# Lecture note on wind turbine

## Prepared by Minoru Taya, Jan. 26, 2016

Main reference: "Wind energy explained:theory, design and application", by F. Manwell, J.G. McGowan and A.L. Rogers, Joh Wiley and Sons, LTD, 2002

Sub reference, "Electronic Composites", by M. Taya, Cambridge University Press, 2005.

**1. Fundamentals of wind energy harvesting**

*Linear momentum model*

$$\underline{P} = m\underline{v}$$

$m$: mass
$v$: velocity

For steady-state, conservation of mass

$$(\rho A u)_1 = (\rho A u)_4 = \dot{m} \tag{1}$$

Then,

$$T = U_1(\rho A u)_1 - U_4(\rho A u)_4 = \dot{m}(U_1 - U_4) \tag{2}$$

where, $T$: thrust

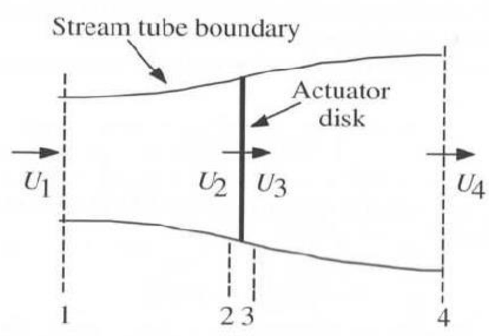


Fig.1: Actuator disk model of a wind turbine; U-mean air velocity, 1,2,3 and 4 indicate locations

Applying Bernoulli theory at control section 1, 2 (both upstream of turbine)

$$P_1 + \frac{1}{2}\rho {U_1}^2 = P_2 + \frac{1}{2}\rho {U_2}^2 \tag{3}$$

For downstream,

$$P_3 + \frac{1}{2}\rho {U_3}^2 = P_4 + \frac{1}{2}\rho {U_4}^2 \tag{4}$$

Assuming $P_1 = P_4$, $U_2 = U_3$

Thrust $T$ can be expressed as

$$T = A_2(P_2 - P_3) \tag{5}$$

Using the assumption of $P_1 = P_4$, $U_2 = U_3$

$$P_2 = P_1 + \frac{1}{2}\rho\, {U_1}^2 - \frac{1}{2}\rho {U_2}^2$$

$$P_3 = P_4 + \frac{1}{2}\rho\, {U_4}^2 - \frac{1}{2}\rho {U_3}^2$$

$$P_2 - P_3 = \frac{1}{2}\rho({U_1}^2 - {U_4}^2) \tag{6}$$

Substituting (6) into (5)

$$T = \frac{1}{2}\rho A_2({U_1}^2 - {U_4}^2) \tag{7}$$

Equating (7) to (2)

$$T = \dot{m}(U_1 - U_4) = \frac{1}{2}\rho A_2({U_1}^2 - {U_4}^2)$$

$$\rho A_2 U_2(U_1 - U_4) = \frac{1}{2}\rho A_2(U_1 - U_4)(U_1 + U_4)$$

$$\therefore U_2 = \frac{1}{2}(U_1 + U_4) \tag{8}$$

Define $a$ as the fractional decrease in wind velocity between free stream and rotor plane, then

$$a \equiv \frac{(U_1 - U_2)}{U_1} \tag{9}$$

$$U_2 = U_1(1 - a) \tag{10}$$

From (8) and (10)

$$\frac{1}{2}(U_1 + U_4) = U_1(1 - a)$$

$$U_4 = U_1(1 - 2a) \tag{11}$$

$aU_1$: induction velocity

Power of wind turbine, $P$ is equal to thrust ($T$) times velocity ($U_2$), from (7)

$$P = \frac{1}{2}\rho A_2\left({U_1}^2 - {U_4}^2\right)U_2 = \frac{1}{2}\rho A_2(U_1 - U_4)(U_1 + U_4)U_2 \tag{12}$$

Substituting (10), (11) into (12)

$$P = \frac{1}{2}\rho A U^3 4a(1-a)^2 \tag{13}$$

where $A_2 = A$, $U_1 = U$

Power coefficient, $C_p$ is given by

$$C_p = \frac{P}{\frac{1}{2}\rho U^3 A} = \frac{Rotor\ power}{Power\ in\ the\ wind} \tag{14}$$

