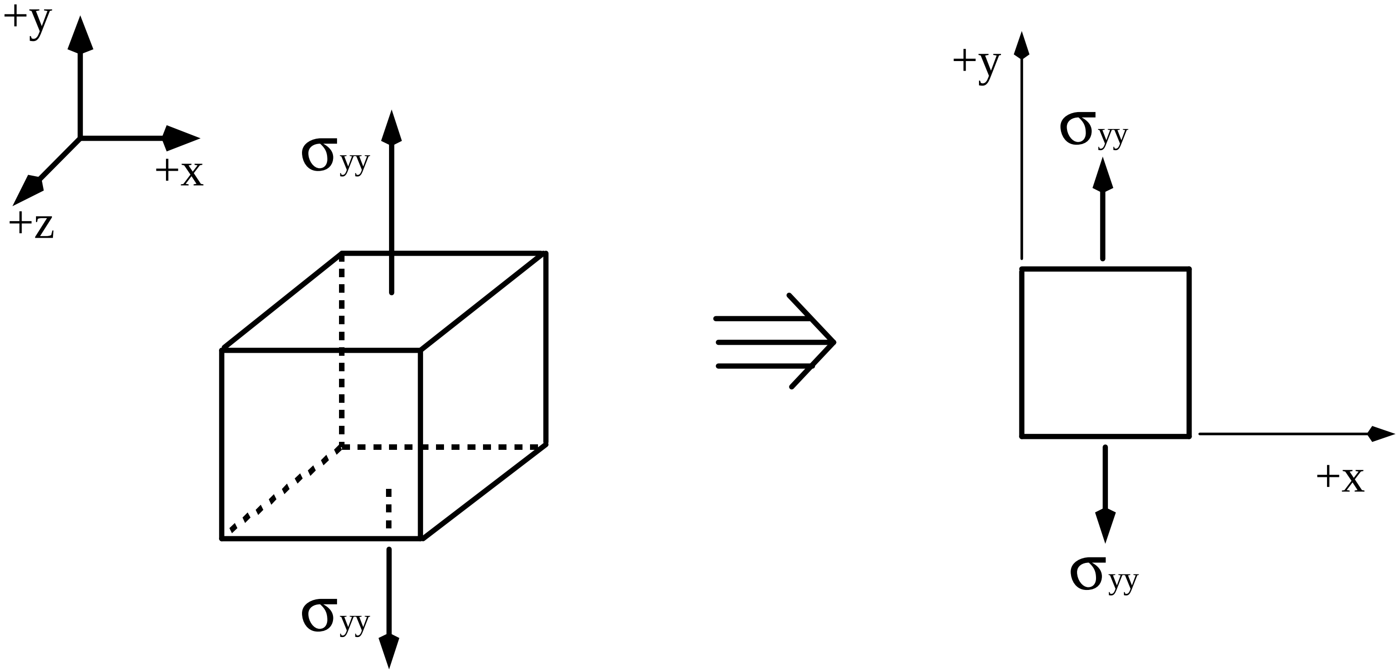
Plane Stress

- If all non-zero stress components exist in a single plane (i.e., if $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$), the state of stress is called "plane stress"

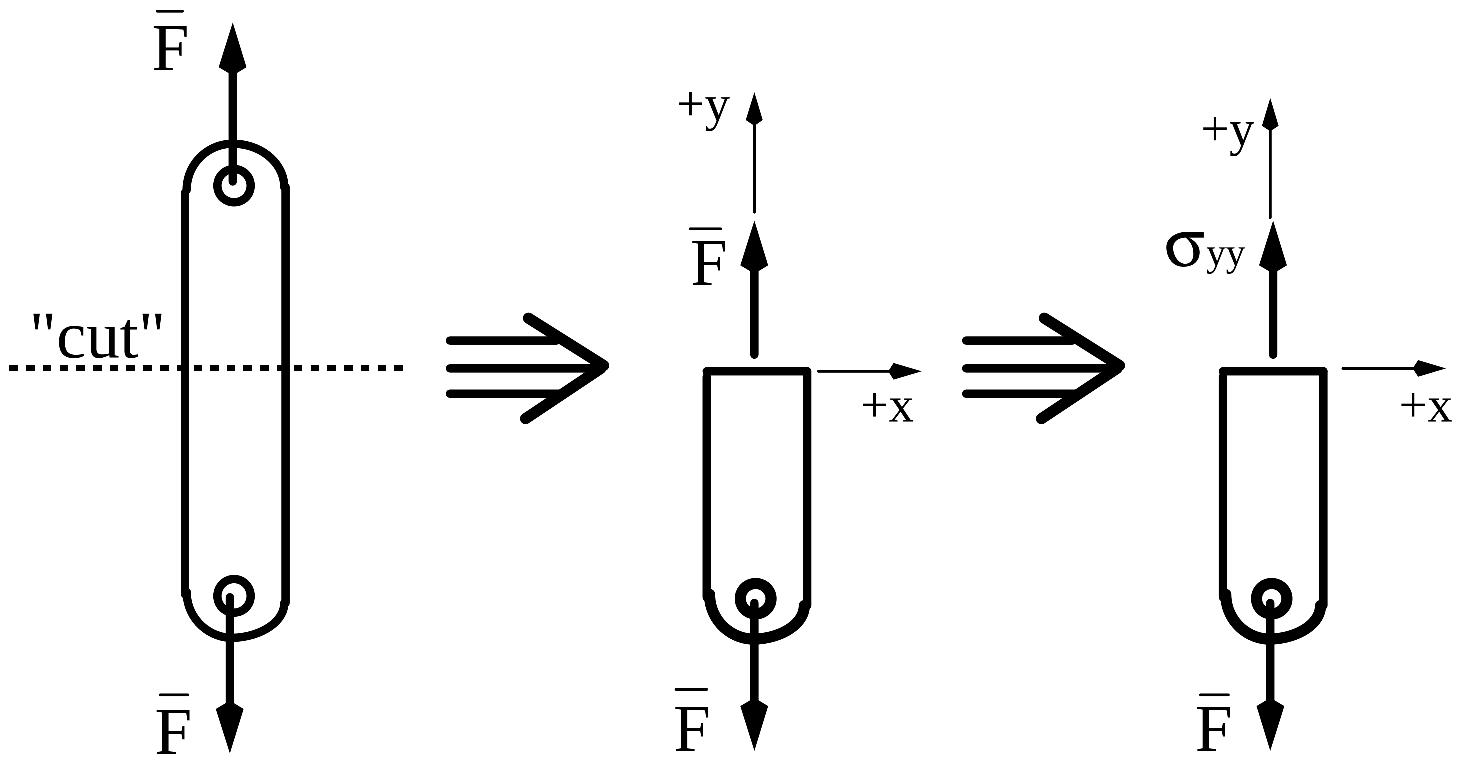


Uniaxial Stress

- If only one normal stress exists (if $\sigma_{xx} = \sigma_{zz} = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0$), the state of stress is called a "uniaxial stress"



Free Body Diagram *Defines* the Coordinate System



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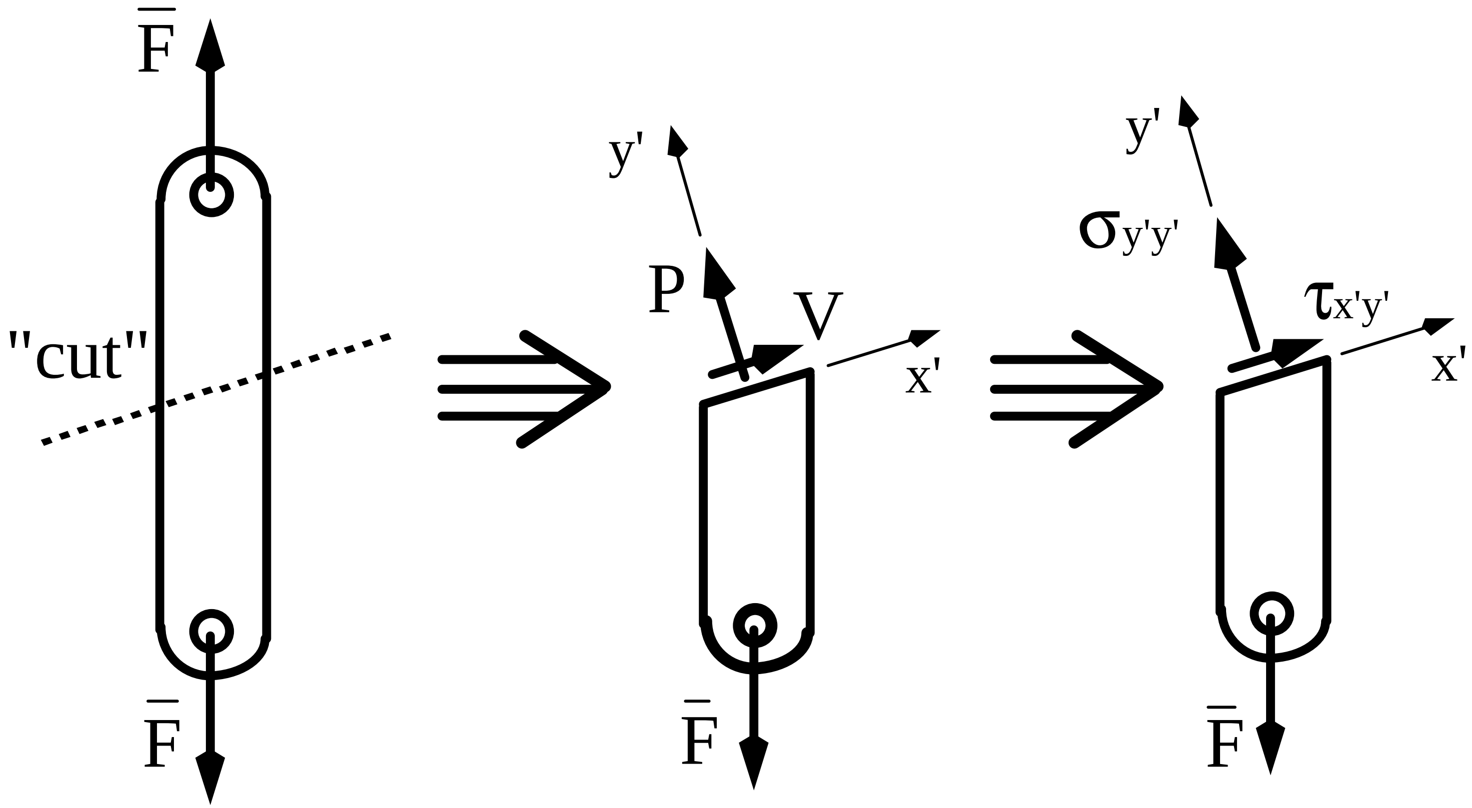
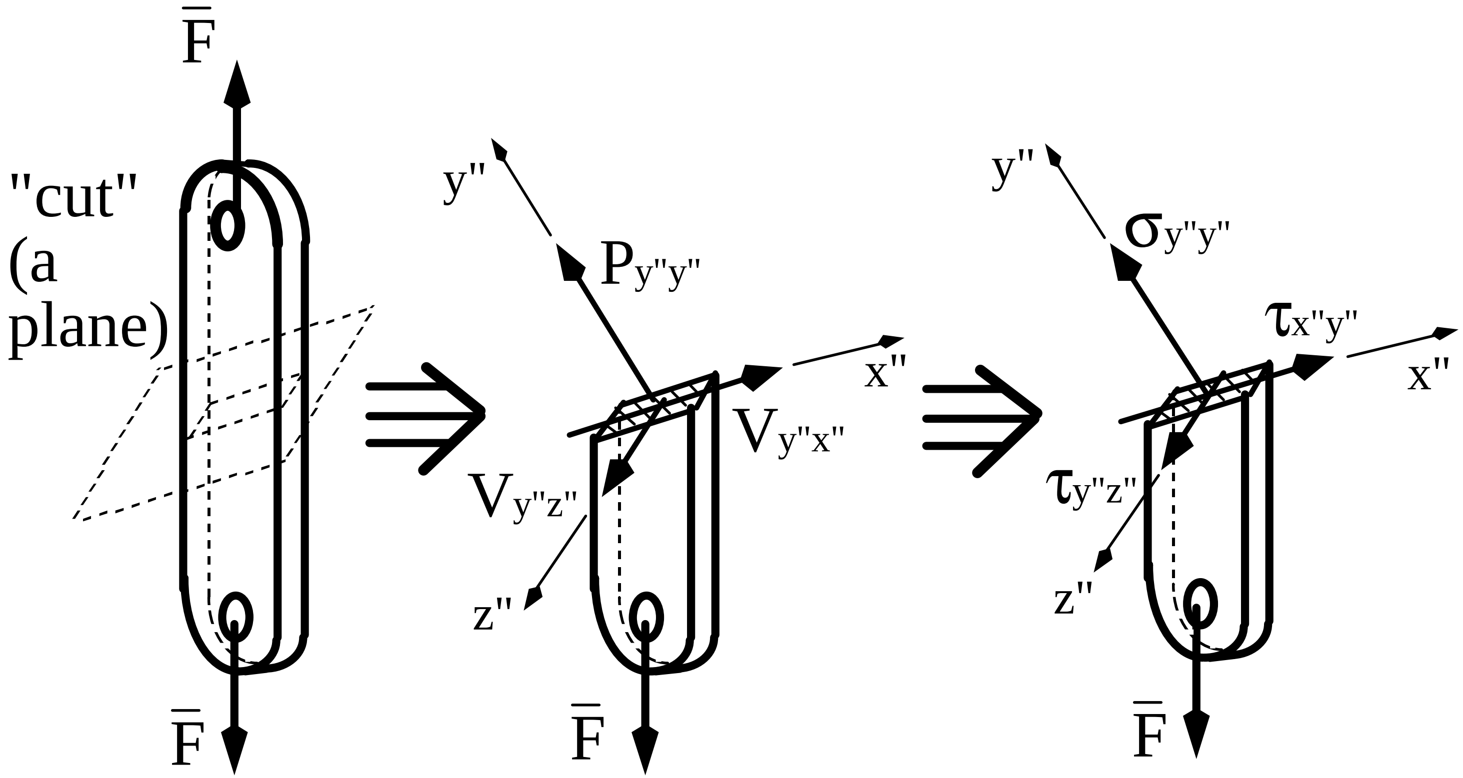
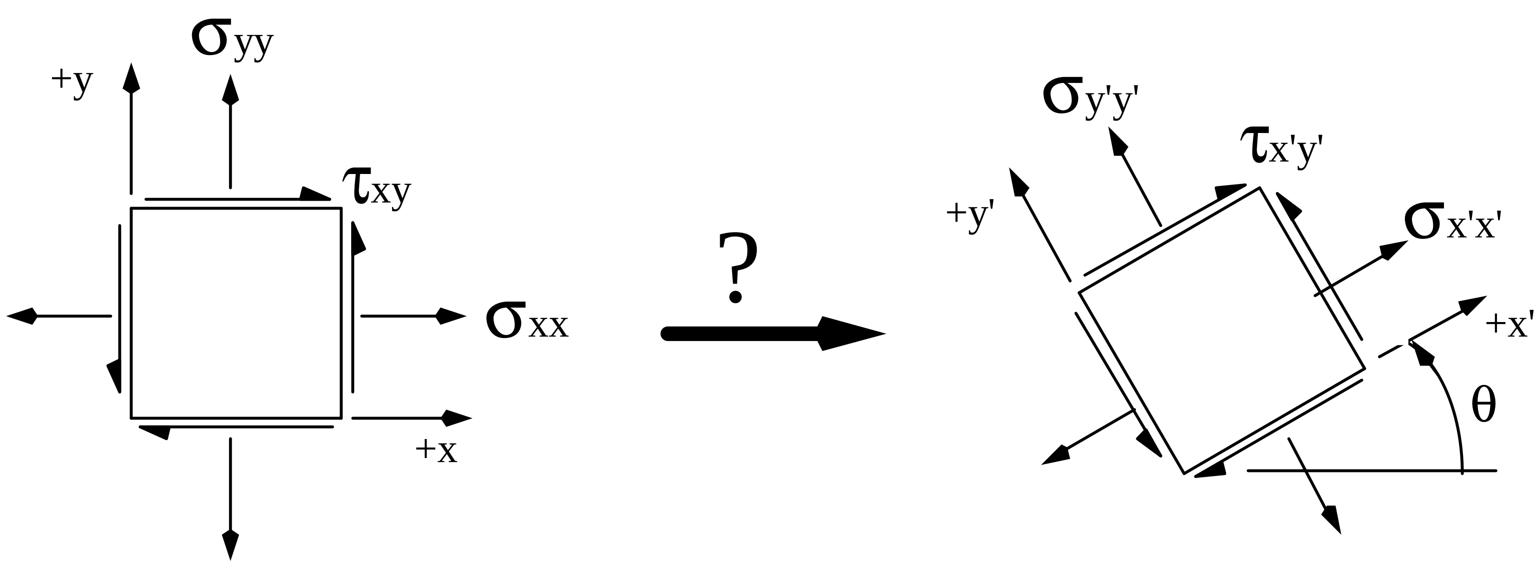


