Algebraic Combinatorics Lecture Notes

Lecturer: Sara Billey, et al; written and edited by Josh Swanson

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Abstract

The following notes were taking during a course on Algebraic Combinatorics at the University of Washington in Spring 2014. Please send any corrections to `jps314@uw.edu`. Thanks!

Contents

March 31st, 2014: Diagram Algebras and Hopf Algebras Intro

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1 Remark

Students will present something this quarter; see web site for topics. Will roughly focus on “diagram algebras” and Hopf algebras bumping in to representation theory and topology. Homework for the course will essentially be studying and presenting papers, possibly in small groups. Sara will give background for the first few weeks.

2 Remark

The rough idea is that objects from enumerative combinatorics index bases for algebras, and conversely important algebraic bases are indexed by combinatorial objects.

Definition 3. A $\boxed{\text{diagram algebra}}$ (not necessarily standard terminology) is as follows.

(i) The quitessential example is the group algebra $\mathbb{C}[S_n]$, with basis given by permutations. Multiplication can be thought of by concatenating string diagrams. The generators are the adjacent transpositions $(i, i+1)$, whose string diagrams have a single crossing. Their relations are $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, with $s_i s_j = s_j s_i$ if $|i - j| > 1$, and $s_i^2 = 1$.

(ii) For another example, consider the $\boxed{\text{Hecke algebra}}$0-Hecke algebra. The basis again will be indexed by permutations. Use generators $T_1, T_2, \dots, T_{n-1}$ with relations as with the s_i except for $T_i^2 = T_i$. A real-life example of this algebra comes from sorting a list by putting adjacent pairs in order. Diagram multiplication can be done similarly, but a crossing concatenated with itself is somehow just itself.

(iii) A generalization of the previous examples, roughly: the $\boxed{\text{Hecke algebra}}$ is given by $T_i^2 = qT_{\text{id}} + (1-q)T_i$. Setting $q = 0$ gives the 0-Hecke algebra; $q = 1$ gives the symmetric group.

(iv) The $\boxed{\text{Temperly-Lieb algebra}}$ $\text{TL}_n(\mathbb{C})$. The basis is given by non-crossing matchings on $2n$ vertices arranged in two columns. (That means we connect each vertex to precisely one vertex by a string.) Multiplication is given by concatenation. One fiddle is you can get "islands" when concatenating sometimes; in that case, formally multiply the diagram by δ for each island and erase the islands. What are the generators? U_i is the diagram going straight across except we connect i and $i+1$; $i+n$ and $i+n+1$. One checks

$$U_iU_{i+1}U_i = U_i, \qquad U_iU_{i-1}U_i = U_i, \qquad U_i^2 = \delta U_i, \qquad U_iU_j = U_jU_i$$

for $|i-j| > 1$. This comes up in topological quantum field theory.

How large is the basis? Match 1 with i and separate the vertexes into two "halves"; this gives the Catalan number recurrence $C_{n+1} = \sum_{i=0}^{n} C_iC_{n-i}$.

(v) The $\boxed{\text{Brauer algebra}}$ B_n: same as the Temperly-Lieb algebra, but allow non-crossing matchings. Ignore loops, i.e. set $\delta = 1$. Multiplication remains by concatenation. This algebra comes up in the representation theory of O_n. (There is a q-deformation where we don't require $\delta = 1$.)

Definition 4. Vauge first definition: a $\boxed{\text{Hopf algebra}}$ is an algebra, so it has addition and multiplication, and it has a coalgebra structure, so it has a coproduct and a counit, and it has an antipode. Some motivation: given an algebra A, multiplication is a map $A \otimes A \to A$. The coproduct is $\Delta\colon A \to A \otimes A$ going the other way. A good pair of motivating examples is the following.

(i) Hopf algebra on words in some alphabet, say $\{a, b, c\}$. (Maybe infinite, maybe not.) The basis is given by words in the alphabet (with an empty word), multiplication given by concatenation, eg. "base $\times$ ball = baseball", so $m(y_1 \cdots y_k \otimes z_i \cdots z_j) = y_1 \cdots y_k z_1 \cdots z_j$. A good comultiplication is $\Delta(y_1 \cdots y_k) = \sum_{j=0}^{k} y_1 \cdots y_j \otimes y_{j+1} \cdots y_k$. Doing this to "baseball" gives $1 \otimes \text{baseball} + \text{b} \otimes \text{aseball} + \cdots + \text{baseball} \otimes 1$. (1 is the empty word.) Note: This comultiplication was incorrect; see the remark at the start of the next lecture.

In this case the counit is given by setting $\epsilon(y_1 \cdots y_k)$ to 1 on the empty word and 0 elsewhere (i.e. it sets $y_1 = \cdots = y_k = 0$). In our example, this gives

$$(1 \otimes \epsilon)\Delta(\text{baseball}) = \text{baseball} \otimes 1.$$

An antipode will be $s(y_1 \cdots y_k) = (-1)^k y_1 \cdots y_k$, which is an involution $A \to A$.