From (13) and (14)

$$C_p = 4a(1-a)^2 \tag{15}$$

The maximum of $C_p$ is obtained by taking its derivative $\frac{dC_p}{da}$ and setting it to be zero.

$$\frac{dC_p}{da} = 4\{(1-a)^2 + 4a2(1-a)(-1)\} = 0$$

$$a_{max} = \frac{1}{3} \tag{16}$$

Substituting $a = \frac{1}{3}$ in (15)

$$C_{p,max} = \frac{16}{27} = 0.5926 \tag{17}$$

Substituting (10), (11) into (7)

$$T = \frac{1}{2}\rho A_2\left({U_1}^2 - {U_4}^2\right) = \frac{1}{2}\rho A_2\left\{{U_1}^2 - {U_1}^2(1-2a)^2\right\}$$
$$= \frac{1}{2}\rho A_2 {U_1}^2\{1 - (1-2a)^2\} = \frac{1}{2}\rho A {U_1}^2[4a(1-a)] \tag{18}$$

where $A_2 = A$

Thrust coefficient, $C_T$ is given by

$$C_T = \frac{T}{\frac{1}{2}\rho U^2 A} = \frac{Thrust\ force}{Dynamic\ force} \tag{19}$$

$$= 4a(1-a)$$

The maximum of $C_T$ is obtained from

$$\frac{dC_T}{da} = 0 \rightarrow\ a = \frac{1}{2} \tag{20}$$

$C_T$ becomes the maximum, $C_T = 1$ at $a = \frac{1}{2}$

At $a = \frac{1}{2}$,

$$U_4 = U_1(1 - 2a) = 0$$

At maximum power output ($a = \frac{1}{3}$)

$$C_T = \frac{8}{9} \tag{21}$$

It is noted from Fig. 2, the above model (Betz) is valid only for $a \leq \frac{1}{2}$

At $a = \frac{1}{2}$, $C_{p,max} = \frac{16}{27} = 0.5926$, is theoretically the maximum rotor power efficiency.

Practical turbine efficiency, $\eta_{overall}$ is given by

$$\eta_{overall} = \frac{P_{out}}{\frac{1}{2}\rho A U^3} = \eta_{mech} C_p$$

$$\therefore P_{out} = \frac{1}{2}\rho A U^3(\eta_{mech} C_p) \tag{22}$$

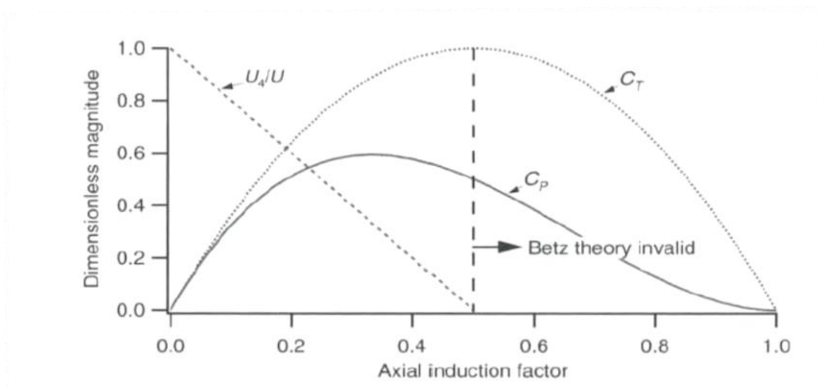


Fig. 2: Operating parameters for a Betz turbine: $U$-velocity of undisturbed air, $U_4$-air velocity behind th e rotor, $C_p$-pwer coefficient, and $C_T$-thrust coefficient

The previous model did not account for the rotating turbine blades which induces angular momentum in the wake region, see Fig. 3.

*Conservation of angular momentum*

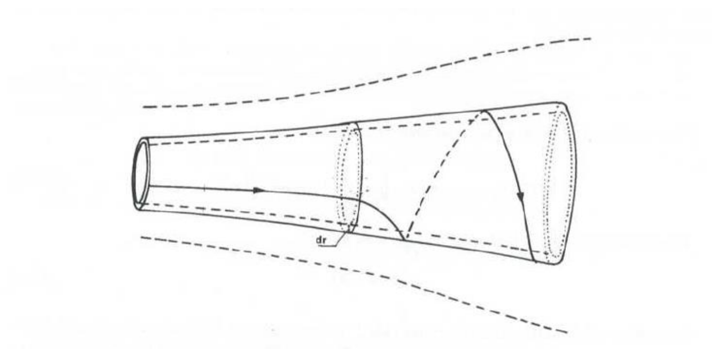


Fig. 3: Stream tube model of flow behind rotating wind turbine blade.