Free Body Diagram *Defines* the Coordinate System



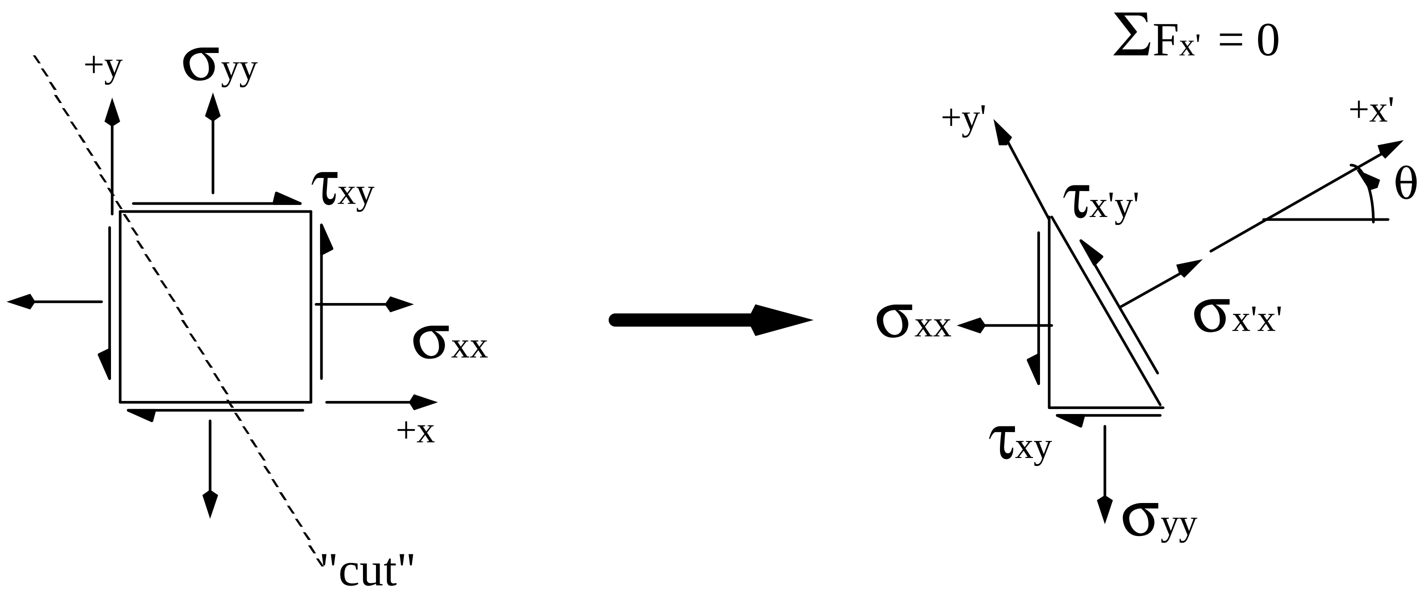
Stress Transformations

- Given stress components in the x-y coordinate system (σ_{xx}, σ_{yy}, τ_{xy}), what are the corresponding stress components in the x'-y' coordinate system?



Stress Transformations

- Stress components in the x'-y' coordinate system may be related to stresses in the x-y coordinate system using a free body diagram and enforcing $\sum \bar{F} = 0$



Stress Transformation Equations

- By enforcing $\Sigma F_{x'} = 0$, $\Sigma F_{y'} = 0$, it can be shown:

$$\sigma_{x'x'} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\cos\theta\sin\theta$$

$$\sigma_{y'y'} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\cos\theta\sin\theta$$

$$\tau_{x'y'} = \left(\sigma_{yy} - \sigma_{xx}\right)\cos\theta\sin\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$

Important note: angle θ is positive counter-clockwise (CCW) *from* the +x-direction *towards* the +y'-direction

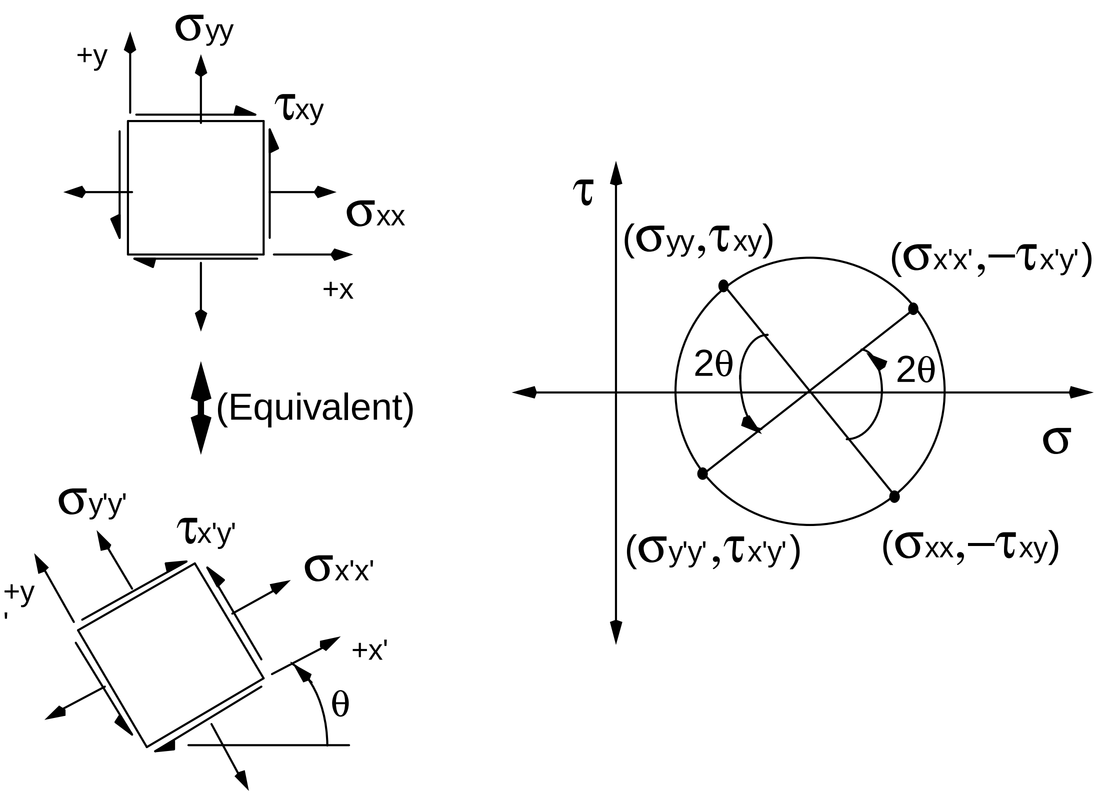
Stress Transformation Equations

- Using matrix notation, these can also be written:

$$\begin{Bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \tau_{x'y'} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & (\cos^2\theta - \sin^2\theta) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}$$

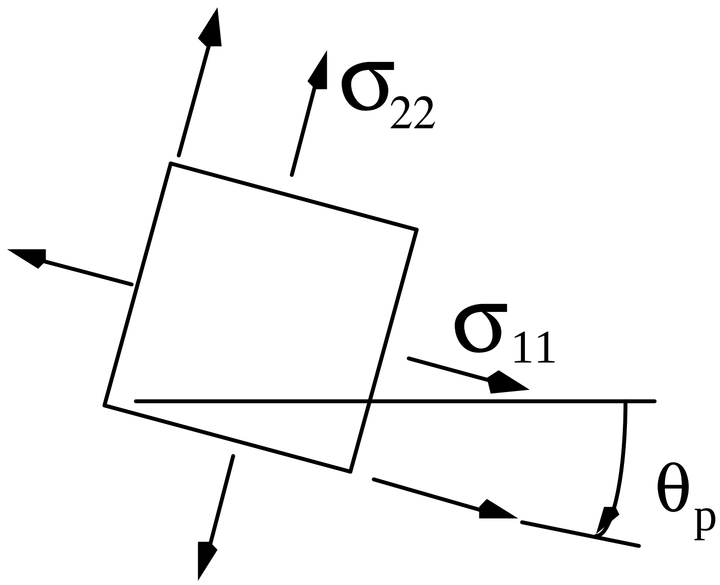
Stress Transformation Equations

- These transformation equations can also be visualized using Mohr's circle of stress:

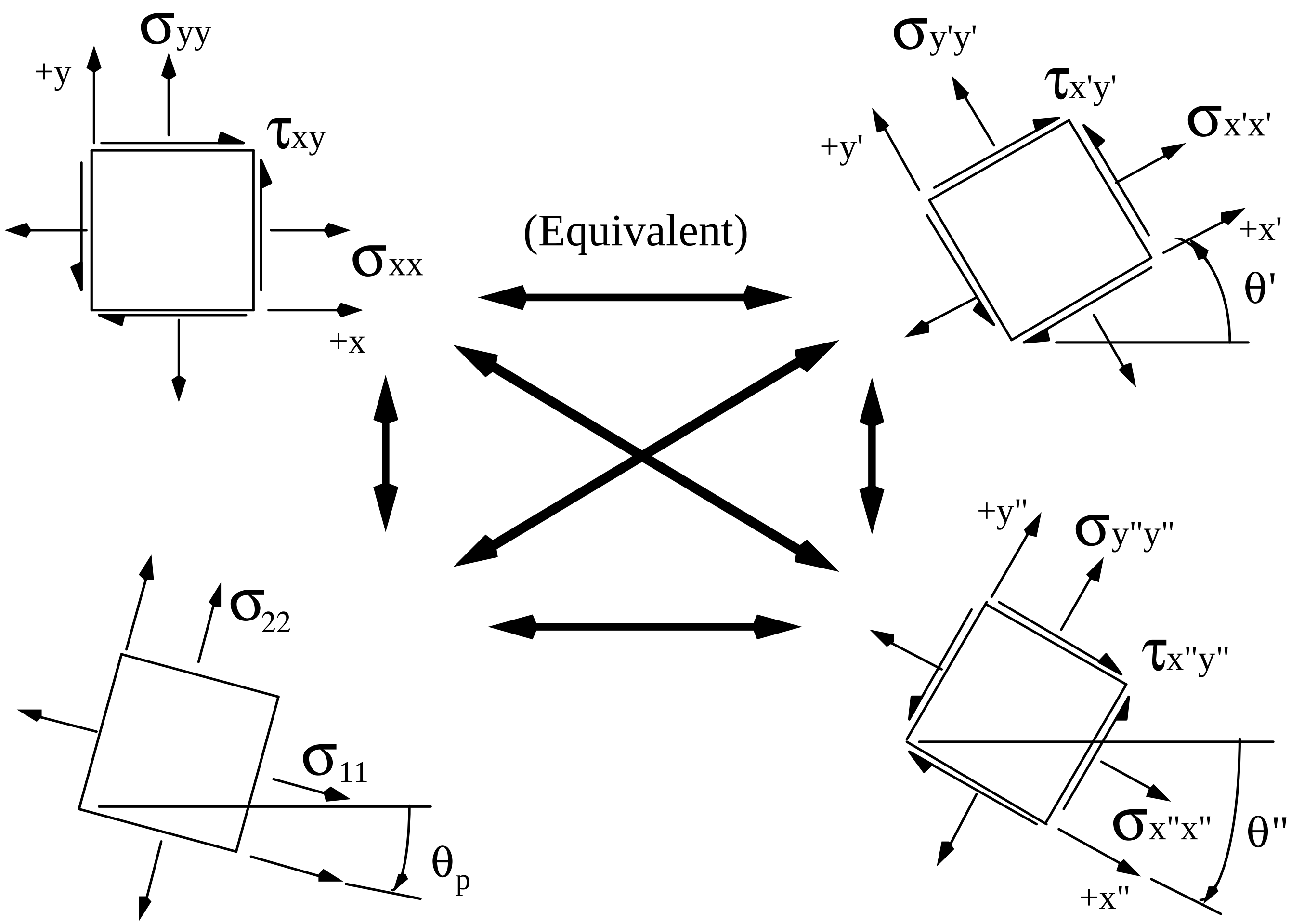


Principal Stresses

- In the principal stress coordinate system the shear stress is zero, and the normal stresses are max/min...



“Transformation” of Stress



"Stress": Summary of Key Points

- Normal and shear stresses are both defined as a (force/area)
- Six components of stress must be known to specify the "state of stress" at a point
- Stress is a tensorial quantity; values of individual stress components depend on the coordinate system used
- Stress is defined strictly on the basis of static equilibrium; definition is independent of:
  - material properties
  - strain
  - temperature

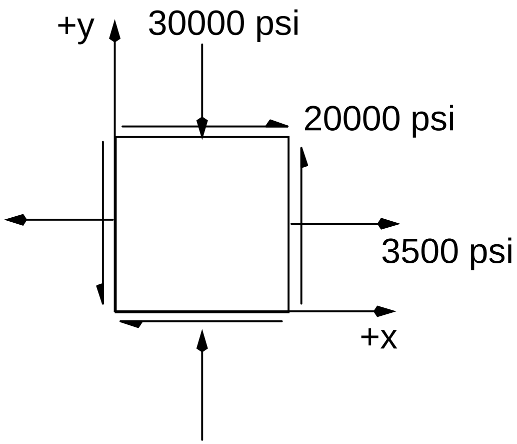
Sample Problem

- Given the following stress components (where the x-axis is horizontal and positive to the right, and the y-axis is vertical and positive upwards):

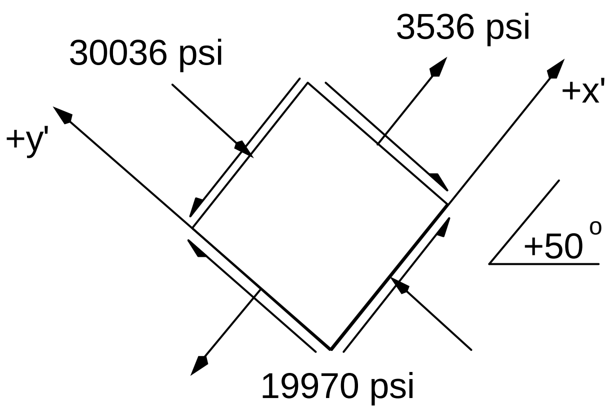
σ_{xx} = 3500 psi σ_{yy} = -30000 psi τ_{xy} = 20000 psi

(a) Sketch the stress element in the x-y coordinate system
(b) Sketch the stress element in the x'-y' coordinate system, oriented 50°CCW
(c) Sketch the stress element in the principal stress coordinate system

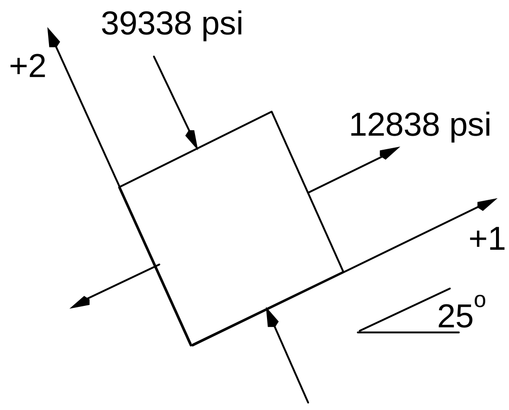
Sample Problem (answers)



Part (a)



Part (b)



Part (c)