(ii) The symmetric functions SYM. This is an algebra sitting inside $\mathbb{C}[[x_1, x_2, \dots]]$. Take all permutations and take their direct limit S_∞ (gives permutations $\mathbb{P} \to \mathbb{P}$ fixing all but finitely many naturals). Define an S_∞ action by

$$s_i \cdot f(\dots, x_i, x_{i+1}, \dots) := f(\dots, x_{i+1}, x_i, \dots).$$

(Variables in ... fixed.) Let SYM be the set of power series with bounded degree in these variables which are fixed by all s_i.

An easy source of elements of SYM are the $\boxed{\text{elementary symmetric functions}}$

$$e_k := \sum x_{j_1} \cdots x_{j_k},$$

where the sum is over k-subsets of the positive integers $\mathbb{P}$ with j_i strictly increasing.

5 Theorem
SYM is $\mathbb{C}[e_1, e_2, \ldots]$.

(Similarly we define the $\boxed{\text{complete homogeneous symmetric functions}}$

$$\boxed{h_k} := \sum x_{j_1} \cdots x_{j_k},$$

where the sum is over k-multisubsets of the positive integers $\mathbb{P}$ with *weakly* increasing j_i.)

Generalize the elementary symmetric functions to integer partition indexes as follows. Given a partition $\lambda = (\lambda_1, \cdots, \lambda_k)$, set

$$e_\lambda := e_{\lambda_1} \cdots e_{\lambda_k}.$$

SYM is a Hopf algebra.

- Multiplication: formal power series multiplication.
- Coproduct: $\Delta(e_k) := \sum_{i+j=k} e_i \otimes e_j$ (with $\Delta(e_0) = 1$). Alternatively, $\Delta(h_k) = \sum h_i \otimes h_j$. There are other bases, which give $\Delta(P_k) = 1 \otimes P_k + P_k \otimes 1$, or $\Delta(s_\lambda) = \sum_{\mu \leq \lambda} s_\mu \otimes s_{\lambda/\mu}$; maybe more on these later.
- Antipode: $S(e_k) = (-1)^k h_k$, $S(h_k) = (-1)^k e_k$.

6 Remark
We have a "grand vision" associating diagram algebras, Hopf algebras, and topology:

Diagram Algebra	Hopf Algebra	Topology
$\mathbb{C}[S_n]$	SYM	cohomology of $\mathrm{Gr}(k, n)$ "in the limit as $n \to \infty$, $k \to \infty$"
0-Hecke algebra	QSYM / NCSYM	unknown
Temperly-Lieb algebra	unknown	unknown
Brauer algebra	unknown	unknown
unknown	combinatorial Hopf algebras; graphs, posets, polytopes	unknown

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April 2nd, 2014: Group Representations Summary; S_n-Representations Intro

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7 Remark
Correction: the coproduct for the Hopf algebra on words example was missing terms. Say we have some alphabet $\{x, y, \ldots\}$; we require $\Delta(x) = 1 \otimes x + x \otimes 1$ (called "primitive"), extended to be a homomorphism. For instance,

$$\begin{aligned}\Delta(xy) &= [(1 \otimes x) + (x \otimes 1)][(1 \otimes y) + (y \otimes 1)] \\ &= 1 \otimes (xy) + y \otimes x + x \otimes y + (xy) \otimes 1,\end{aligned}$$

which gives a term $y \otimes x$ not seen in the previous formula. When we were doing the coproduct of "baseball" last time, we get more terms than we wrote; in general there would be 8 terms for a word of length 3.

8 Remark
Outline of the course:

1. S_n-representation theory
2. SYM
3. Hopf algebra
4. QSYM
5. James Zhang's view
6. Student lectures
7. Monty McGovern lectures
8. Edward Witten lecture on Jones polynomials, connection with topological quantum field theory.

9 Remark

Today: S_n representation theory in a nutshell. There's a very nice book by Bruce Sagan, "The Symmetric Group". (Unfortunately, there doesn't seem to be a free version. SpringerLink doesn't give us access either.)

10 Theorem

Let V be a finite dimensional vector space over $\mathbb{C}$, let G be a finite group, with $\mathbb{C}[G]$ the group algebra of G, which is a vector space itself. V is a G-module if $g(v) \in V$ is linear in v and if it respects composition, which is the same as saying there's a group homomorphism $\chi\colon G \to \mathrm{GL}(V)$. Equivalently, V is a $\mathbb{C}[G]$-module: given χ, let $(\sum c_i g_i) \cdot v = \sum c_i(\chi(g_i))(v)$; given V as a $\mathbb{C}[G]$-module, let $\chi(g_i)(v) = g_i \cdot v$.

Here's a summary of basic non-modular representation theory of finite groups.

(A) Every G-module V can be decomposed into irreducible G-modules. G has finitely many distinct irreducible representations (up to isomorphism), say $V^{(1)}, V^{(2)}, \ldots, V^{(n)}$, and

$$V \cong c_1 V^{(1)} \oplus \cdots \oplus c_n V^{(n)}.$$

Hence $\chi(g)$ is block diagonal, with c_1 blocks of size $\dim V^{(1)}$, etc. Note that $c_i \in \mathbb{N}$.

(B) The trivial representation $\chi\colon G \to (1)$ is always among the irreducibles.