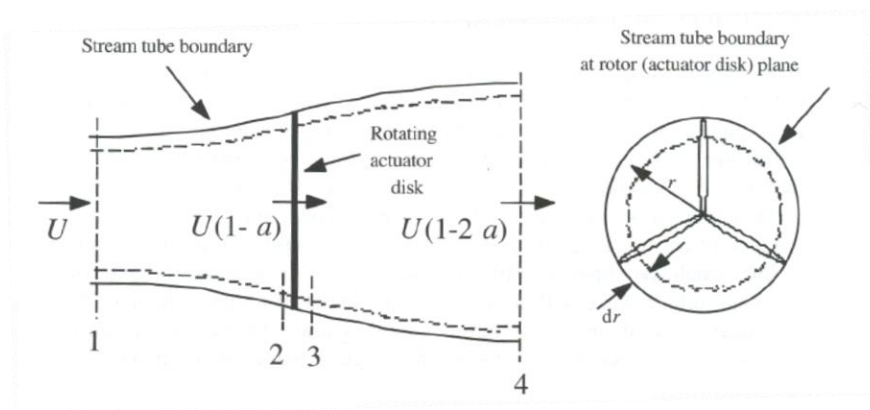


Fig. 4: Geometry for rotor analysis; $U$ – velocity of undisturbed air, $a$ - induction factor, $r$ – radius

$\omega$: angular velocity of the air flow in the wake

$\Omega$: angular velocity of turbine blade

Assume $\Omega \gg \omega$

Then, the pressure gap before and after the rotor location is estimated as

$$P_2 - P_3 = \rho\left(\Omega + \frac{1}{2}\omega\right)\omega r^2 \tag{23}$$

For the annual element ($dr$), thrust, $dT$ is given by

$$dT = (P_2 - P_3)dA = \rho\left(\Omega + \frac{1}{2}\omega\right)\omega r^2 \cdot 2\pi r\, dr \tag{24}$$

Define angular induction factor, $a'$ as

$$a' = \frac{\omega}{2\Omega} \tag{25}$$

Induced velocity at the rotor is sum of axial component $Ua$, and $r\Omega a'$.

By using (25), (24) is reduced to

$$dT = 4a'(1+a')\frac{1}{2}\rho\Omega^2 r^2 2\pi r dr \tag{26}$$

From (18) applied to the annual area element ($2\pi r dr$)

$$dT = 4a(1-a)\frac{1}{2}\rho U^2 2\pi r dr \tag{27}$$

Equating (26) to (27), we obtain

$$\frac{a(1-a)}{a'(1+a)} = \frac{\Omega^2 r^2}{U^2} \equiv \lambda_r^2 \tag{28}$$

where $\lambda r$ is local speed ratio. Tip speed ratio, $\lambda$ is given by

$$\lambda = \frac{\Omega R}{U} \tag{29a}$$

Intermediate speed ratio (at $r$), $\lambda_r$ is given by

$$\lambda_r = \frac{\Omega_r}{U} = \frac{\lambda r}{R} \tag{29b}$$

Let us consider the conservation of angular momentum focusing on the annual element. The torque ($Q$) expected on the rotor is equal to the change in the angular momentum;

$$dQ = d\dot{m}(\omega r)r = (\rho U_2 2\pi r dr)(\omega r)r \tag{30}$$

Since $U_2 = U(1-a)$, i.e., Eq (10) and $a' = \frac{\omega}{2\Omega}$, Eq (25),

$$dQ = 4a'(1-a)\frac{1}{2}\rho U\Omega r^2 2\pi r dr \tag{31}$$

The power generated in each element, $dP$ is

$$dP = \Omega dQ \tag{32}$$

Substituting (31) into (32) and using (29)

We arrive at

$$dP = \frac{1}{2}\rho A U^3\left[\frac{8a'(1-a)\lambda_r^3}{\lambda^2}d\lambda_r\right] \tag{33}$$

where $A = \pi R^2$

Incremental contribution to the power coefficient ($dC_p$) from annual ring is

$$dC_p = \frac{dP}{\frac{1}{2}\rho A U^3} \tag{34}$$

Thus, total power coefficient ($C_p$) is given by

$$C_p = \frac{1}{\lambda^2}\int_0^\lambda a'(1-a){\lambda_r}^3 d\lambda_r \tag{35}$$