Strain

Fundamental Definitions

- "Strain" is a measure of the deformation of a solid body
- There are two "types" of strain; normal strain (ε) and shear strain (γ)

$$\varepsilon = \frac{\text{(change in length)}}{\text{(original length)}} \qquad \text{units} = \text{in/in, m/m, etc}$$

$$\gamma = \text{(change in angle)} \qquad \text{units} = \text{radians}$$

The Strain Tensor

- Strain is a 2nd-order tensor, and in the most general case, six components of strain exist "at a point":

$$\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}$$

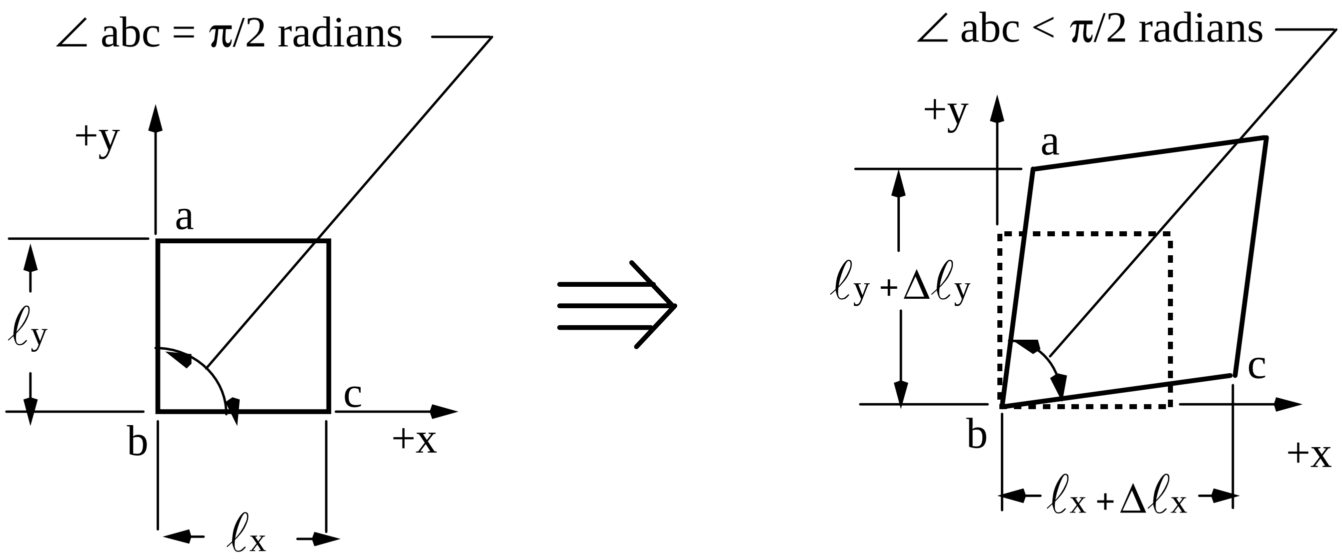
- Since strain is a tensorial quantity, the values of the individual strain components which define the "state of strain" depend on the coordinate system used…

- This review will primarily involve strains which exist within a single plane

Strain Within a Plane

- We often encounter two distinct conditions that result in problems involving "strains within a plane":
  - Plane Stress: All non-zero *stress* components lie within a single plane (e.g., σ_{xx}, σ_{yy}, $\tau_{xy} \neq 0$, $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$). If the material is isotropic, the plane stress condition induces four non-zero strain components: ε_{xx}, ε_{yy}, ε_{zz}, and γ_{xy}
  - Plane Strain: All non-zero *strain* components lie within a single plane (e.g., ε_{xx}, ε_{yy}, $\gamma_{xy} \neq 0$, $\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$). By definition, the plane strain condition involves three non-zero strain components: ε_{xx}, ε_{yy}, and γ_{xy}

Strain Within a Plane



Original Shape

Deformed Shape

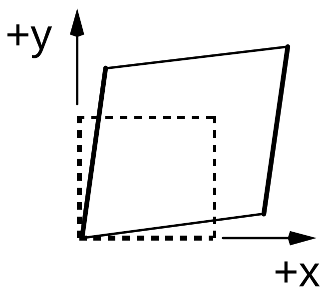
$$\varepsilon_{xx} = \lim_{\ell_x \to 0} (\Delta \ell_x / \ell_x)$$

$$\varepsilon_{yy} = \lim_{\ell_y \to 0} (\Delta \ell_y / \ell_y)$$

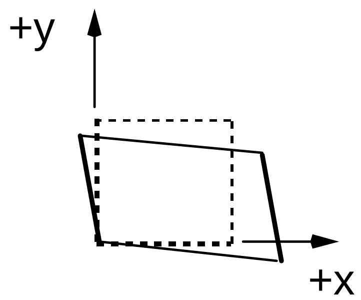
$$\gamma_{xy} = \lim_{\ell_x, \ell_y \to 0} (\Delta \angle abc)$$

Strain Sign Convention

- A positive (tensile) normal strain is associated with an increase in length
- A shear strain is positive if the angle between two positive faces (or two negative faces) decreases



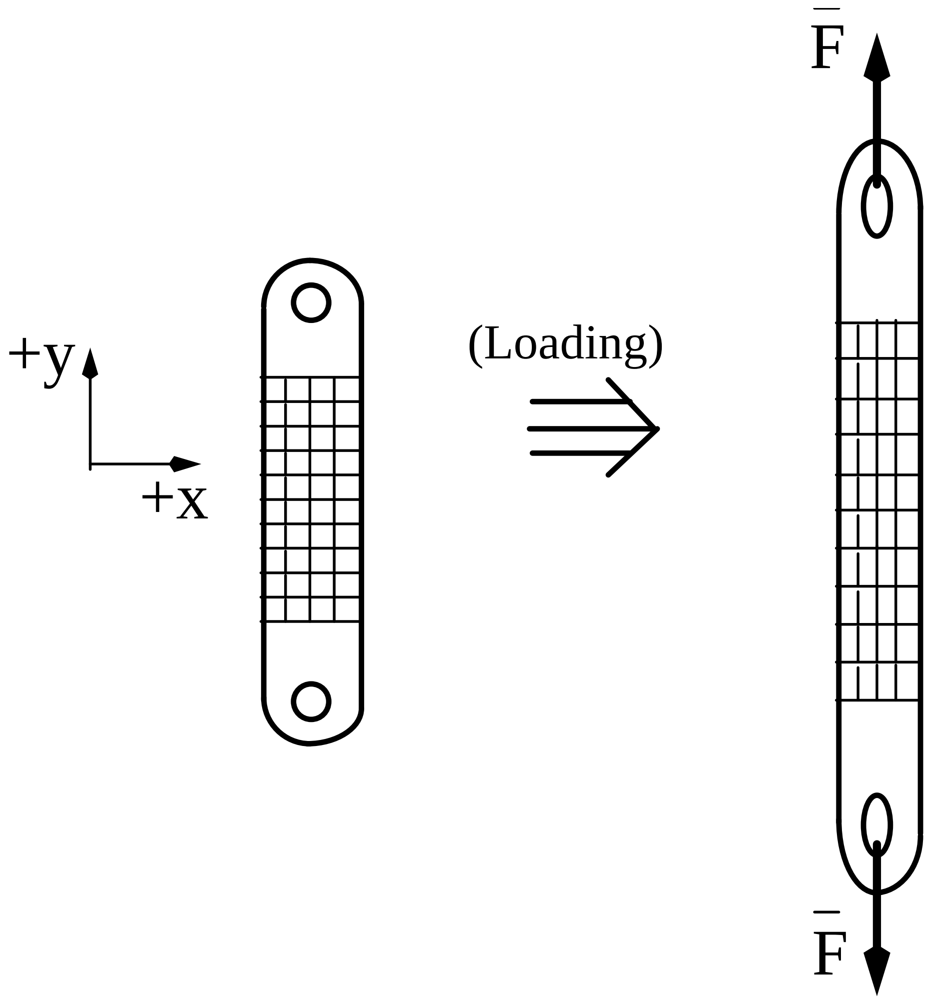
All Strains Positive



ε_{xx} Positive

ε_{yy} and γ_{xy} Negative

Visualization of Strain
(Assuming link made of an isotropic material)



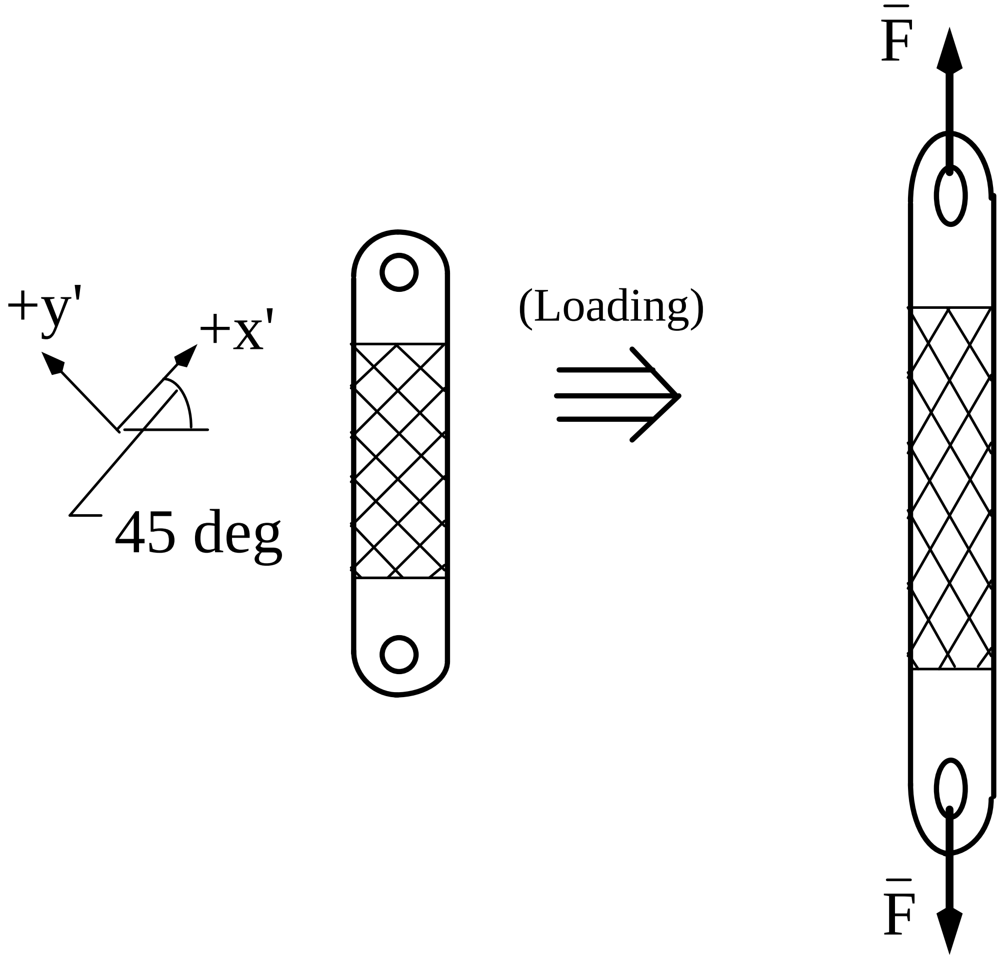
$\varepsilon_{yy} > 0$

$\varepsilon_{xx} < 0$

$\gamma_{xy} = 0$

Visualization of Strain

(Assuming link is made of an isotropic material)

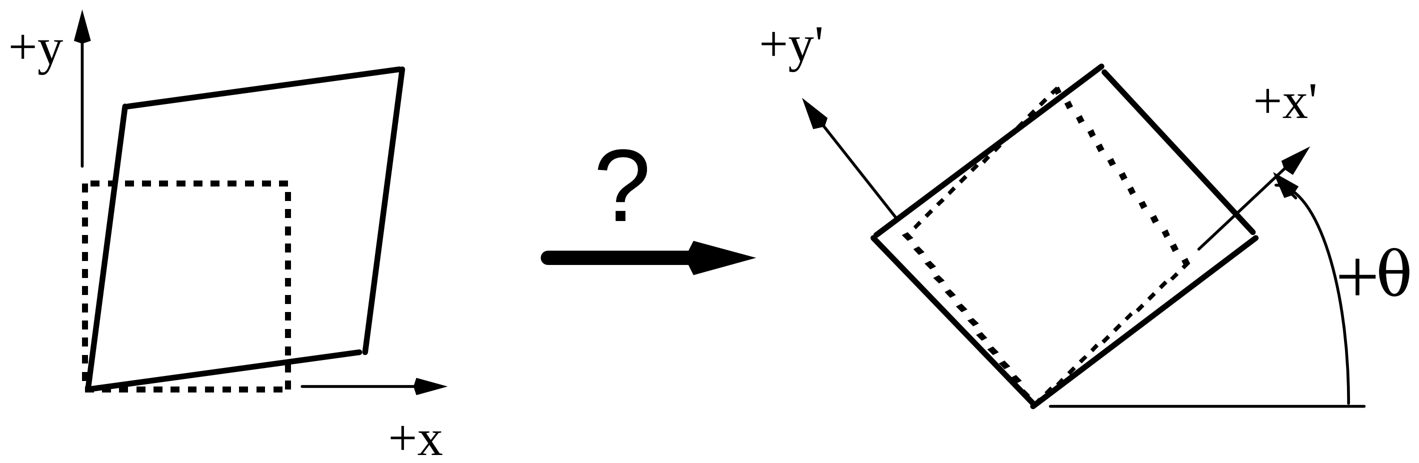


$\varepsilon_{x'x'} = \varepsilon_{y'y'} > 0$

$\gamma_{x'y'} > 0$

Strain Transformations

- Given strain components in the x-y coordinate system, what are the corresponding strain components in the x'-y' coordinate system?



Given: ε_{xx}, ε_{yy}, and γ_{xy} Find: $\varepsilon_{x'x'}$, $\varepsilon_{y'y'}$, and $\gamma_{x'y'}$

Strain Transformation Equations

- Based strictly on geometry, it can be shown:

$$\varepsilon_{x'x'} = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \left(\frac{\gamma_{xy}}{2}\right)2\cos\theta\sin\theta$$

$$\varepsilon_{y'y'} = \varepsilon_{xx}\sin^2\theta + \varepsilon_{yy}\cos^2\theta - \left(\frac{\gamma_{xy}}{2}\right)2\cos\theta\sin\theta$$

$$\frac{\gamma_{x'y'}}{2} = (\varepsilon_{yy} - \varepsilon_{xx})\cos\theta\sin\theta + \left(\frac{\gamma_{xy}}{2}\right)(\cos^2\theta - \sin^2\theta)$$

Important note: angle θ is positive counter-clockwise (CCW) *from* the +x-direction *towards* the +y-direction

Well, I'll Be Darned!!!