$\boxed{\text{Characters}}$ of G-modules: $\chi_V\colon G \to \mathbb{C}$ is defined by $\chi_V(g) := \mathrm{tr}\chi(g)$. Note trace is invariant under conjugation, so the characters are class functions, i.e. constant on conjugacy classes.

The dimension of the space of (linear) class functions is the number of conjugacy classes. (Obvious basis.)

(C) There is a G-invariant inner product on the characters

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}).$$

Great property: if $\chi^{(i)}, \chi^{(j)}$ are characters of irreducible representations $V^{(i)}, V^{(j)}$, then $\langle \chi^{(i)}, \chi^{(j)} \rangle = \delta_{ij}$.

(D) The characters corresponding to the irreducibles $\{\chi^{(1)}, \cdots, \chi^{(n)}\}$ are a basis for the class functions. In particular, the number of distinct irreducibles for the group is always equal to the number of conjugacy classes, which is a nice ennumerative fact.

(E) $V = c_1 V^{(1)} \oplus \cdots \oplus c_n V^{(n)}$ iff $\chi_V = c_1\chi^{(1)} + \cdots + c_n\chi^{(n)}$.

(F) If we can decompose $\mathbb{C}[G]$, then we get all irreducible representations period.

11 Example

$\mathbb{C}[G]$ is a G-module (the $\boxed{\text{regular representation}}$). How does it decompose into irreducibles? What's the multiplicity of the irreducibles, in terms of the characters? Say this representation has character χ, and say the irreducibles have characters $\chi^{(i)}$. From the inner product formula,

$$\langle \chi, \chi^{(i)} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\chi^{(i)}(g^{-1})$$

Note that $\chi(g)$ is a permutation matrix (in the obvious basis), and that $gh = h$ iff $g = \mathrm{id}$, so its matrix is either the identity or a derangement. In particular, $\chi(g) = |G|$ if g is the identity, and is 0 otherwise. The inner product then ends up being

$$\langle \chi, \chi^{(i)} \rangle = \chi^{(i)}(\mathrm{id}) = \dim V^{(i)}.$$

It follows that the multiplicity of $V^{(i)}$ in the decomposition of $\mathbb{C}[G]$ is $\dim V^{(i)}$.

Definition 12. Given a subgroup H of G, a representation $\chi\colon G \to \mathrm{GL}(V)$, we trivially get a representation $\boxed{\chi\downarrow^G_H\colon H \to \mathrm{GL}(V)}$ by restriction. Indeed, we can also lift representations in the following way. Suppose we have $Y\colon H \to \mathrm{GL}(V)$; we construct the induced representation $\boxed{Y\uparrow^G_H\colon G \to \mathrm{GL}(V)}$. Consider cosets of G/H. Say $t_1, \ldots, t_\ell$ is a complete set of representatives, i.e.

$$G = t_1H \cup t_2H \cup \cdots \cup t_\ell H,$$

where these cosets are pairwise disjoint. Hence G acts on the vector space spanned by cosets t_iH, with $g(t_iH) = t_jH$ if $gt_i \in t_jH$. Thus g permutes cosets. Now define

$$Y\uparrow^G_H(g) := [Y(t_i^{-1}gt_j)],$$

where we set a block to 0 if $t_i^{-1}gt_j \notin H$. This is

$$\begin{bmatrix} Y(t_1^{-1}gt_1) & Y(t_1^{-1}gt_2) & \cdots & Y(t_1^{-1}gt_\ell) \\ Y(t_2^{-1}gt_1) & \cdots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

There is exactly one nonzero block in each row or column, and each such block is invertible, hence the whole thing is indeed invertible.

Note: $X\downarrow^G_H$ or $Y\uparrow^G_H$ to irred. rep. does not mean that X, Y are necessarily irreducible. (? Probably means if we start with an irreducible representation, we don't necessarily end up with one.)

13 Example

Fix $G = S_n$. We know the number of conjugacy classes of S_n is $p(n)$, the number of partitions of the number n. (Cycle types determine the conjugacy classes.) Let $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ denote the permutations which permute $[1, \ldots, \lambda_1]$, $[\lambda_1 + 1, \cdots, \lambda_1 + \lambda_2]$, etc. in blocks. What is $1\uparrow^{S_n}_{S_\lambda}$?

To apply the previous construction, we need to pick representatives of cosets S_n/S_λ. Note that for a particular coset, every element's first 1 through λ_1 entries are the same, and similarly with the other blocks. So, a nice representative is to pick a permutation whose "constant blocks" are ordered to be strictly increasing. (They may have descents between blocks.)

We can write permutations as $\boxed{\text{tabloids}}$ which are rows of numbers with each row corresponding to a block. For instance, with $\lambda = (4, 2, 2)$ for $n = 8$, a tabloid might be

$$\begin{array}{l} 1234 \\ 56 \\ 78 \end{array}$$

We can rearrange the rows however we want without changing cosets/tabloids.