From (28), we can express $a'$ in terms of $a$,

$$a' = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{{\lambda_r}^2} a(1-a)} \tag{36}$$

Integrand in (35) is

$$f(a) = (1-a)\left\{-\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{{\lambda_r}^2} a(1-a)}\right\} \tag{37}$$

The maximum of $f(a)$ gives the maximum of $C_p$,

$\frac{df(a)}{da} = 0$ gives us

$${\lambda_r}^2 = \frac{(1-a)(4a-1)^2}{(1-3a)} \tag{38}$$

Substituting (38) into (28) for maximum power in each annual ring

$$a' = \frac{1-3a}{4a-1} \tag{39}$$

By taking derivatives of (38), we obtain

$$2\lambda_r d\lambda_r = \frac{6(4a-1)(1-2a)^2}{(1-3a)^2} da \tag{40}$$

Substituting (38)~(40) into (35), the maximum power coefficient, $C_{p,max}$ is obtained as

$$C_{p,max} = \frac{24}{\lambda^2} \int_{a_1}^{a_2} \left[\frac{(1-a)(1-2a)(1-4a)}{(1-3a)}\right]^2 da \tag{41}$$

where $a_1$corresponds to axial induction factor for $\lambda_r = 0$, $a_2$ to the axial induction factor for $\lambda_r = \lambda$.

From (38),

$$\lambda^2 = \frac{(1-a_2)(1-4a_2)^2}{(1-3a_2)} \tag{42}$$

Note in (42), $a_1 = 0.25,\ \lambda_r = 0$

Note in (42), $a_2 = \frac{1}{3},\ \lambda \to \infty$, thus $a_2 = \frac{1}{3}$ is the upper limit of axial induction factor

Integral in (41) is performed by setting[1] $x = (1-3a)$, then, $C_{p,max}$ is expressed as

$$C_{p,max} = \frac{8}{729\lambda^2}\left[\frac{64}{5}x^5 + 72x^4 + 124x^3 + 38x^2 - 63x - 12\ln x - \frac{4}{x}\right]_{x=(1-3a_2)}^{x=0.25} \tag{43}$$

[1] Eggleston, D. M. and Stoddard, F. S., 1987, Wind Turbine Engineering Design, Van Nostrand Reinhold, New York.

The relation between $\lambda, a_2, C_{P,max}$ is given in Table 1.

| $\lambda$ | $a_2$ | $C_{P,max}$ |
|---|---|---|
| 0.5 | 0.2983 | 0.289 |
| 1.0 | 0.3170 | 0.416 |
| 1.5 | 0.3245 | 0.477 |
| 2.0 | 0.3279 | 0.511 |
| 2.5 | 0.3297 | 0.533 |
| 5.0 | 0.3324 | 0.570 |
| 7.5 | 0.3329 | 0.581 |
| 10 | 0.3330 | 0.585 |

Table 1

The numerical results of (43) are shown in Fig. 5, and induction factors: axial ($a$), rotational ($a'$) are plotted as a function of $\left(\frac{r}{R}\right)$ in Fig. 6.

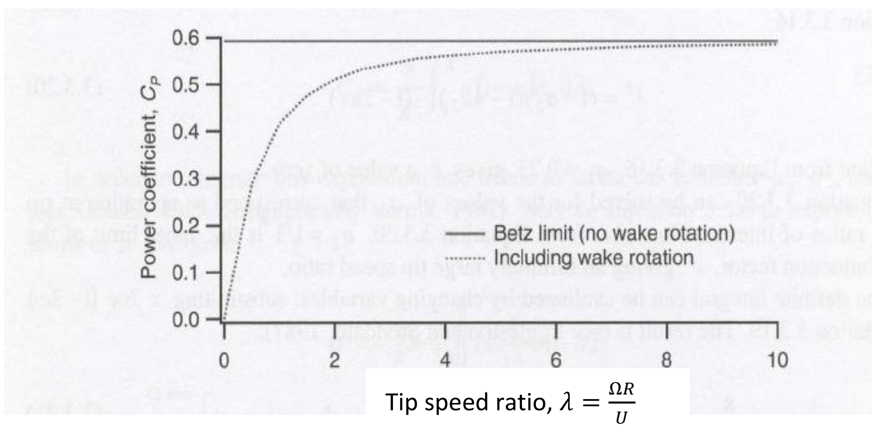


Fig. 5: Theoretical maximum power coefficient as a function of tip speed ratio for an ideal horizontal axis wind turbine, with and without wake rotation

$$\lambda = \frac{\Omega R}{U}$$

$U$: entering wind speed
$\Omega$: angular velocity of blade
$R$: radius of rotor