- The stress transformation equations are based strictly on the equations of static equilibrium
- The strain transformation equations are based strictly on geometry
- Nevertheless, the stress and strain transformation equations are nearly identical!! (…because both stress and strain are 2^{nd}-order tensors…)

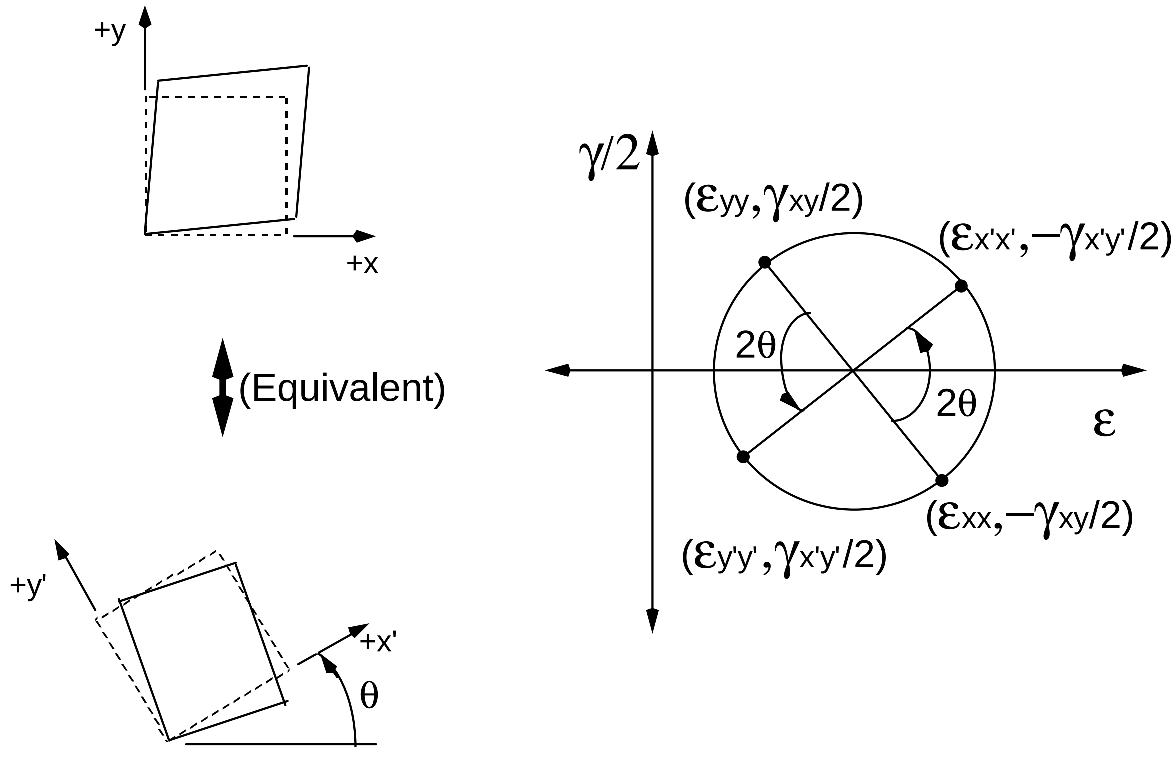
Strain Transformation Equations

- Using matrix notation, these can also be written

$$\begin{Bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ \gamma_{x'y'}/2 \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & (\cos^2\theta - \sin^2\theta) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy}/2 \end{Bmatrix}$$

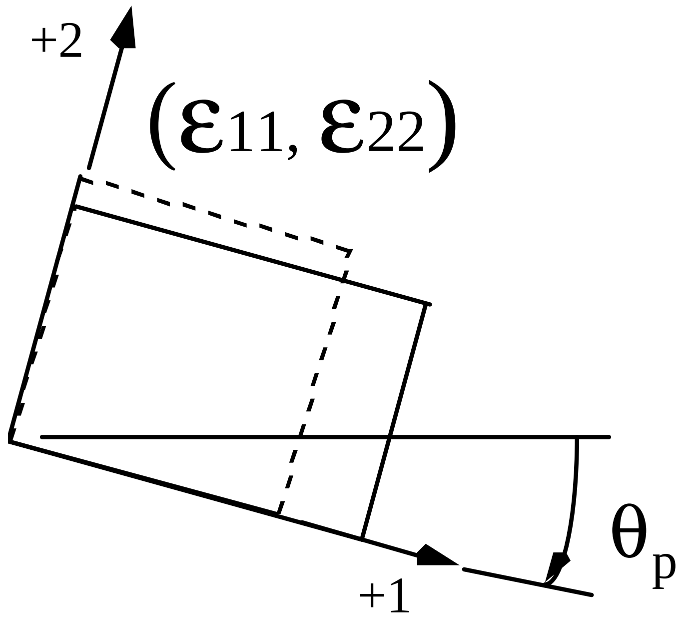
Strain Transformation Equations

- These transformation equations can also be visualized using Mohr's circle of strain:

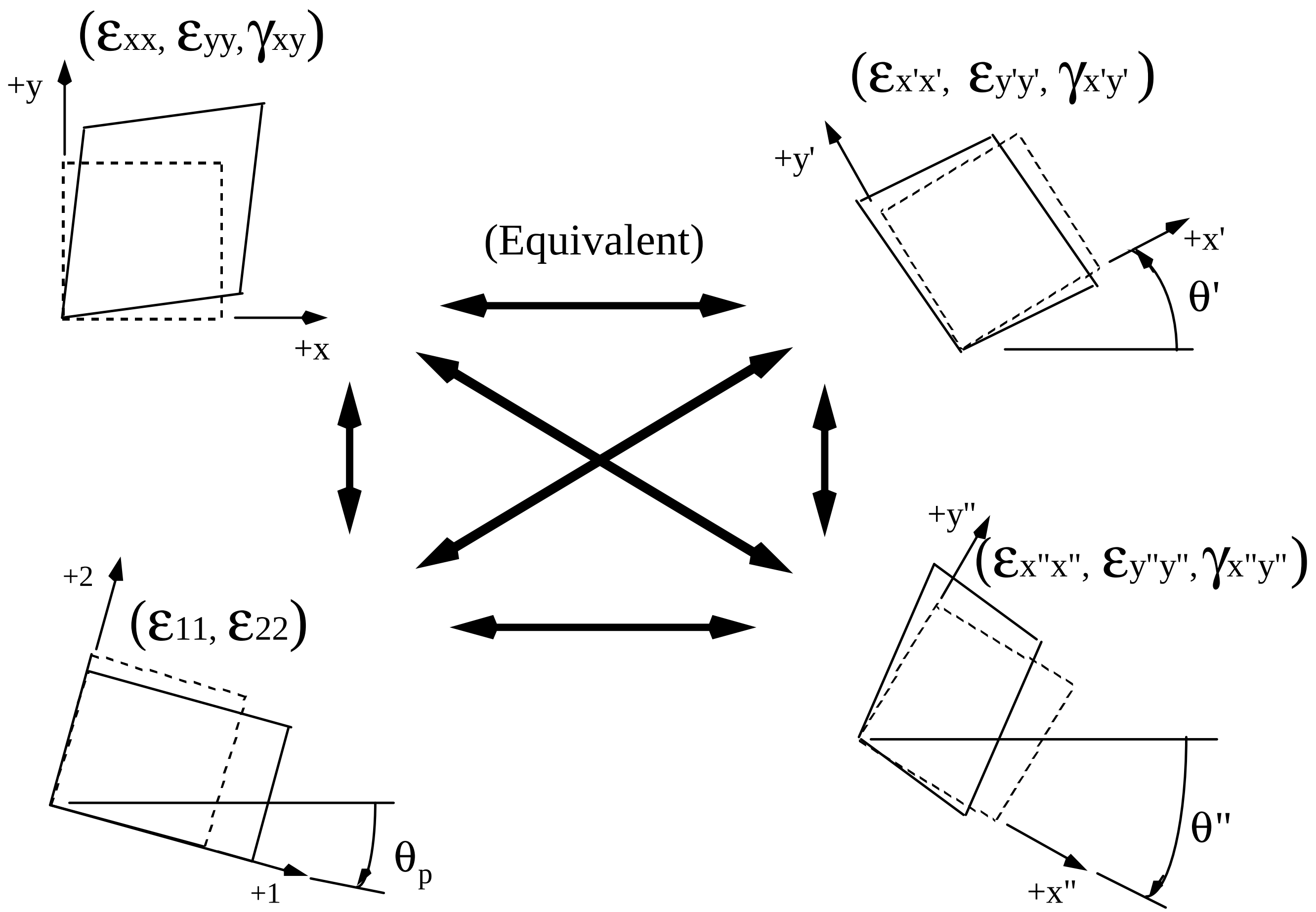


Principal Strains

- In the principal strain coordinate system the shear strain is zero and the normal strains are max/min...



“Transformation” of Strain



"Strain": Summary of Key Points

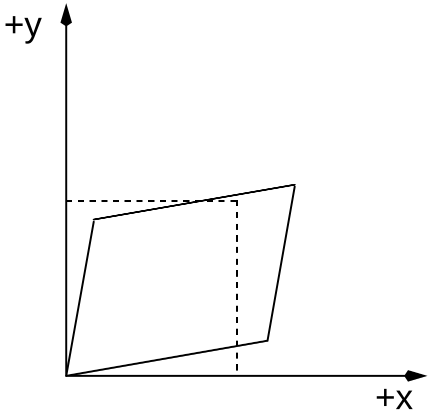
- ε = (Δ length)/(original length) γ = (Δ angle)
- Six components of strain specify the "state of strain"
- Strain is a tensorial quantity; numerical values of individual strain components depend on the coordinate system used
- Strain is defined strictly on the basis of a change in shape; definition is independent of:
  - material properties
  - stress
  - temperature
- "Suprisingly," the stress and strain transformation equations are nearly identical

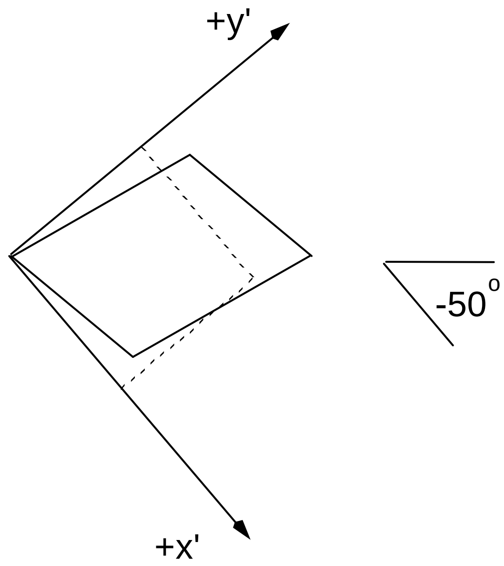
Sample Problem

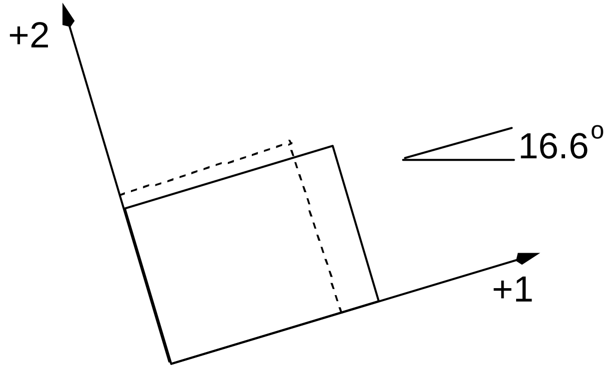
- Given the following strain components (where the x-axis is horizontal and positive to the right, and the y-axis is vertical and positive upwards):
  ε_{xx} = 2000 μin/in ε_{yy} = -1350 μin/in γ_{xy} = 2200 μrad

- (a) Sketch (not to scale) the strain element in the x-y coordinate system
  (b) Sketch (not to scale) the strain element in the x'-y' coordinate system, oriented 50°CW
  (c) Sketch (not to scale) the strain element in the principal strain coordinate system

Sample Problem (answers)







Part (a):
ε_{xx} = 2000 μin/in
ε_{yy} = -1350 μin/in
γ_{xy} = 2200 μrad

Part (b):
$\varepsilon_{x'x'}$ = -1049 μin/in
$\varepsilon_{y'y'}$ = 1699 μin/in
$\gamma_{x'y'}$ = 2917 μrad

Part (c):
ε_{11} = 2329 μin/in
ε_{22} = -1679 μin/in
θ_p = 16.6°

Hooke's Law

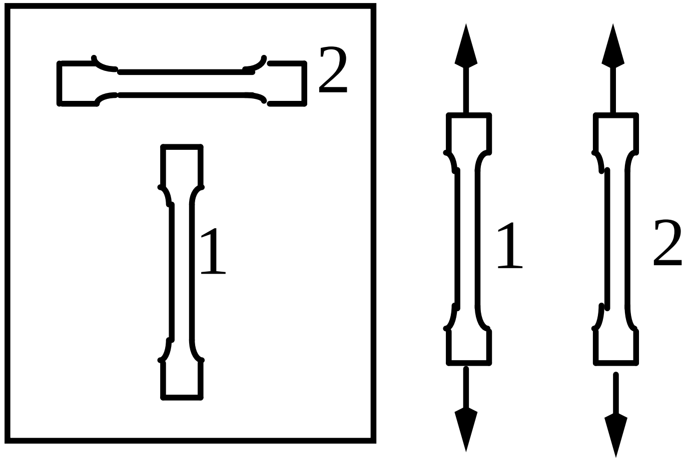
- The structural engineer is typically interested in the state of stress induced in a structure during service
- The state of stress cannot be measured directly….
- The state of strain can be measured directly….
- Hence, we must develop a relationship between the stress tensor and the strain tensor…this relationship is called a "constitutive model", and the most common is "Hooke's Law"

Hooke's Law (Cont'd)

- The form of Hooke's depends on whether the material is isotropic or anisotropic:
  - Isotropic materials: same properties in all directions
  - Anisotropic materials: properties vary with direction
- More than one "type" of anisotropic behavior. Three will be mentioned in this review:
  - Transversely isotropic
  - Orthotropic
  - Generally anisotropic

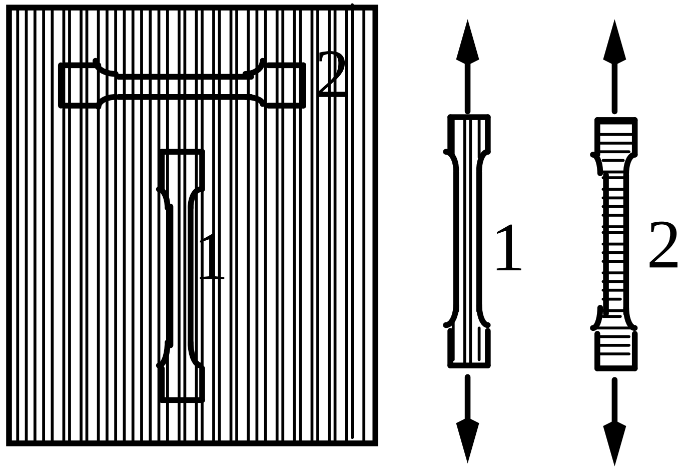
Isotropic vs Anisotropic Materials

Anisotropy occurs because of some type of order in the microstructure



$E_1 \cong E_2$

Isotropic Materials



$E_1 > E_2$

Anistropic Materials

Hooke's Law

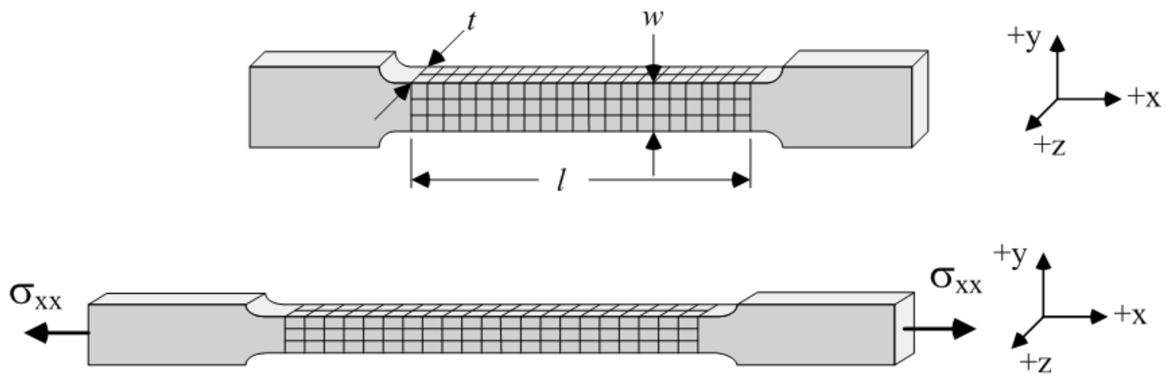
- Hooke's Law will be reviewed/discussed in the following order:
  - Isotropic materials
  - Anisotropic
    - Transversely isotropic
    - Orthotropic
    - Generally anisotropic