Definition 14. We can make a vector space $\boxed{M^\lambda}$ spanned by the tabloids of shape λ. S_n acts on M^λ by just permuting the numbers in the rows,

$$\begin{aligned} \sigma(T) = {} & \sigma(t_{11})\sigma(t_{12})\cdots \\ & \sigma(t_{21})\sigma(t_{22})\cdots \\ & \cdots \end{aligned}$$

You can check that M^λ is the S_n-module that goes with $1\uparrow_{S_\lambda}^{S_n}$. (In general, for a subgroup H of G, 1_H^G is the representation for the action of G on the left cosets of H, as we can see that the i, j entry will be 1 if $t_i^{-1} g t_j \in H$ and 0 otherwise. From the construction above, we can then see that $M^\lambda = 1\uparrow_{S_\lambda}^{S_n}$).

15 Proposition

Let χ be the character of M^λ. For every $\sigma \in S_n$, $\chi(\sigma)$ is the number of tabloids of shape λ fixed by σ (immediate from definitions).

For a tabloid to be fixed by σ, we require each row to be a union of cycles in the cycle decomposition of σ. Let σ have cycle type $\mu = 1^{m_1} 2^{m_2} \cdots n^{m_n}$. Suppose a tabloid fixed by σ has $r(p, q)$ cycles of length q in row p—how many other fixed tabloids have the same $r(p, q)$'s? For fixed q, we can permute the cycles of length q however we wish, giving $m_q!$ choices, except permuting them in the same row does nothing, so this overcounts by $r(1, q)!$ for the first row, etc. These choices are independent for each q, and it follows that

$$\chi(\sigma) = \sum \prod_{q=1}^{n} \frac{m_q!}{r(1, q)! r(2, q)! \cdots r(n, q)!}$$

where the sum is over solutions to $r(p, 1) + 2r(p, 2) + \cdots + nr(p, n) = \lambda_p$, $r(1, q) + r(2, q) + \cdots + r(n, q) = m_q$. (The first constraint says the number of entries in row p is λ_p; the second says the number of cycles of length q overall is m_q.)

April 4th, 2014: M^λ Decomposition and Specht Modules

Summary Last time: did background on finite group representation theory. $\mathbb{C}[S_n] = m_1 V^{(1)} \oplus \cdots \oplus m_k V^{(k)}$. Now

$$n! = \sum_{i=1}^{k} (\dim V^{(i)})^2.$$

We defined M^λ as the $\mathbb{C}$ vector space spanned by λ-tabloids, which is $1_{S_{\lambda_1} \times \cdots \times S_{\lambda_k}}^{S_n}$.

Today: we'll decompose M^λ and talk about Sprecht modules.

16 Example

$n = 3$; $3! = 6 = a^2 + b^2 + c^2$; have three representations, $a = 1$ gives trivial, $c = 1$ is sign representation. $M^{(1,1,1)} = \mathbb{C}[S_3]$, and more generally λ given by singleton rows gives $M^\lambda = \mathbb{C}[S_n]$. $M^{(3)}$ is one dimensional, indeed the trivial representation. $M^{(2,1)}$ has three tabloids 12/3, 13/2, 23/1—the bottom row is all that matters. Calling these b_3, b_2, b_1, then $\sigma(\sum c_i b_i) = \sum c_i b_{\sigma(i)}$. The M_λ's are not in general irreducible. How do we decompose this one?

We have an obvious invariant subspace $V = \mathrm{Span}\{b_1 + b_2 + b_3\}$. What's the orthogonal complement? $W = \{\sum c_i b_i : \sum c_i = 0\}$. Hence $M^{(2,1)} = V \oplus W$.

Take basis for W given by $b_3 - b_1, b_2 - b_1$. Note that $[2, 1, 3](b_3 - b_1) = (b_3 - b_1) - (b_2 - b_1)$, and similarly we can compute the diagonal elements of the matrix representations of elements of S_3 in this basis. This gives the character: $\chi^W([1, 2, 3]) = 2$, on the transpositions it's 0, on the three-cycles it's -1.

We know the characters of the irreducibles form an orthonormal basis:

$$\langle \chi^w, \chi^w \rangle = \frac{1}{|G|} \sum_{\sigma \in S_n} \chi^w(\sigma) \chi^w(\sigma^{-1})$$

which in this case simplifies to just taking the dot product (since inverses stay in the same conjugacy class). Doing this computation here gives

$$\frac{1}{6}(2,-1,-1,0,0,0)\cdot(2,-1,-1,0,0,0) = (4+1+1)/6 = 1.$$

This shows W is actually irreducible! Hence $M^{(1,1,1)} = V^{(1)} \oplus V^{(2)} \oplus 2V^{(3)}$ (letting W be the third summand). Indeed, $M^{(n-1,1)} = V^{(1)} \oplus V^{(2)}$ in general.

17 Remark

Let χ^λ be the character of M^λ and let $\sigma \in S_n$ be of cycle type $\mu = 1^{m_1}2^{m_2}\cdots n^{m_n}$. Recall $\chi^\lambda(\sigma)$ is the number of tabloids of shape λ fixed by σ. See the proposition at the end of the previous lecture for the computation, notation, and formula.

What's a generating function for these numbers? Note the innermost fraction is a multinomial coefficients, suggesting

$$(x_1^q + \cdots + x_n^q)^{m_q}$$

which has the correct multinomial coefficient on $x_1^{qr(1,q)}x_2^{qr(2,q)}\cdots x_n^{qr(n,q)}$.