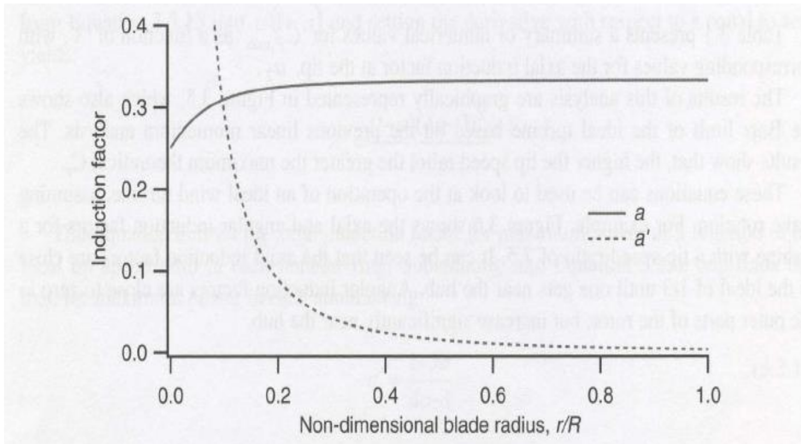


**Figure 3.6** Induction factors for an ideal wind turbine with wake rotation; tip speed ratio, $\lambda$ = 7.5; $a$, axial induction factor; $a'$, angular induction factor; $r$, radius; $R$ rotor radius

Fig. 6: Induction factors for an ideal wind turbine with wake rotation: tip speed ratio, $\lambda = 7.5$; $a$-axial induction factor, $a'$-angular induction factor, $r$-radius, and $R$-rotor radius

### *Effect of Drag and Blade number on optimum performance*

Wilson *et.al.* (1976)[2] obtain the formula for $C_{p,max}$

$$C_{p,max} = \left(\frac{16}{27}\right)\lambda\left\{\lambda + \frac{1.32 + \left(\frac{\lambda - 8}{20}\right)^2}{B^{\frac{2}{3}}}\right\}^{-\mathbf{1}} - \frac{0.57\lambda^2}{\frac{C_l}{C_d}\left(\lambda + \frac{1}{2B}\right)}$$

where, $\lambda$ is speed ratio $(\frac{\Omega R}{U})$ , B is number of blades, $C_l$ is lift coefficient of a blade, $C_d$ is the drag coefficient of blade. If no drag is considered, maximum achievable power coefficient is given in Fig. 7.

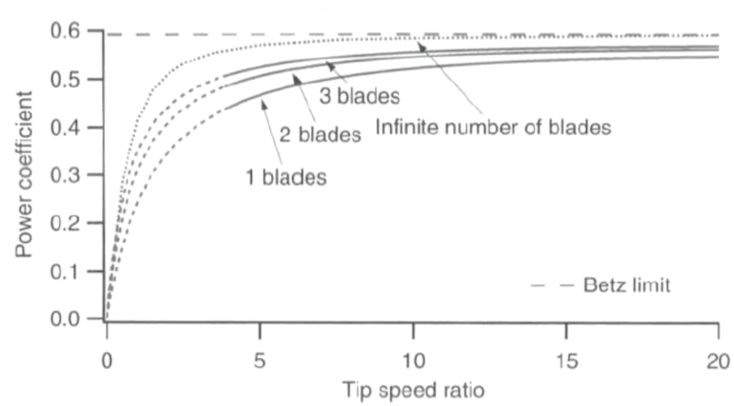


Fig. 7: Maximum achievable power coefficient as a function of number of blades, no drag

[2] Wilson, R. E., Lissaman, P. B. S. and Walker, S. N., 1976, "Aerodynamic performance of wind turbines," Energy research and development administration, ERDA/NSF/04014-7611.

Fig. 8 shows the maximum power achievable as a function of tip speed ratio for 3-blade wind turbine if the lift to drag ratio, $\frac{C_l}{C_d}$ is considered.