Isotropic Material Properties

The Uniaxial Tensile Test

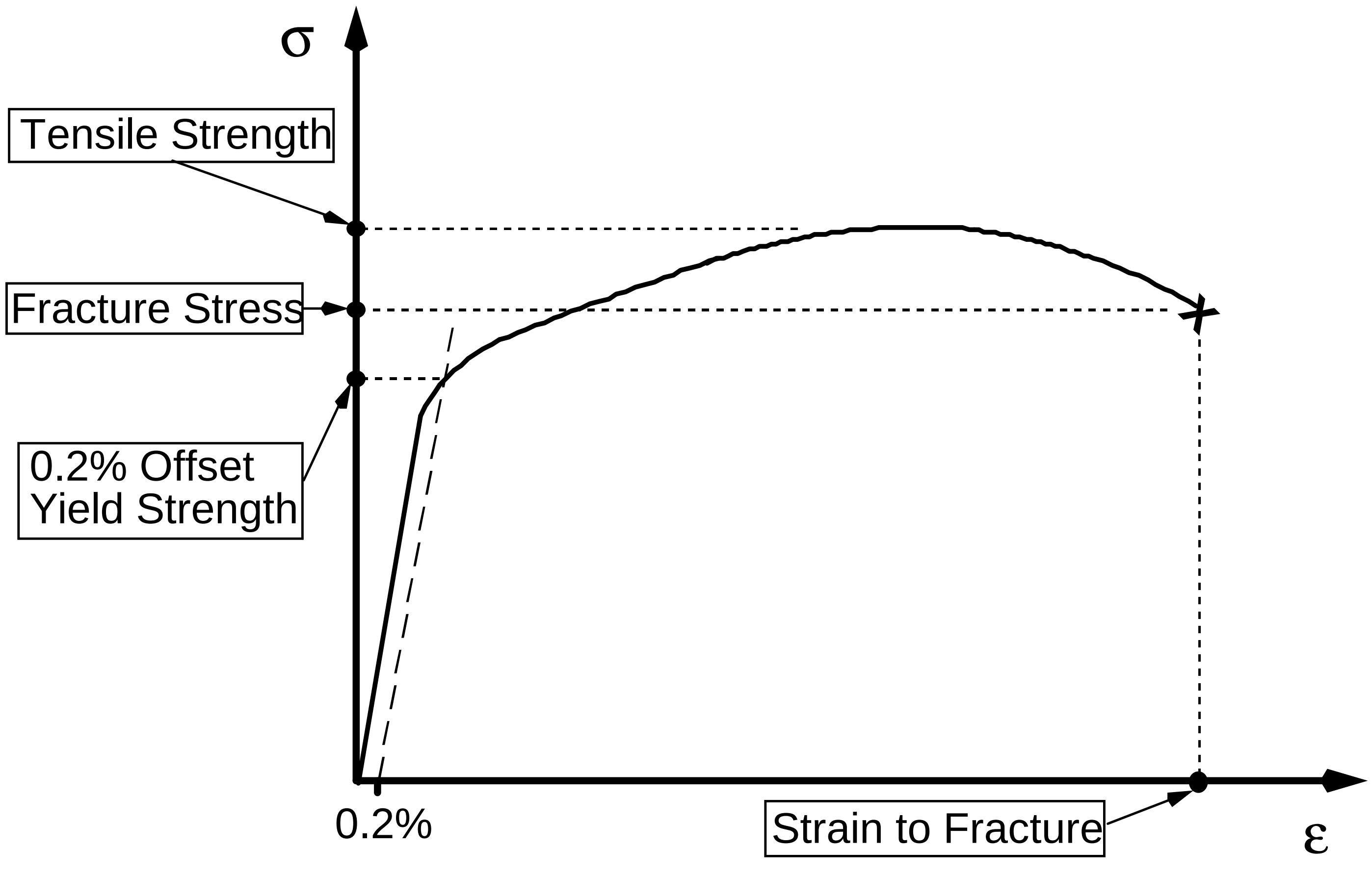
- Specimen is subjected to axial tensile force, inducing a ***uniaxial*** state of stress in the "gage" region
- Stress is increased until fracture occurs; corresponding axial and transverse strains are measured throughout the test (all shear strains = 0)

$$\varepsilon_{xx} = \Delta l/l \quad \varepsilon_{yy} = \varepsilon_{zz} = \Delta w/w = \Delta t/t \quad \gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0$$

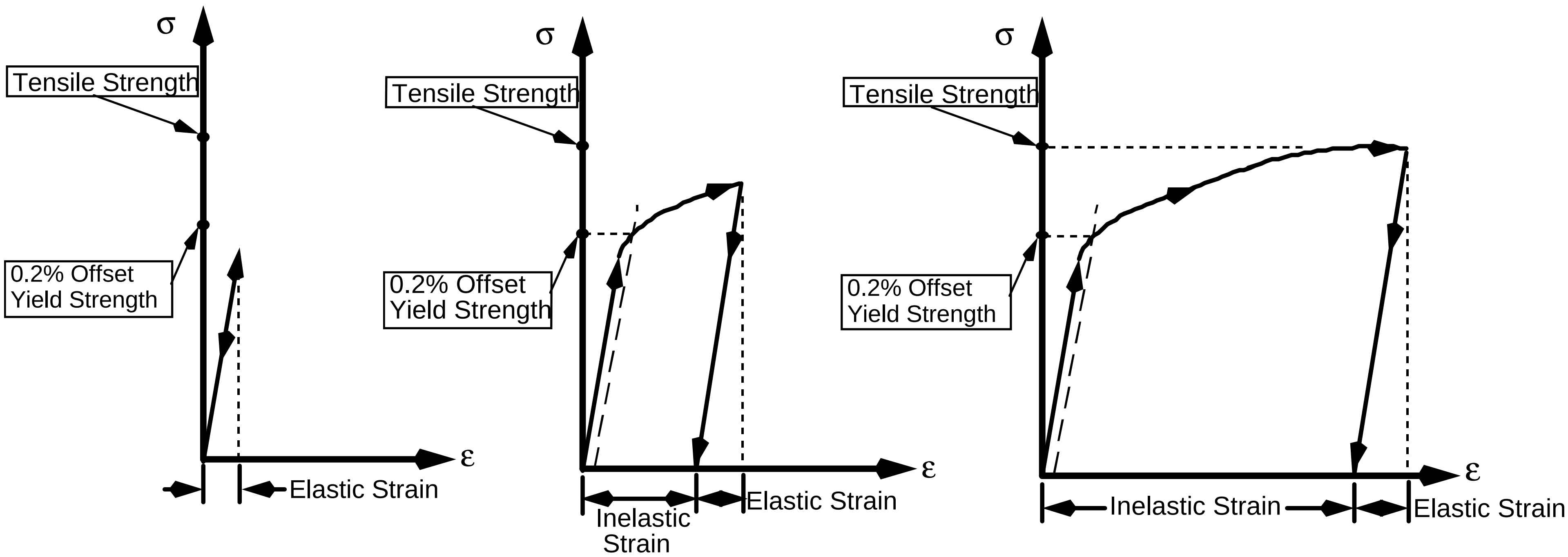


The Tensile Stress-Strain Curve

A plot of axial stress vs axial strain



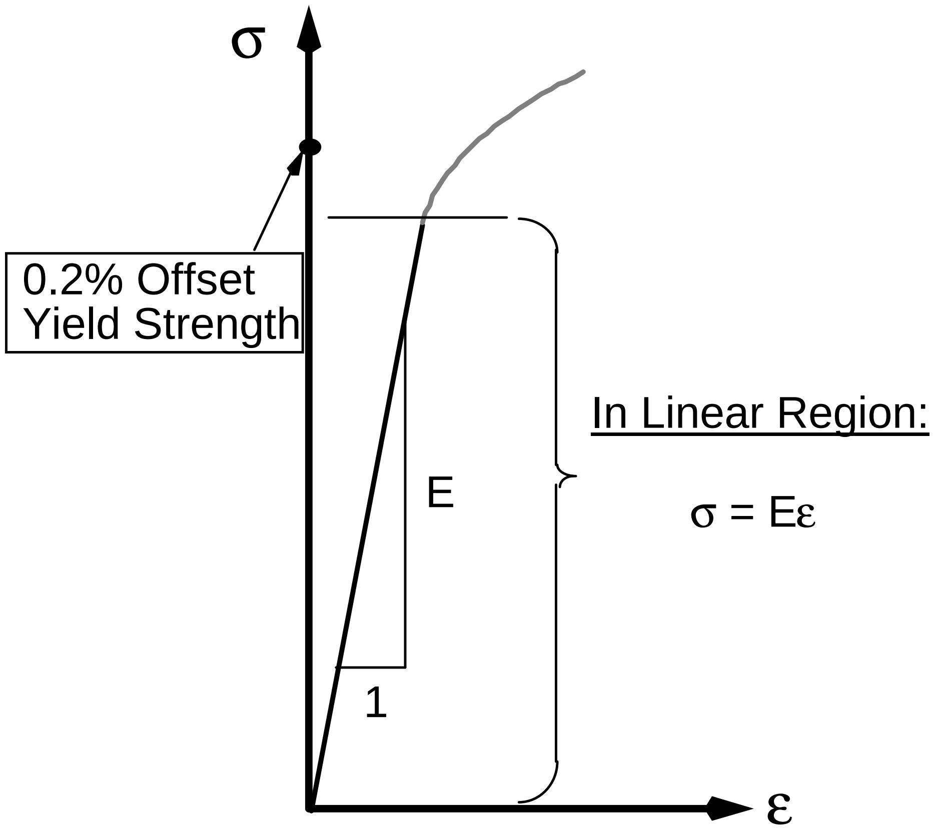
Load-Unload Cycles



Material Property:
Young's Modulus

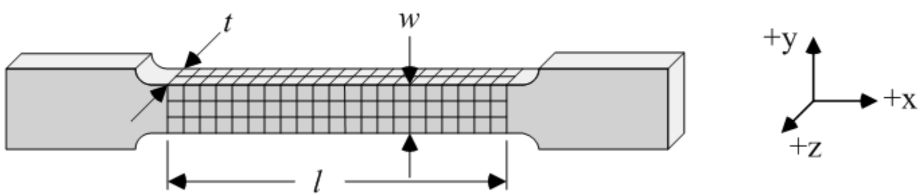
- At stress levels below the yield stress the response is called "linear elastic"
- The slope of the linear region is called "Young's modulus" or the "modulus of elasticity", E
- In the linear region and *for a uniaxial stress-state* (<u>only</u>!!!):

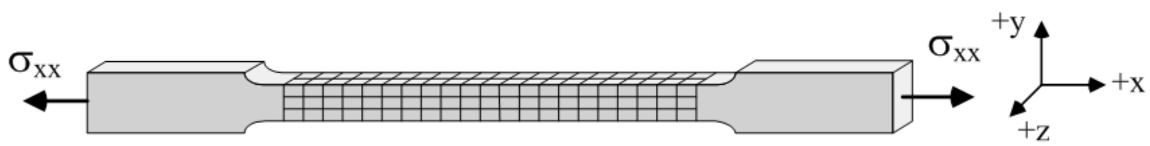
$\sigma = E\varepsilon$ (or) $\varepsilon = \sigma/E$



Material Property:
Poisson's Ratio

- Poisson's ratio is based on the ratio of two normal strains cause by a *uniaxial* stress: $\nu = -(\varepsilon_t/\varepsilon_a)$
- Poisson’s ratio is a measure of the *coupling* between σ_{xx} and ε_{yy}, ε_{zz}
- In this case: $\nu = (-\varepsilon_{yy}/\varepsilon_{xx}) = (-\varepsilon_{zz}/\varepsilon_{xx})$





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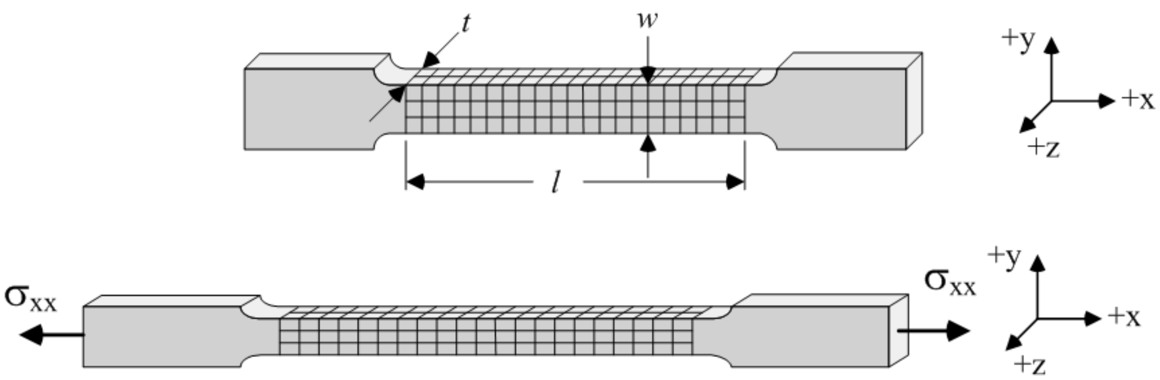
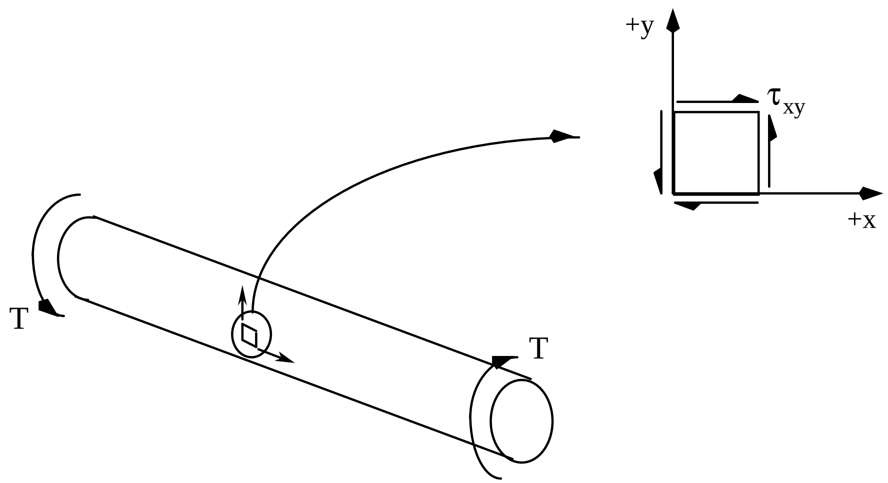


Material Properties

The Torsion Test

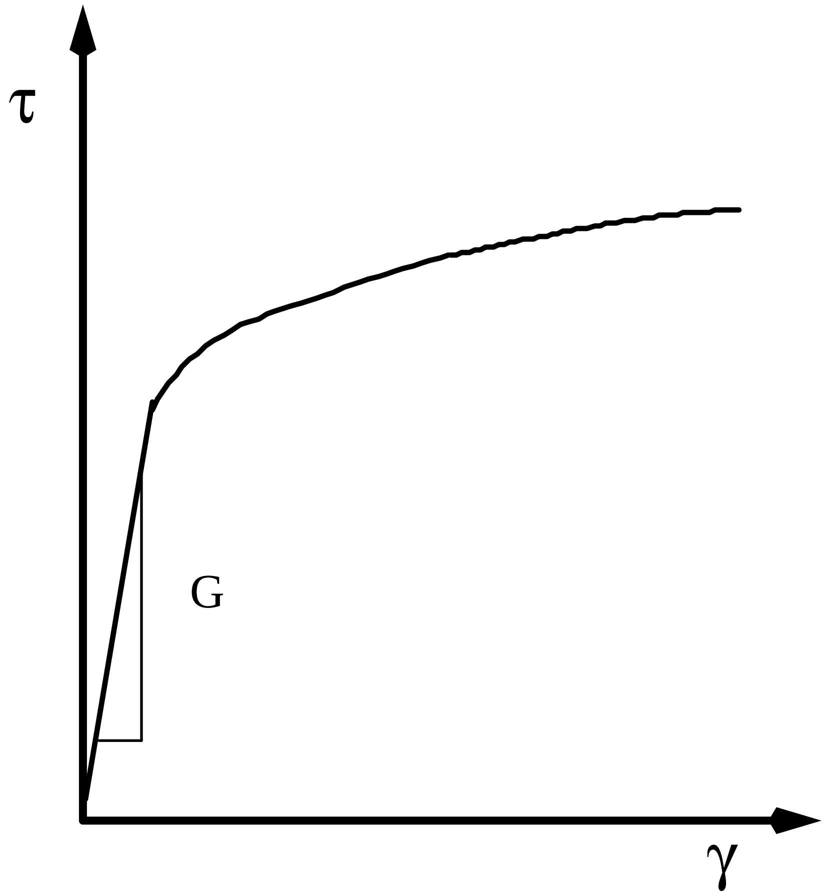
- Thin-walled cylindrical specimen subjected to a torque, inducing a uniform shear stress τ_{xy} in the gage region of the specimen
- Shear stress (i.e., torque) increased until fracture occurs; shear strain measured throughout test



The Shear Stress- Shear Strain Curve

- At linear levels, the slope of the shear stress -shear strain curve is called the “shear modulus”

- In the linear region (only!!) $\tau_{xy} = G\gamma_{xy}$ (or) $\gamma_{xy} = \tau_{xy}/G$