Hence we have

18 Proposition

The generating function for $\chi^\lambda(\sigma)$ where σ has cycle type μ is given by

$$\boxed{\xi_\mu^\lambda} := \prod_{q=1}^n (x_1^q + \cdots + x_n^q)^{m_q}|_{x^\lambda}.$$

Definition 19. Let $\boxed{p_i} := (x_1^i + x_2^i + \cdots + x_n^i)$, $\boxed{p_\mu} := p_{\mu_1}\cdots p_{\mu_k}$ for a partition μ. The first is a $\boxed{\text{power sum}}$, the second is a $\boxed{\text{power symmetric function}}$. Hence

$$\xi_\mu^\lambda = p_\mu|_{x^\lambda}.$$

Definition 20. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k > 0)$. Define the $\boxed{\text{monomial symmetric functions}}$

$$\boxed{m_\lambda} := \sum_{(i_1,i_2,\ldots,i_k)} x_{i_1}^{\lambda_1}\cdots x_{i_k}^{\lambda_k},$$

where the sum is over sequences of distinct values in $[1,\ldots,n]$. Informally, this is x^λ symmetrized.

21 Theorem

$p_\mu(x) = \sum_\lambda \xi_\mu^\lambda m_\lambda$.

Proof This is immediate from the fact that ξ_μ^λ is the coefficient of x^λ and that $p_\mu(x)$ is symmetric.

Definition 22. Preliminaries for Specht modules: they'll be irreducible submodules of M^λ, λ a partition of n; first we need the following.

Start with a bijective filling T of a Ferrers diagram of λ (Sagan calls this a Young tableau); eg. $T = 154/23$ is a bijective filling of the shape $(2,3)$. Call these $\boxed{\text{lambda-tableaux}}$ λ-tableaux. As a reality check, the number of λ-tableaux is $n!$. Define

- $\boxed{\{T\}}$ is the tabloid given by T. Recall this is the equivalence class of λ-tableaux containing T, where two tableau are equivalent if they can be obtained from one another by permuting the *rows*.
- $\boxed{R(T)}$ is the row stabilizer of T, meaning the rows as sets are preserved

- $\boxed{C(T)}$ is the column stabilizer of T, meaning the columns as sets are preserved
- $\boxed{a_T} := \sum_{\pi \in R(T)} \pi$
- $\boxed{b_T} := \sum_{\pi \in C(T)} \operatorname{sgn}(\pi)\pi$

For example, $C(154/23) = S_{\{12\}} \times S_{\{35\}} \times S_{\{4\}}$. Note $\mathbb{C}[S_n] \cdot \{T\} = M^\lambda$ roughly by definition, if T is a λ-tableau. b_T will be a very important element in our construction of the Specht modules.

Definition 23. Given a Young tableau T of shape λ, define

$$\boxed{e_T} := b_T \cdot \{T\} = \sum_{\pi \in C(T)} \operatorname{sgn}(\pi)\pi\{T\}.$$

Check:

$$\sigma \cdot e_T = e_{\sigma(T)}.$$

Proof Set $w_\sigma = \sigma\pi\sigma^{-1}$ and observe

$$\begin{aligned} \sigma \cdot e_T &= \sum_{\pi \in C(T)} \operatorname{sgn}(\pi) \cdot \sigma\pi\{T\} \\ &= \sum_{w_\sigma \in \sigma^{-1}C(T)\sigma} \operatorname{sgn}(\pi) w_\sigma\{\sigma T\} \end{aligned}$$

Note $\sigma C(T)\sigma^{-1} = C(\sigma T)$. Hence the right-hand side is precisely $e_{\sigma(T)}$.

Definition 24. Define the $\boxed{\text{Specht module}}$

$$S^\lambda := \operatorname{Span}\{e_T : T \text{ is a } \lambda\text{-tableau}\} = \mathbb{C}[S_n] \cdot e_T.$$

(The second equality follows from the observation in the previous definition.)

25 Example
Let $\lambda = (1^n)$, $T = 1/2/3/\cdots/n$. Hence $C(T) = S_n$. Now

$$\begin{aligned} \sigma \cdot e_T = e_{\sigma(T)} &= \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\pi\sigma\{T\} \\ &= \sum_w \operatorname{sgn}(w)\operatorname{sgn}(\sigma) w\{T\} \\ &= \operatorname{sgn}(\sigma) e_T. \end{aligned}$$

Hence we recovered the sign representation!

Note: the $\pi\{T\}$ appearing above are distinct as $\pi \in C(T)$ varies, so the coefficients really are ± 1.

Definition 26. We define a poset on $\{\lambda \text{ a partition of } n\}$ called $\boxed{\text{dominance order}}$. Set

$$\lambda \leq_D \mu \Leftrightarrow \lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$$

for all i. Note we get some non-comparable things, eg. $(2,2,2)$ and $(3,1,1,1)$. Maximal element: the horizontal stick (n).

27 Lemma
Lexicographic order is a linear extension of dominance order.

PROOF By contrapositive: if $\lambda <_L \mu$ (in Lexicographic order), then there exists $\lambda_j < \mu_j$ and $\lambda_i = \mu_i$ for all $i < j$. Then we can't have $\mu < \lambda$ in dominance, since it just went over at the jth spot.