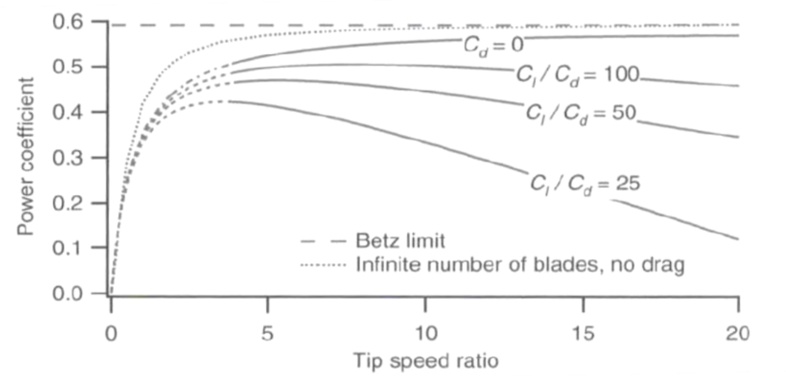


Fig. 8: Maximum achievable power coefficients of a 3-blade optimum rotor as a function of the lift to drag ratio, $\frac{C_l}{C_d}$

**2. Fundamental of electromagnetic motor**

Magnetic flux density B ($Wb/m^2$) is related to magnetic field H ($A/m$) by

$$\underline{B} = \mu \underline{H} \tag{1}$$

where $\mu$ is magnetic permeability, and it is conveniently expressed by

$$\mu = \mu_r \mu_0 \tag{2}$$

$\mu_0 = 4\pi \times 10^{-7}\ Wb/(A \cdot m)$, for free space (or in vacuum) $\mu_r$ is relative permeability, thus it is non-dimensional.

Non-magnetic materials, $\mu_r = 1$, and ferromagnetic materials, $\mu_r$ is very large, $10^3 \sim 10^5$. There are two kinds of ferromagnetic materials, soft and hard (or permanent) magnet, see Fig. 1.

For ferromagnetic materials, Eq. (3) is normally used.

$$\underline{B} = \mu_0 \left(\underline{H} + \underline{M}\right) \tag{3a}$$

or

$$= \mu_0 \underline{H} + \underline{M} \tag{3b}$$

(3a) is newer expression then the unit of $\underline{M}$is the same as $\underline{H}$ (i.e., $A/m$), and (3b) is the older expression, still used by materials scientists. Then the unit of $\underline{M}$ is $\frac{Wb}{m^2} = Tesla$. In Eq. (3), $\underline{M}$ is called as magnetization vector (magnetic dipole moment per unit volume), and it is related to $\underline{H}$ as

$$\underline{M} = \chi_m \underline{H} \tag{4}$$

where $\chi_m$ is magnetic susceptibility. From (1), (2), (3a) and (4),

$$\underline{B} = \mu_0\left(\underline{H} + \chi_m \underline{H}\right)$$
$$= \mu_0(1 + \chi_m)\underline{H} \qquad (5)$$

Thus, $\mu = \mu_0(1 + \chi_m)$

$$\mu_r = \frac{\mu}{\mu_0} = 1 + \chi_m \qquad (6)$$

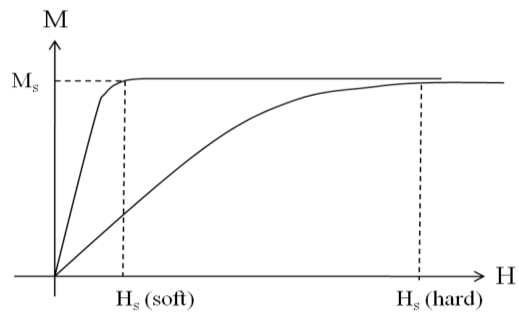


Fig. 1: M-H curve of ferromagnetic materials. Soft magnet has very small $H_c$, hard magnet has large $H_c$

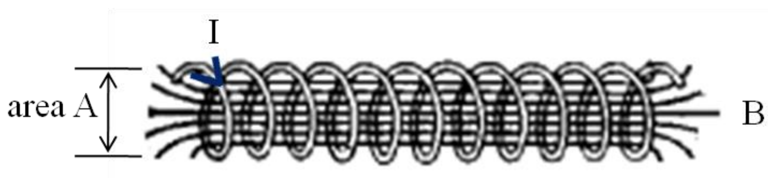


Fig. 2: Solenoid with number of turn, N

Let us consider simple solenoid which has N turns. If electric current (I) flows through the solenoid, magnetic flux density $\underline{B}$ is induced and its magnitude $B$ is given by

$$B = \mu I \frac{N}{L} \qquad (7)$$

where $L$ is inductance [Henry=$H$].