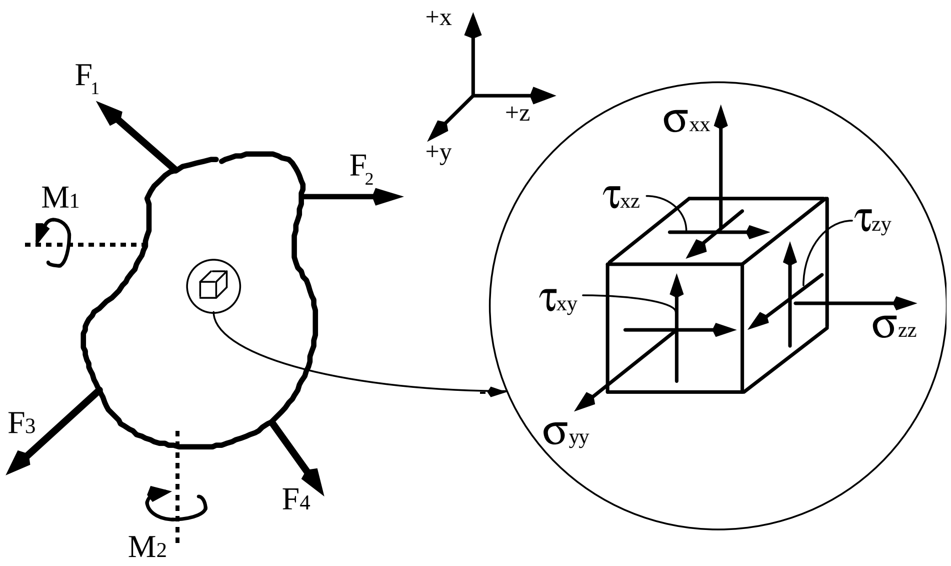
Number of Independent Material Properties

- Three material properties have been defined; E, ν, and G
- For an isotropic material, only two of these three properties are independent…it can be shown:

$$G = \frac{E}{2(1+\nu)}$$

Derivation of Hooke’s Law

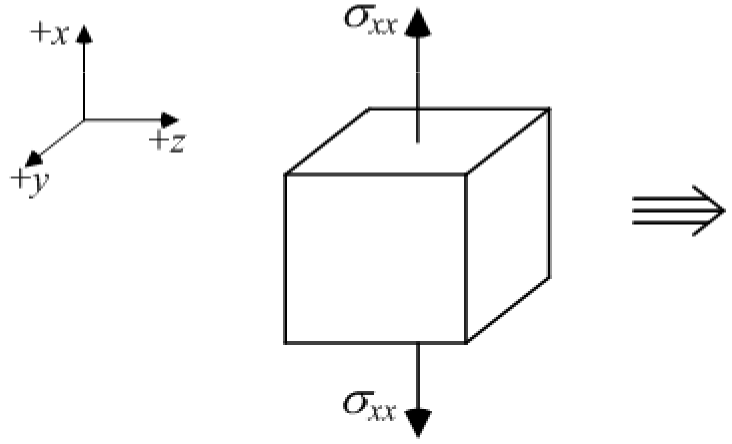
For an isotropic material subjected to general 3D streses



- We assume the strain tensor is linearly related to the stress tensor….(when is this a bad assumption?)
- Assuming the linear assumption is appropriate, the principle of superposition can be used to develop a Hooke’s law:

Hooke's Law (cont'd)

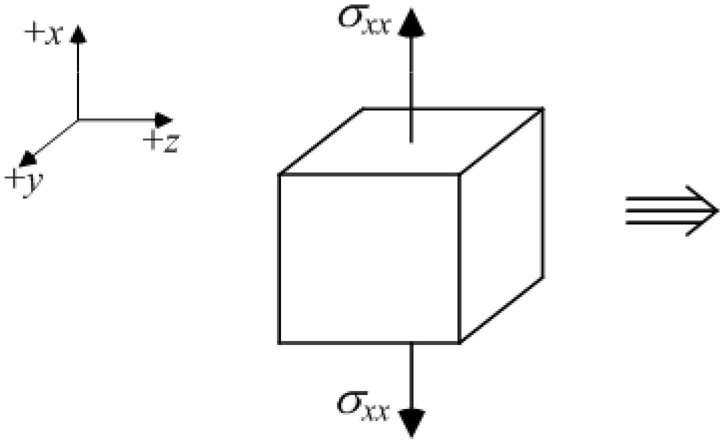
Strains caused by σ_{xx} only:



(What strains are induced by σ_{xx} only?)

Hooke's Law (cont'd)

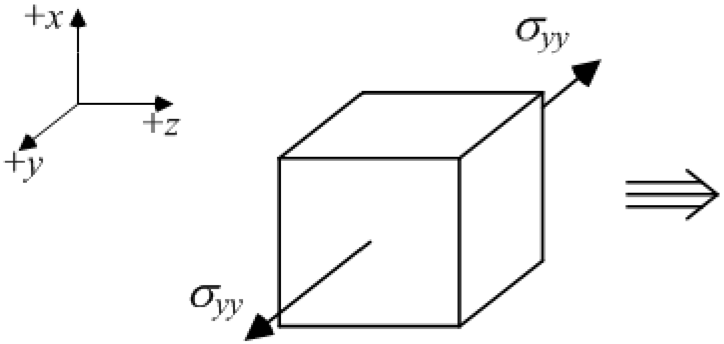
Strains caused by σ_{xx} only:



$$\varepsilon_{xx} = k_{11}\,\sigma_{xx} = (1/E)\,\sigma_{xx}$$
$$\varepsilon_{yy} = k_{21}\,\sigma_{xx} = (-\nu/E)\,\sigma_{xx}$$
$$\varepsilon_{zz} = k_{31}\,\sigma_{xx} = (-\nu/E)\,\sigma_{xx}$$
$$\gamma_{xy} = k_{41}\,\sigma_{xx} = 0$$
$$\gamma_{yz} = k_{51}\,\sigma_{xx} = 0$$
$$\gamma_{zx} = k_{61}\,\sigma_{xx} = 0$$

Hooke's Law (cont'd)

Strains caused by σ_{yy} only:



$\varepsilon_{xx} = k_{12}\ \sigma_{yy} = (-\nu/E)\ \sigma_{yy}$

$\varepsilon_{yy} = k_{22}\ \sigma_{yy} = (1/E)\ \sigma_{yy}$

$\varepsilon_{zz} = k_{32}\ \sigma_{yy} = (-\nu/E)\ \sigma_{yy}$

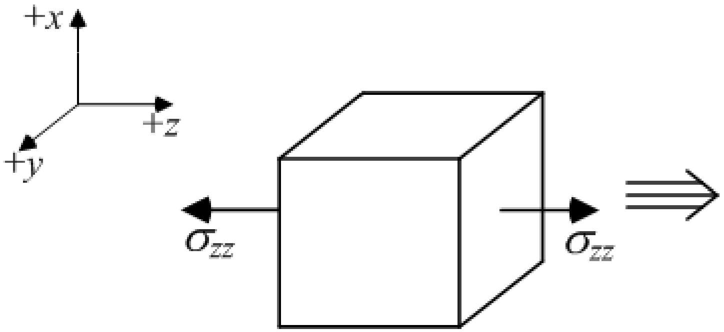
$\gamma_{xy} = k_{42}\ \sigma_{yy} = 0$

$\gamma_{yz} = k_{52}\ \sigma_{yy} = 0$

$\gamma_{zx} = k_{62}\ \sigma_{yy} = 0$

Hooke's Law (cont'd)

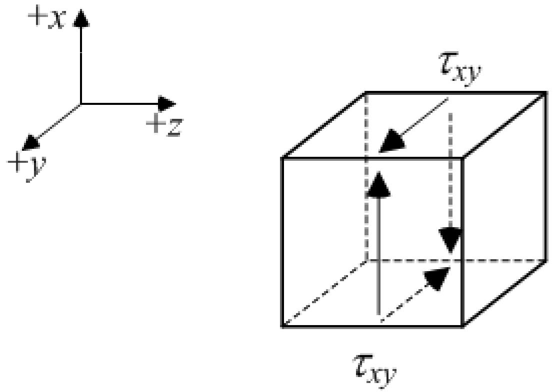
Strains caused by σ_{zz} only:



$$\varepsilon_{xx} = k_{13}\,\sigma_{zz} = (-\nu/E)\,\sigma_{zz}$$
$$\varepsilon_{yy} = k_{23}\,\sigma_{zz} = (-\nu/E)\,\sigma_{zz}$$
$$\varepsilon_{zz} = k_{33}\,\sigma_{zz} = (1/E)\,\sigma_{zz}$$
$$\gamma_{xy} = k_{43}\,\sigma_{zz} = 0$$
$$\gamma_{yz} = k_{53}\,\sigma_{zz} = 0$$
$$\gamma_{zx} = k_{63}\,\sigma_{zz} = 0$$

Hooke's Law (cont'd)

Strains caused by τ_{xy} only:

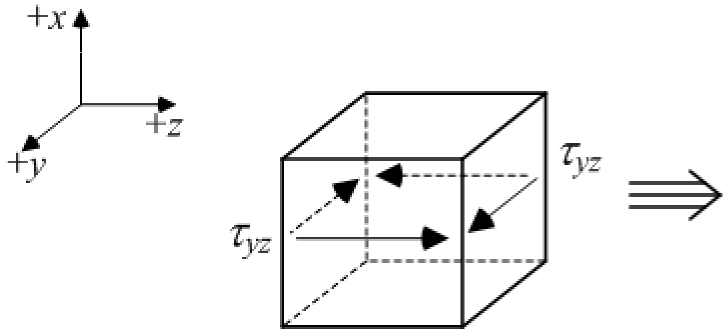


$$\Longrightarrow$$

$$\varepsilon_{xx} = k_{14}\,\tau_{xy} = 0$$
$$\varepsilon_{yy} = k_{24}\,\tau_{xy} = 0$$
$$\varepsilon_{zz} = k_{34}\,\tau_{xy} = 0$$
$$\gamma_{xy} = k_{44}\,\tau_{xy} = (1/G)\tau_{xy} = [2(1+\nu)/E]\,\tau_{xy}$$
$$\gamma_{yz} = k_{54}\,\tau_{xy} = 0$$
$$\gamma_{zx} = k_{64}\,\tau_{xy} = 0$$

Hooke's Law (cont'd)

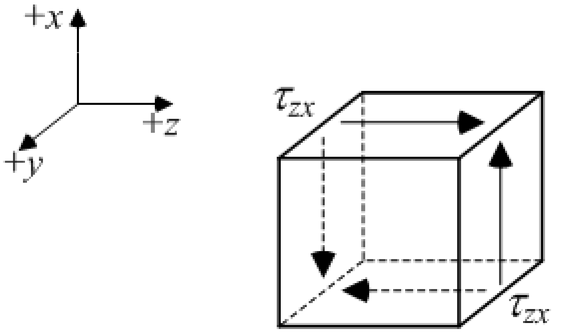
Strains caused by τ_{yz} only:



$$\varepsilon_{xx} = k_{15}\,\tau_{yz} = 0$$
$$\varepsilon_{yy} = k_{25}\,\tau_{yz} = 0$$
$$\varepsilon_{zz} = k_{35}\,\tau_{yz} = 0$$
$$\gamma_{xy} = k_{45}\,\tau_{yz} = 0$$
$$\gamma_{yz} = k_{55}\,\tau_{yz} = (1/G)\tau_{yz} = [2(1+\nu)/E]\,\tau_{yz}$$
$$\gamma_{zx} = k_{65}\,\tau_{yz} = 0$$

Hooke's Law (cont'd)

Strains caused by τ_{zx} only:

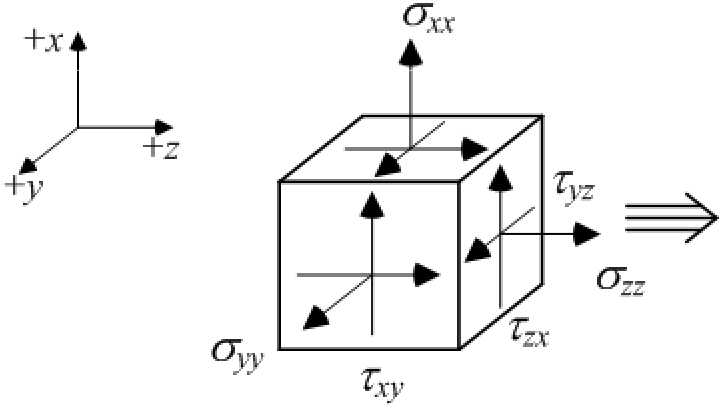


$\Longrightarrow$

$$\varepsilon_{xx} = k_{16}\ \tau_{zx} = 0$$
$$\varepsilon_{yy} = k_{26}\ \tau_{zx} = 0$$
$$\varepsilon_{zz} = k_{36}\ \tau_{zx} = 0$$
$$\gamma_{xy} = k_{46}\ \tau_{zx} = 0$$
$$\gamma_{yz} = k_{56}\ \tau_{zx} = 0$$
$$\gamma_{zx} = k_{66}\ \tau_{zx} = (1/G)\tau_{zx} = [2(1+\nu)/E]\ \tau_{zx}$$

Hooke's Law (cont'd)

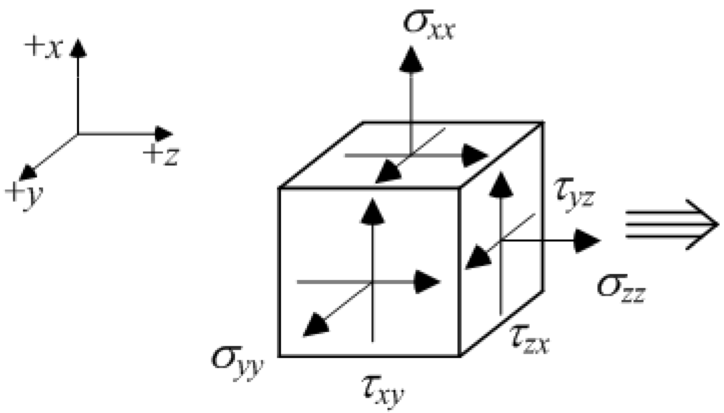
Strain ε_{xx} caused by all stress components acting simultaneously:



$\varepsilon_{xx} = ?$

Hooke's Law (cont'd)

Strain ε_{xx} caused by all stress components acting simultaneously:

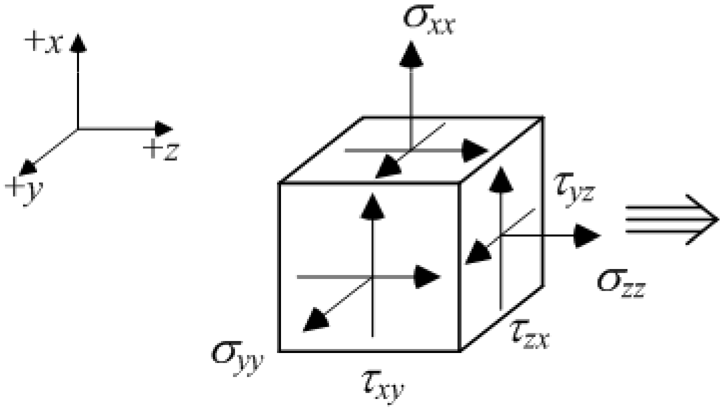


$\varepsilon_{xx} = ?$

(since strain-stress relation assumed linear, we can apply the principle of superposition and simply add up contribution of each stress component):

Hooke's Law (cont'd)

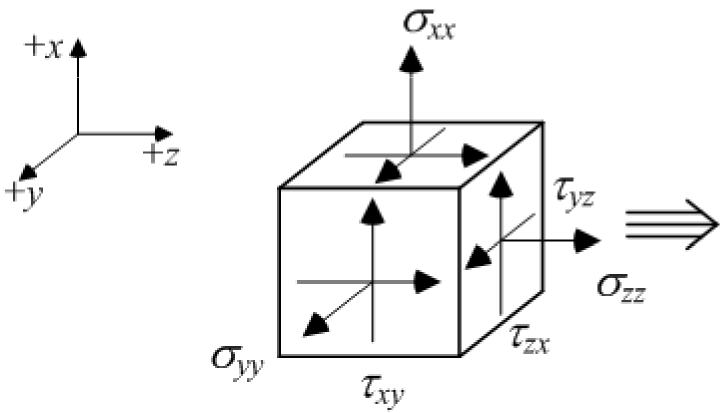
Strain ε_{xx} *caused by all stress components acting simultaneously:*



$$\varepsilon_{xx} = k_{11}\,\sigma_{xx} + k_{12}\,\sigma_{yy} + k_{13}\,\sigma_{zz} + k_{14}\,\tau_{xy} + k_{15}\,\tau_{yz} + k_{16}\,\tau_{zx}$$

Hooke's Law (cont'd)