28 Lemma

Say T, T' are Young tableaux with shapes λ, λ', respectively. Then $b_T\{T'\}$ is as follows:

- *$b_T\{T'\} = \pm e_T$ if $\lambda = \lambda'$ and no $t_{ij} \in R(T') \cap C(T)$ (i.e. no two numbers simultaneously appear in the same row of T' and the same column of T).*
- *$b_T\{T'\} = 0$ if $\lambda <_L \lambda'$ or $\lambda = \lambda'$, $t_{ij} \in R(T') \cap C(T)$.*

Note $b_T\{T'\} \neq 0$ implies $\lambda \geq_D \lambda'$.

PROOF If $i < j$ appear in the same row of T' and the same column of T, consider $b_T \cdot t_{ij}$. This is $-b_T$ since it just permutes the sum (and flips the signs). We'll finish this next time.

April 7th, 2014: Fundamental Specht Module Properties and Branching Rules

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Summary Recall: last time, we defined Specht modules

$$S^\lambda := \text{Span}\{e_T : T \text{ bijective filling of } \lambda\} \subset M^\lambda.$$

Today's goals:

A) S^λ is irreducible

B) $S^\lambda \cong S^\mu$ implies $\lambda = \mu$

C) Induction and restriction work on S^λ

Definition 29. Recall dominance order on partitions of size n, where

$$\lambda \leq_D \mu \Leftrightarrow \lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i, \quad \forall i.$$

Last time we said lexicographic extends dominance order.

30 Lemma

(See last lemma from previous lecture.) Say T, T' are Young tableau of shapes $\lambda, \lambda' \vdash n$. Then $b_T \cdot \{T'\} = 0$ if there exists $t_ij \in R(T') \cap C(T)$, i.e. if there are two numbers which are simultaneously in the same row of T' and the same column of T.

PROOF $t_{ij} \in C(T)$ implies $b_T t_{ij} = -b_T$; $t_{ij} \in R(T)$ implies $t_{ij} \cdot \{T'\} = \{T'\}$. Hence $b_T \cdot \{T'\} = b_T \cdot t_{ij}\{T'\} = -b_T\{T'\}$, so $b_T\{T'\} = 0$.

31 Corollary

If $\lambda, \lambda' \vdash n$, T is a λ-tableau, T' is a λ'-tableau, then $b_T \cdot \{T'\} \neq 0$ implies $\lambda \geq_D \lambda'$.

PROOF For each box in T, imagine annotating that box with the index of the row in which that box's entry appears in T' using red ink. We can permute T by some $\sigma \in C(T)$ at the cost of changing b_T by at most a sign, so we can assume the red annotations appear in weakly increasing order in each column. From the lemma, there is no i, j appearing in the same row of T' and the same column of T, so the red annotations actually increase strictly. By the pigeonhole principle, the first k rows of T contain all red annotations numbered $1, \ldots, k$, i.e. there is an injection from the first k rows of T' to the first k rows of T. But this just says

$$\lambda_1 + \cdots + \lambda_k \geq \lambda'_1 + \cdots \lambda'_k,$$

so $\lambda \geq_D \lambda'$.

Note: this proof holds for the weaker hypothesis that $\nexists t_{ij} \in R(T') \cap C(T)$.

32 Lemma

If T, T' are Young tableau of the same shape λ and no $t_{ij} \in R(T') \cap C(T)$ exists, then $b_T \cdot \{T'\} = \pm e_T$.

Proof By hypothesis, every value in row 1 of T' is in a distinct column of T. So, there exists a permutation $\pi^{(1)} \in C(T)$ such that $\pi^{(1)}T$ and T' have the same first row. Induct on the remaining rows to get some permutation $\pi \in C(T)$ such that $\pi T = T'$. Then $b_T \cdot \{T'\} = b_T \cdot \{\pi T\} = \operatorname{sgn}(\pi) e_T$.

33 Corollary

For T a Young tableau of shape λ,

$$b_T M^{\lambda'} = b_T S^{\lambda'} = 0 \text{ if } \lambda <_L \lambda'$$

and

$$b_T M^{\lambda} = b_T S^{\lambda} = \operatorname{Span}\{e_T\} \neq 0.$$

34 Theorem

S^λ is irreducible for each partition λ of n. (Note: it's important that we're working over $\mathbb{C}$.)

Proof Suppose $S^\lambda = V \oplus W$, T a Young tableau of shape λ. By the corollary above,

$$\operatorname{Span} e_T = b_T S^\lambda = b_T V \oplus b_T W.$$

Since $b_T V, b_T W, \operatorname{Span} e_t$ are all vector spaces over $\mathbb{C}$ and $\operatorname{Span} T$ is one dimensional, e_T must be in $b_T V \subset V$ or $b_T W \subset W$, with the other 0. Hence $e_T \in V$ or $e_T \in W$, so $\mathbb{C}[S_n] \cdot e_T = S^\lambda \subset V$ or W, giving the result.

35 Theorem

$S^\lambda \cong S^\mu \Rightarrow \lambda = \mu$.