Magnetic flux ($\underline{\Phi}$) is defined by

$$\underline{\Phi} = \int \underline{B} \cdot d\underline{A} \qquad (8)$$

where $d\underline{A}$ is area element vector. The magnitude of $\underline{\Phi}$ is

$$\Phi = BA$$
$$= \frac{\mu INA}{L} \qquad (9)$$

**<u>Faraday's law</u>**

Electromagnetic force (or voltage) E is given by

$$E = -\frac{d\Phi}{dt} \tag{10}$$

A current flowing in conductor in the presence of magnetic field results in an induced force, acting on the conductor. This is the fundamental property of motors. A conductor which is forced to move through a magnetic field will have a current induced in the conductor. This is the fundamental property of generators. The force in a conductor of incremental length $dl$, the current ($I$), magnetic flux density $d\underline{B}$ are related by

$$d\underline{F} = \underline{I}dl \times d\underline{B} \tag{11}$$

It is noted in (11) '×' is vector (or cross) product between two vectors. If vectors $\underline{I}dl$ and $d\underline{B}$ are perpendicular, the force vector which is perpendicular to both $\underline{I}dl$ and $d\underline{B}$, becomes the maximum.

**Ampere's law**

Current flowing in a conductor induces a magnetic field $\underline{H}$ in the vicinity of the conductor, which is called Ampere's law.

$$\oint \underline{H} \cdot d\underline{l} = I \tag{12}$$

If conductor is solenoid with N turns,

$$V_m = \oint H_s ds = NI \tag{13}$$

$H_s$ is the magnetic field along the magnetic circuit, and $V_m$ is the electromagnetic force. (13) is reduced to

$$V_m = \oint H_s ds = \oint \frac{B}{\mu} ds = \oint \frac{\Phi}{\mu S} ds = \Phi \oint \frac{ds}{\mu S} = \Phi R_m \tag{14}$$

where $R_m$ is magnetic resistance,

$$R_m = \oint \frac{ds}{\mu S} \tag{15}$$

$S$ is the cross sectional area through which Φ passes.

Let us consider electromagnetic (Fig. 3), from (13) and (14),

$$V_m = NI = R_m \Phi$$

$$\therefore \Phi = \frac{NI}{R_m} \tag{16}$$

For Fig. 3 circuit,

$$R_m = \frac{l_y}{\mu_r \mu_0 S_y} + \frac{2g}{\mu_0 S_g} \tag{17}$$

where $l_y$ and $S_y$ are the length of magnetic circuit in the yoke, cross sectional area of the yoke, $g$ is the gap distance, $S_g$ is the cross sectional area of the gap.

For a rotor rotating with angle $\theta$, Fig. 4,

$$R_m = \frac{l_y}{\mu_r \mu_0 S_y} + \frac{2g}{\mu_0 S_g \left(\frac{r\theta}{L}\right)} \quad (18)$$

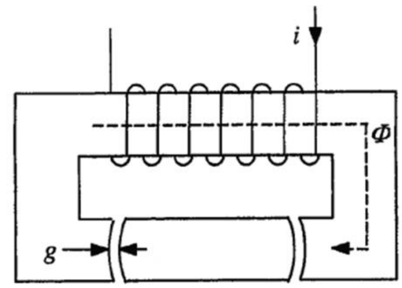


Fig. 3: Simple magnetic device; $g$-width of air gap, $i$-current and Φ-magnetic flux

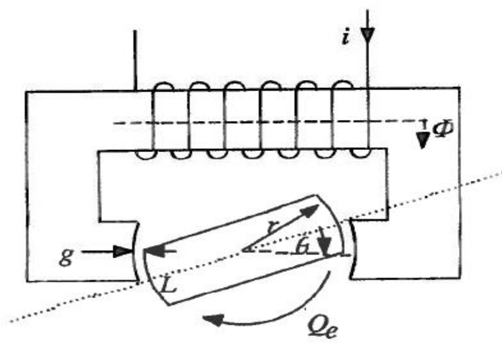


Fig. 4: Simple magnetic torque device; $g$-width of air gap, $i$-current, Φ-magnetic flux, $L$-length of the face of the poles, $Q_e$-electrical torque, $r$-radius and $\theta$-rotation angle.