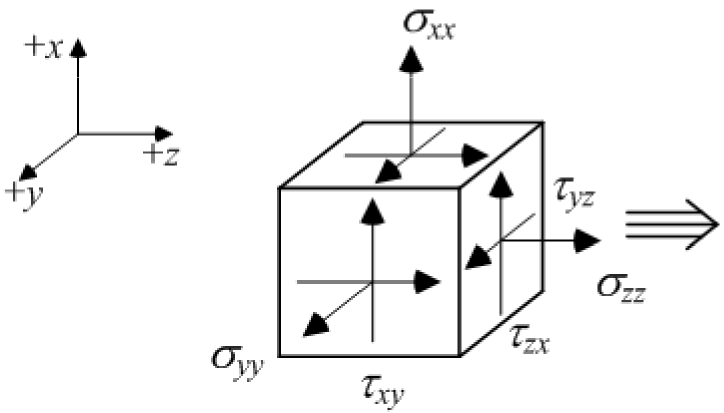
Strain ε_{xx} caused by all stress components acting simultaneously:



$$\varepsilon_{xx} = (1/E)\,\sigma_{xx} + (-\nu/E)\sigma_{yy} + (-\nu/E)\,\sigma_{zz} + (0)\,\tau_{xy} + (0)\,\tau_{yz} + (0)\,\tau_{zx}$$

Hooke's Law (cont'd)

Rearranging:



$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right]$$

Hooke's Law

Repeating process for all six strain components

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \qquad \gamma_{xy} = \frac{2(1+\nu)\tau_{xy}}{E}$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] \qquad \gamma_{xz} = \frac{2(1+\nu)\tau_{xz}}{E}$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] \qquad \gamma_{yz} = \frac{2(1+\nu)\tau_{yz}}{E}$$

Hooke's Law

Matrix Notation

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}$$

Hooke's Law

Inverting the six equations leads to a more convenient form for experimental analysis…

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu(\varepsilon_{yy} + \varepsilon_{zz})\right] \qquad \tau_{xy} = \frac{E\gamma_{xy}}{2(1+\nu)}$$

$$\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{yy} + \nu(\varepsilon_{xx} + \varepsilon_{zz})\right] \qquad \tau_{xz} = \frac{E\gamma_{xz}}{2(1+\nu)}$$

$$\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{zz} + \nu(\varepsilon_{xx} + \varepsilon_{yy})\right] \qquad \tau_{yz} = \frac{E\gamma_{yz}}{2(1+\nu)}$$

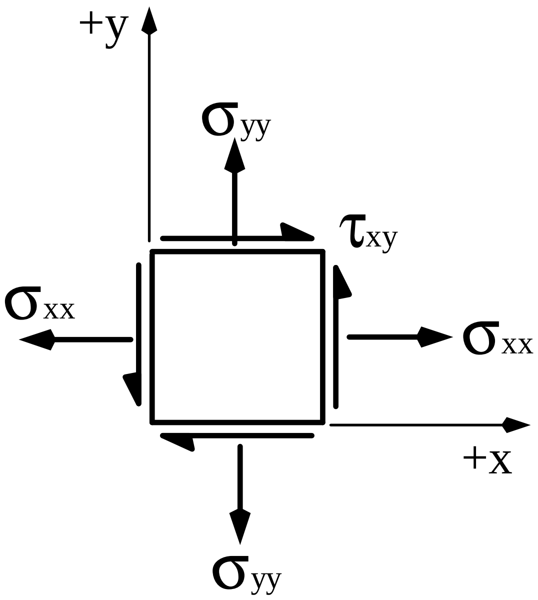
Hooke's Law

Matrix Notation

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ \nu & (1-\nu) & \nu & 0 & 0 & 0 \\ \nu & \nu & (1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

Hooke's Law For Plane Stress

assume $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$



$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right)$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx}\right)$$

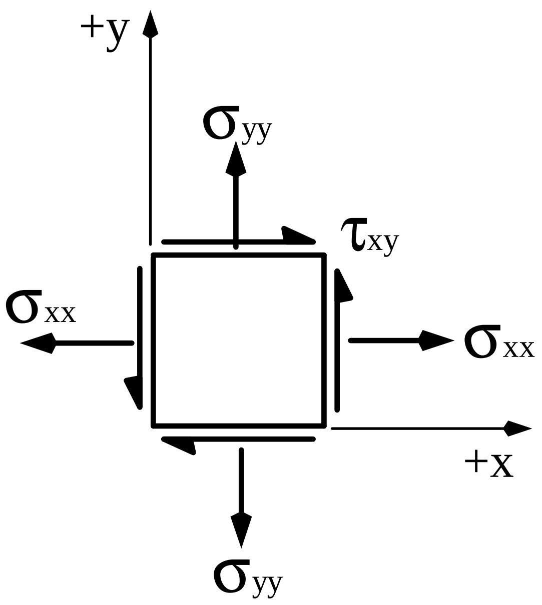
$$\varepsilon_{zz} = \frac{-\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right)$$

$$\gamma_{xy} = \frac{2(1+\nu)\tau_{xy}}{E}$$

$$\gamma_{xz} = \gamma_{yz} = 0$$

Hooke's Law For Plane Stress

assume $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$



$$\sigma_{xx} = \frac{E}{(1-\nu^2)}\left[\varepsilon_{xx} + \nu\varepsilon_{yy}\right]$$

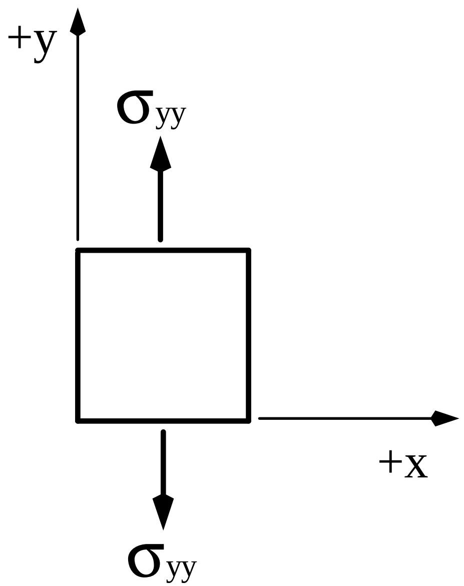
$$\sigma_{yy} = \frac{E}{(1-\nu^2)}\left[\varepsilon_{yy} + \nu\varepsilon_{xx}\right]$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

$$\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$$

Hooke's Law for Uniaxial Stress

If $\sigma_{xx} = \sigma_{zz} = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0$



$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E}$$

$$\varepsilon_{xx} = \varepsilon_{zz} = \frac{-\nu\sigma_{yy}}{E}$$

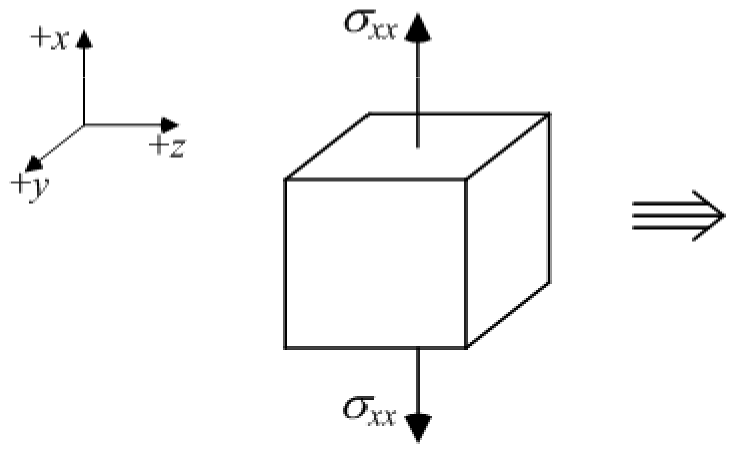
$$\gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0$$

Hooke's Law
Anisotropic materials

- As before, assume stress is linearly related to strain....
- Anisotropic material exhibit two "unusual" features (as compared to isotropic materials):
  - Properties differ with direction (e.g, in general $E_{xx} \neq E_{yy} \neq E_{zz}$)
  - This can lead to unusual "coupling" effects:
    - A *normal* stress may cause *shear* strains
    - A *shear* stress may cause *normal* strains

Hooke's Law – Anisotropic Materials

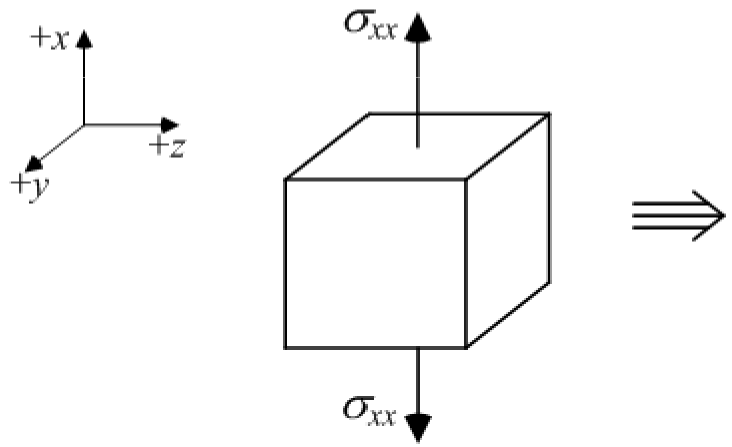
Strains caused by σ_{xx} *only:*



(What strains are induced by σ_{xx} only?)

Hooke's Law – Anisotropic Materials

Strains caused by σ_{xx} *only:*

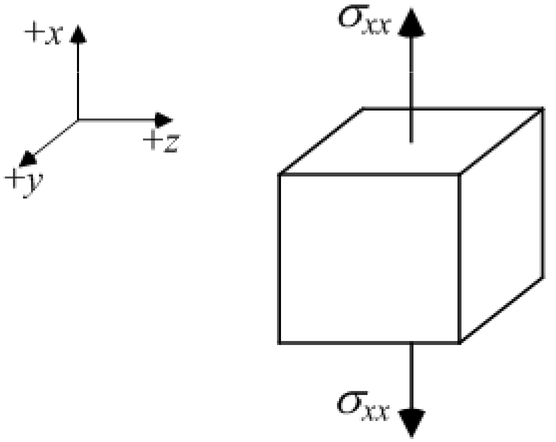


(What strains are induced by σ_{xx} only?)

…for generally anisotropic materials, σ_{xx} will induce <u>*six*</u> components of strain (i.e., σ_{xx} will induce a 3-D strain tensor)

Hooke's Law – Anisotropic Materials

Strains caused by σ_{xx} only:

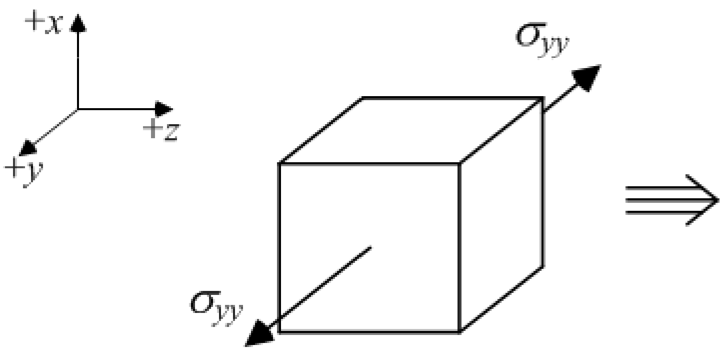


$\Longrightarrow$

$\varepsilon_{xx} = k_{11}\ \sigma_{xx}$
$\varepsilon_{yy} = k_{21}\ \sigma_{xx}$
$\varepsilon_{zz} = k_{31}\ \sigma_{xx}$
$\gamma_{xy} = k_{41}\ \sigma_{xx}$
$\gamma_{yz} = k_{51}\ \sigma_{xx}$
$\gamma_{zx} = k_{61}\ \sigma_{xx}$

Hooke's Law – Anisotropic Materials

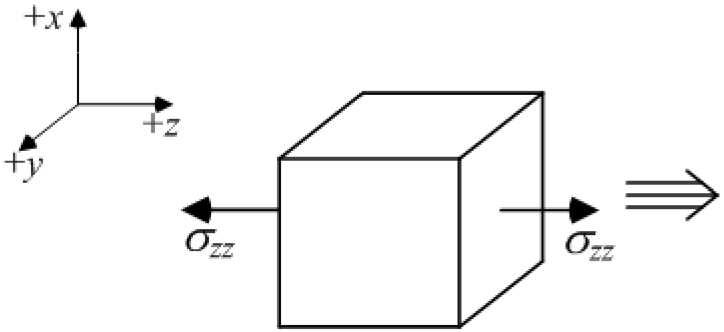
Strains caused by σ_{yy} only:



$$\varepsilon_{xx} = k_{12}\,\sigma_{yy}$$
$$\varepsilon_{yy} = k_{22}\,\sigma_{yy}$$
$$\varepsilon_{zz} = k_{32}\,\sigma_{yy}$$
$$\gamma_{xy} = k_{42}\,\sigma_{yy}$$
$$\gamma_{yz} = k_{52}\,\sigma_{yy}$$
$$\gamma_{zx} = k_{62}\,\sigma_{yy}$$

Hooke's Law – Anisotropic Materials

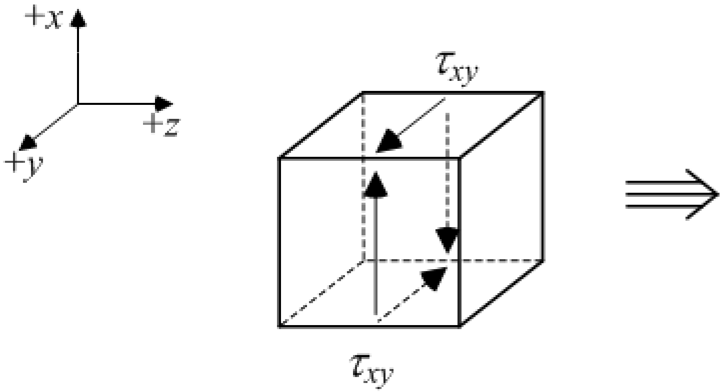
Strains caused by σ_{zz} only:



$$\varepsilon_{xx} = k_{13}\,\sigma_{zz}$$
$$\varepsilon_{yy} = k_{23}\,\sigma_{zz}$$
$$\varepsilon_{zz} = k_{33}\,\sigma_{zz}$$
$$\gamma_{xy} = k_{43}\,\sigma_{zz}$$
$$\gamma_{yz} = k_{53}\,\sigma_{zz}$$
$$\gamma_{zx} = k_{63}\,\sigma_{zz}$$

Hooke's Law – Anisotropic Materials

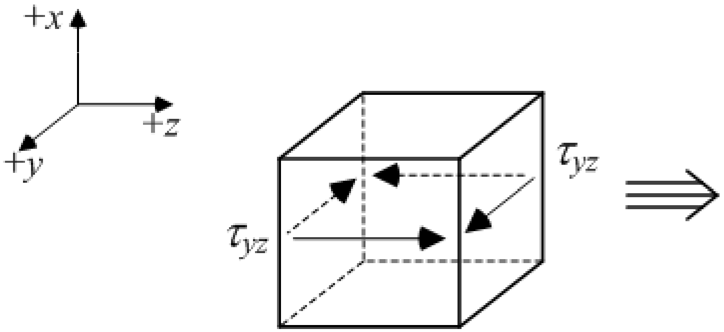
Strains caused by τ_{xy} only:



$$\varepsilon_{xx} = k_{14}\,\tau_{xy}$$
$$\varepsilon_{yy} = k_{24}\,\tau_{xy}$$
$$\varepsilon_{zz} = k_{34}\,\tau_{xy}$$
$$\gamma_{xy} = k_{44}\,\tau_{xy}$$
$$\gamma_{yz} = k_{54}\,\tau_{xy}$$
$$\gamma_{zx} = k_{64}\,\tau_{xy}$$