Proof $S^\lambda \cong S^\mu$ implies there is a non-zero homomorphism $\Theta \colon S^\lambda \to M^\mu$. Extend $\Theta \colon M^\lambda \to M^\mu$ by

$$w \in (S^\lambda)^\perp \Rightarrow \Theta(w) = 0.$$

Now Θ non-zero implies there is some $e_T \in S^\lambda$ such that $\Theta(e_T) \neq 0$, so

$$0 \neq \Theta(e_T) = \Theta(b_T \cdot \{T\}) = b_T \Theta(\{T\}) = b_T \cdot (\sum_i c_i \{S_i\}),$$

where S_i are distinct tabloids of shape μ. At least one c_i is non-zero, so $b_T \cdot \{S_i\} \neq 0$, so $\lambda \geq_D \mu$ by the corollary above. By symmetry of this argument, $\lambda = \mu$.

36 Example

Let T be a Young tableau of shape λ. Note that we can write $b_T \cdot \{T\} = \sum \{S\}$ where the sum is over some S which are column-increasing. For instance, let $T = 462/35/1$, $S = 152/36/4$. These S don't form a basis, unfortunately; too much redundancy.

Definition 37. T is a $\boxed{\text{standard Young tableau}}$, $\mathrm{SYT}(\lambda)$, if T is a bijective filling of λ with rows and columns increasing. Set $\boxed{f^\lambda} := \#SYT(\lambda)$. Recall the hook length formula,

$$f^\lambda = \frac{n!}{\prod_{c \in \lambda} h_c}.$$

38 Theorem
$\{S^\lambda : \lambda \vdash n\}$ is the full set of distinct irreducible representations of S_n.

PROOF Recall the RSK correspondence, mapping $w \in S_n$ to $(P, Q) \in \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda)$ for some $\lambda \vdash n$ bijectively. Then

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2 \geq \sum_{\lambda \vdash n} (\dim S^\lambda)^2,$$

where the inequality comes from the previous two theorems, since each S^λ is a distinct irreducible representation of S_n. Below we show $\dim S^\lambda \leq f^\lambda$, so equality holds, and the result follows.

Definition 39. We next define a $\boxed{\text{partial order on tabloids}}$ of shape $\lambda \vdash n$. First we associate to $\{T\}$ a nested sequence $\alpha^T(1) \subset \alpha^T(2) \subset \cdots \subset \alpha^T(n)$, where $\alpha^T(k)$ is a composition of k (i.e. a partition of k without the weakly decreasing condition). Construct $\alpha^T(k)$ by removing all boxes with label $> k$ from T and interpreting the result as a composition. Equivalently, $\alpha^T(k) = (a_1, \ldots a_n)$, where a_i is the number of cells in T in row i that are $\leq k$.

40 Example
$\{45/32/1\}$ goes to $(001) \subset (011) \subset (021) \subset (121) \subset (221)$.

Define $\{S\} \leq \{T\}$ if $\alpha^S(k) \leq_D \alpha^T(k)$ for all $1 \leq k \leq n$. (Extend the dominance order to compositions in the obvious way.)

41 Lemma
Say $T \in \mathrm{SYT}(\lambda)$ and write $e_T = \sum c_S\{S\}$ for distinct tabloids $\{S\}$ of shape λ. Then $c_S \neq 0$ implies $\{S\} \leq \{T\}$. Hence $\{e_T : T \in \mathrm{SYT}(\lambda)\}$ are independent, and using the ennumerative RSK formula and theorem above, they span S^λ.

PROOF Suppose $t_{ij} \in C(T)$, $i < j$, and $\{S\} = \{\pi T\}$ for some $\pi \in C(T)$ has $c_S \neq 0$. First, if S has no column inversions, we are done. Otherwise, there exists $i < j$ in the same column of S with i the row just below j in S, so the reverse is true in $t_{ij}S$. Since i appears in an earlier row than j in $t_{ij}S$, but only i and j are moved, $\alpha^S(k) < \alpha^{t_{ij}S}(k)$ for all k. Furthermore, this operation reduces the number of column inversions by 1. Hence, $\{S\} \leq \{t_{ij}S\} \leq \{T\}$ by induction on the number of inversions.

Next, we use this to show that $\{e_T \mid T \in \mathrm{SYT}(\lambda)\}$ is linearly independent. The partial order above restricts to a partial order on the Standard Young Tabeaux, so we can list the standard Young Tableaux as

$$T_1, T_2, \ldots, T_{m-1}, T_m$$

in such a way that T_k is maximal among $T_1, \ldots, T_k$. Now, suppose that

$$0 = \sum_{i=1}^m c_i e_{T_i} = \sum_{i=1}^m c_i \sum_{\{S\} \leq \{T_i\}} d_S^i \{S\}.$$

But by maximality of T_m among $T_1, \ldots, T_m$, $\{T_m\}$ only appears in e_{T_m}, which forces $c_m = 0$. Thus

$$0 = \sum_{i=1}^{m-1} c_i e_{T_i}.$$

So, inductively, $c_i = 0$ for all i, and $\{e_T \mid T \in \mathrm{SYT}(\lambda)\}$ is linearly independent.