Hooke's Law – Anisotropic Materials

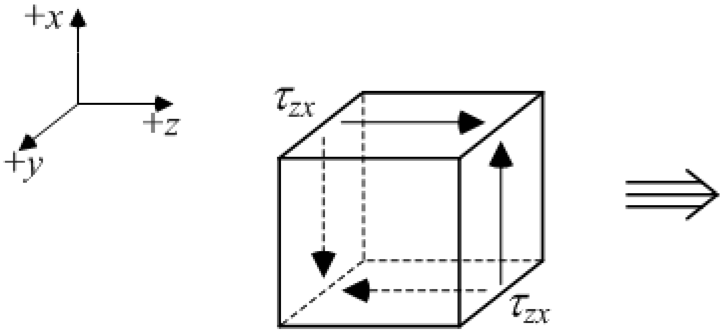
Strains caused by τ_{yz} only:



$$\varepsilon_{xx} = k_{15}\,\tau_{yz}$$
$$\varepsilon_{yy} = k_{25}\,\tau_{yz}$$
$$\varepsilon_{zz} = k_{35}\,\tau_{yz}$$
$$\gamma_{xy} = k_{45}\,\tau_{yz}$$
$$\gamma_{yz} = k_{55}\,\tau_{yz}$$
$$\gamma_{zx} = k_{65}\,\tau_{yz}$$

Hooke's Law – Anisotropic Materials

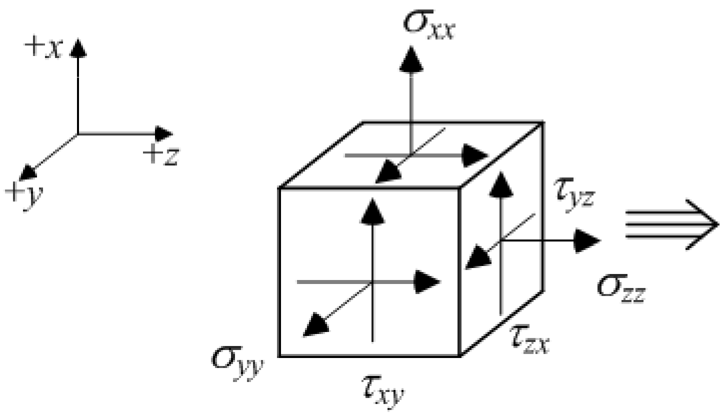
Strains caused by τ_{zx} *only:*



$$\varepsilon_{xx} = k_{16}\ \tau_{zx}$$
$$\varepsilon_{yy} = k_{26}\ \tau_{zx}$$
$$\varepsilon_{zz} = k_{36}\ \tau_{zx}$$
$$\gamma_{xy} = k_{46}\ \tau_{zx}$$
$$\gamma_{yz} = k_{56}\ \tau_{zx}$$
$$\gamma_{zx} = k_{66}\ \tau_{zx}$$

Hooke's Law – Anisotropic Materials

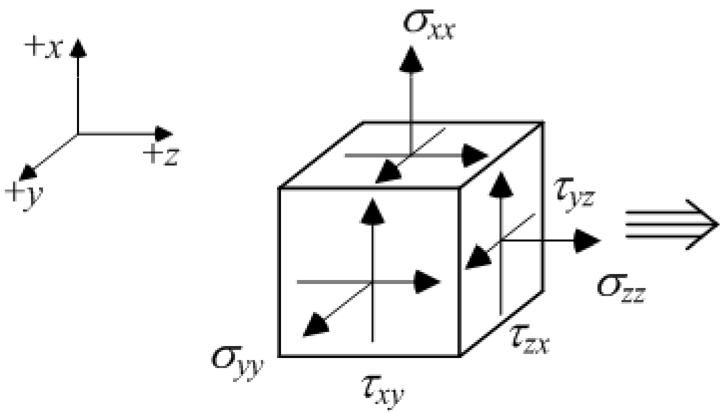
Strain ε_{xx} *caused by all stress components acting simultaneously:*



$\varepsilon_{xx} = ?$

Hooke's Law – Anisotropic Materials

Strain ε_{xx} caused by all stress components acting simultaneously:

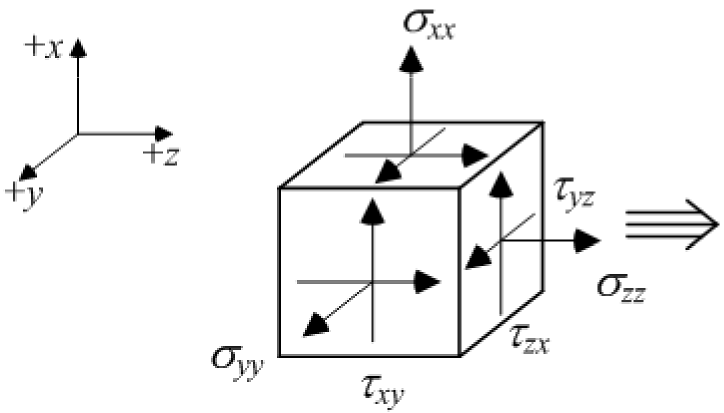


ε_{xx} = ?

(since strain-stress relation assumed linear, we can apply the principle of superposition and simply add up contribution of each stress component):

Hooke's Law – Anisotropic Materials

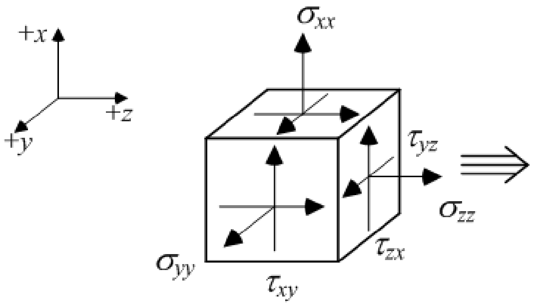
Strain ε_{xx} *caused by all stress components acting simultaneously:*



$$
\begin{aligned}
\varepsilon_{xx} = k_{11}\,\sigma_{xx} & \\
+ k_{12}\,\sigma_{yy} & \\
+ k_{13}\,\sigma_{zz} & \\
+ k_{14}\,\tau_{xy} & \\
+ k_{15}\,\tau_{yz} & \\
+ k_{16}\,\tau_{zx} &
\end{aligned}
$$

Hooke's Law – Anisotropic Materials

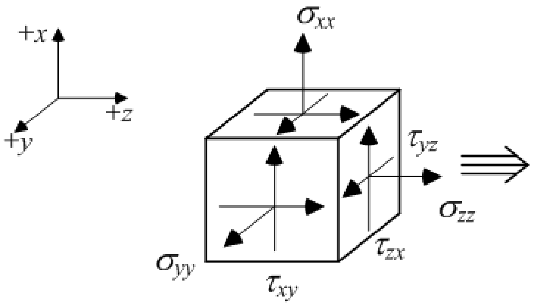
Repeating this process for each strain component results in six simultaneous equations



$$\varepsilon_{xx} = k_{11}\,\sigma_{xx} + k_{12}\,\sigma_{yy} + k_{13}\,\sigma_{zz} + k_{14}\,\tau_{xy} + k_{15}\,\tau_{yz} + k_{16}\,\tau_{zx}$$
$$\varepsilon_{yy} = k_{21}\,\sigma_{xx} + k_{22}\,\sigma_{yy} + k_{23}\,\sigma_{zz} + k_{24}\,\tau_{xy} + k_{25}\,\tau_{yz} + k_{26}\,\tau_{zx}$$
$$\varepsilon_{zz} = k_{31}\,\sigma_{xx} + k_{32}\,\sigma_{yy} + k_{33}\,\sigma_{zz} + k_{34}\,\tau_{xy} + k_{35}\,\tau_{yz} + k_{36}\,\tau_{zx}$$
$$\gamma_{xy} = k_{41}\,\sigma_{xx} + k_{42}\,\sigma_{yy} + k_{43}\,\sigma_{zz} + k_{44}\,\tau_{xy} + k_{45}\,\tau_{yz} + k_{46}\,\tau_{zx}$$
$$\gamma_{yz} = k_{51}\,\sigma_{xx} + k_{52}\,\sigma_{yy} + k_{53}\,\sigma_{zz} + k_{54}\,\tau_{xy} + k_{55}\,\tau_{yz} + k_{56}\,\tau_{zx}$$
$$\gamma_{zx} = k_{61}\,\sigma_{xx} + k_{62}\,\sigma_{yy} + k_{63}\,\sigma_{zz} + k_{64}\,\tau_{xy} + k_{65}\,\tau_{yz} + k_{66}\,\tau_{zx}$$

Hooke's Law – Anisotropic Materials

…the six eq's can be expressed using matrix notation



$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

Hooke's Law – Anisotropic Materials

Inverting the six equations:

$$\sigma_{xx} = K_{11}\,\varepsilon_{xx} + K_{12}\,\varepsilon_{yy} + K_{13}\,\varepsilon_{zz} + K_{14}\,\gamma_{xy} + K_{15}\,\gamma_{yz} + K_{16}\,\gamma_{zx}$$
$$\sigma_{yy} = K_{21}\,\varepsilon_{xx} + K_{22}\,\varepsilon_{yy} + K_{23}\,\varepsilon_{zz} + K_{24}\,\gamma_{xy} + K_{25}\,\gamma_{yz} + K_{26}\,\gamma_{zx}$$
$$\sigma_{zz} = K_{31}\,\varepsilon_{xx} + K_{32}\,\varepsilon_{yy} + K_{33}\,\varepsilon_{zz} + K_{34}\,\gamma_{xy} + K_{35}\,\gamma_{yz} + K_{36}\,\gamma_{zx}$$
$$\tau_{xy} = K_{41}\,\varepsilon_{xx} + K_{42}\,\varepsilon_{yy} + K_{43}\,\varepsilon_{zz} + K_{44}\,\gamma_{xy} + K_{45}\,\gamma_{yz} + K_{46}\,\gamma_{zx}$$
$$\tau_{yz} = K_{51}\,\varepsilon_{xx} + K_{52}\,\varepsilon_{yy} + K_{53}\,\varepsilon_{zz} + K_{54}\,\gamma_{xy} + K_{55}\,\gamma_{yz} + K_{56}\,\gamma_{zx}$$
$$\tau_{zx} = K_{61}\,\varepsilon_{xx} + K_{62}\,\varepsilon_{yy} + K_{63}\,\varepsilon_{zz} + K_{64}\,\gamma_{xy} + K_{65}\,\gamma_{yz} + K_{66}\,\gamma_{zx}$$

(2.11)

Hooke's Law – Anisotropic Materials

Using matrix notation:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

- K_{ij} = [K_{ij}] = "coefficients of elasticity"

Hooke's Law – Anisotropic Materials

Inverting the six equations:

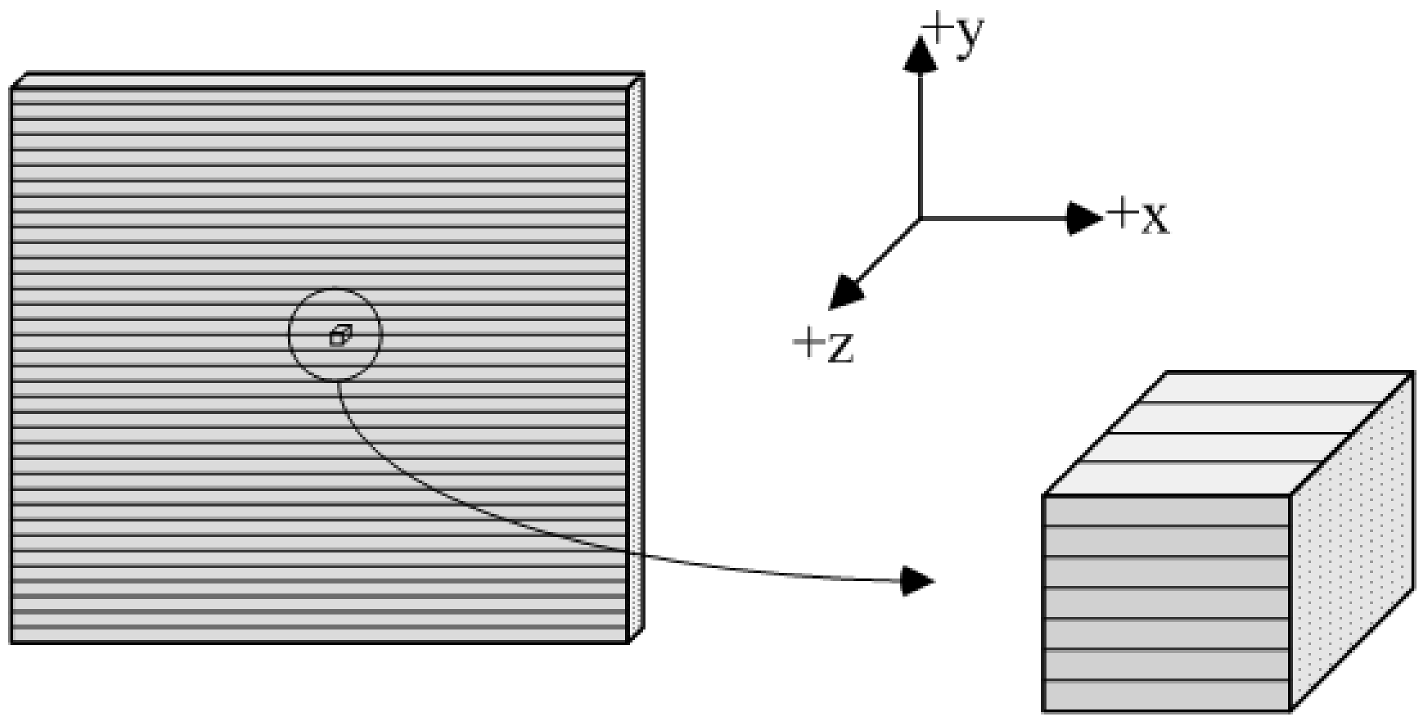
- Apparently, there are 36 coefficients of elasticity, however
- Strain energy considerations show that the $[K_{ij}]$ matrix *must be symmetric*….number of independent coefficients reduces from 36 to 21:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{12} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{13} & K_{23} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{14} & K_{24} & K_{34} & K_{44} & K_{45} & K_{46} \\ K_{15} & K_{25} & K_{35} & K_{45} & K_{55} & K_{56} \\ K_{16} & K_{26} & K_{36} & K_{46} & K_{56} & K_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

Hooke's Law – Anisotropic Materials

Anisotropy originates due to microstructure

• Unidirectional composites possess an inherent "principal material coordinate system", defined by the fiber orientation



Hooke's Law – Anisotropic Materials

Anisotropy originates due to microstructure

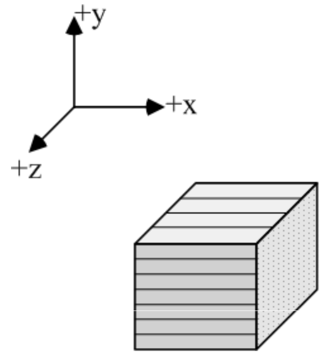
• If the stress and strain tensor are described relative to the principal material coordinate system, then there is no coupling between normal stress and shear strain, and no coupling between shear stress and normal strain:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 \\ K_{13} & K_{23} & K_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{44} & 0 & 0 \\ 0 & 0 & 0 & & K_{55} & 0 \\ & & & & & K_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

Hooke's Law – Anisotropic Materials

Anisotropy originates due to microstructure

- If fiber distribution *differs* in y- and z-directions, then:



$E_{xx} > E_{yy} \neq E_{zz}$

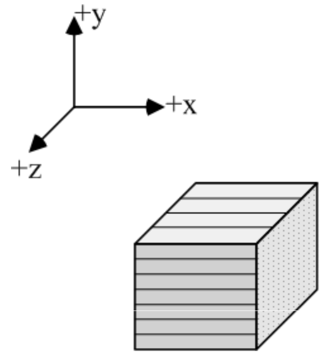
<u>*Orthotropic Material*</u>

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 \\ K_{13} & K_{23} & K_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{44} & 0 & 0 \\ 0 & 0 & 0 & & K_{55} & 0 \\ & & & & & K_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \quad (2.21)$$

Hooke's Law – Anisotropic Materials

Anisotropy originates due to microstructure

- If fiber distribution in y- and z-directions is identical, then:



$E_{xx} > E_{yy} = E_{zz}$

Transversely Isotropic Material

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{12} & 0 & 0 & 0 \\ K_{12} & K_{22} & K_{23} & 0 & 0 & 0 \\ K_{12} & K_{23} & K_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{K_{22}-K_{23}}{2} & 0 & 0 \\ 0 & 0 & 0 & & K_{66} & 0 \\ & & & & & K_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \qquad (2.32)$$

Hooke’s Law
Summary of Key Points

- Hooke’s Law is valid under linear-elastic conditions only
- The mathematical form of Hooke’s Law depends on the problem involved:
  - Isotropic vs anisotropic materials
  - 3-D stress/strains
  - Plane stress states
  - Plane strain states
  - Uniaxial stress