This means that

$$n! = \sum_{\lambda \vdash n} \dim(S^\lambda)^2 \geq \sum_{\lambda \vdash n} (f^\lambda)^2 = n!,$$

which forces $f^\lambda = \dim(S^\lambda)$, and thus $\{e_T \mid T \in SYT(\lambda)\}$ is a basis for S^λ.

42 Corollary

If S is any Young tableau of shape λ, then

$$e_S = \sum_{T \in \mathrm{SYT}(\lambda)} c_T e_T$$

for some constants c_T.

43 Remark

The algorithm to compute the coefficients in the previous corollary is called Rota straightening. It's complicated, so we'll just vaguely touch on it. Assume S is column increasing. As you scan down a pair of adjacent columns, there is a first place where $a_1 > b_n$, say; put $b_1 < b_2 < \cdots < b_n$ as the elements above b_n in its column, and $a_1 < a_2 < \cdots < a_m$ below a_1 in its column. Set $A := \{a_1, \ldots, a_m\}$, $B := \{b_1, \ldots, b_n\}$.

Definition 44. Set

$$g_{A,B} := \sum_{\pi \in \mathfrak{S}(A \cup B)} \mathrm{sgn}(\pi)\pi.$$

These are called $\boxed{\text{Garnir elements}}$.

Claim: $g_{A,B} e_S = 0$. Assuming this,

$$e_S = -\sum_{\pi \neq \mathrm{id}} \mathrm{sgn}(\pi)\pi e_S.$$

The $\pi e_S = e_{\pi S}$ are "closer" to standard. The full argument is in Bruce Sagan's book.

45 Theorem (Branching Rules)

Let $\lambda \vdash n$.

$$S^\lambda\downarrow^{S_n}_{S_{n-1}} \cong \bigoplus_{\mu \subset \lambda, |\mu| = |\lambda| - 1} S^\mu$$

and

$$S^\lambda\uparrow^{S_{n+1}}_{S_n} \cong \bigoplus_{\mu \supset \lambda, |\mu| = |\lambda| + 1} S^\mu.$$

Proof First, use Frobenius reciprocity, which states the following. Suppose $H \subset G$, ψ is an H-character, χ is a G-character. Then

$$\langle \psi\uparrow^G_H, \chi \rangle = \langle \psi, \chi\downarrow^G_H \rangle.$$

It follows that if we've proved one of the above statements, we've proved the other. So, we'll prove the restriction statement. Let $c_1, \ldots, c_k$ be the "outside corners" of the partition λ, ordered with increasing row indexes. These have the property that $\mu_i := \lambda - \{c_i\}$ is still a partition.

Our strategy is to define an S_{n-1}-module filtration of S^λ whose successive quotients are S^{μ_i}. Since $W \cong V \oplus W/V$ for G-modules in general, the filtration gives the desired isomorphism.

For the filtration, let $V^{(i)}$ be the span of e_T where $T \in \mathrm{SYT}(\lambda)$ and n in T is in row $\leq \mathrm{row}(c_i)$. Since n is fixed by S_{n-1}, $V^{(i)}$ is an S_{n-1}-module. This gives a filtration

$$\{0\} = V^{(0)} \subset V^{(1)} \subset V^{(2)} \subset \cdots \subset V^{(k)} = S^\lambda.$$

To see $V^{(i)}/V^{(i-1)} \cong S^{\mu_i}$ as S_{n-1}-modules, define an S_{n-1}-module homomorphism

$$\theta_i \colon M^\lambda \to M^{\mu_i}$$

as follows. If n is in row(c_i) of T, by tabloid equivalence say $\{T\}$ has n in c_i and set $\theta_i(\{T\}) = \{T - n\}$. Otherwise, set $\theta_i(\{T\}) = 0$.

$V^{(i-1)}$ is spanned by e_T which are by definition annihilated by θ_i, so $\ker \theta_i \supset V^{(i-1)}$. For $T\,\mathrm{SYT}(\lambda)$ with n in row c_i, we find $\theta_i(e_T) = e_{T-n}$ as follows. n must be in box c_i, so $\pi \in C(T)$ either (i) leaves it fixed or (ii) moves it up. Hence

$$\theta_i(e_T) = \theta_i \left(\sum_{\pi \in C(T) - C(T-n)} \mathrm{sgn}(\pi)\{\pi T\} \right) + \theta_i \left(\sum_{\pi \in C(T-n)} \mathrm{sgn}(\pi)\{\pi T\} \right) = 0 + e_{T-n},$$

whence $\theta_i(V^{(i)}) = S^{\mu_i}$.

Indeed, the map $V^{(i)} \xrightarrow{\theta_i} S^{\mu_i}$ gives the isomorphism $V^{(i)}/(V^{(i)} \cap \ker \theta_i) \cong S^{\mu_i}$, which is of dimension f^{μ_i}. We'd like $V^{(i)} \cap \ker \theta_i = V^{(i-1)}$, which is true as follows. We can extend our previous filtration to

$$\{0\} \subset V^{(0)} \subseteq V^{(1)} \cap \ker \theta_1 \subset V^{(1)} \subseteq V^{(2)} \cap \ker \theta_2 \subset V^{(2)} \subset \cdots \subset V^{(k)} = S^\lambda.$$

But then the successive quotients $V^{(i)}/(V^{(i)} \cap \ker \theta_i)$ account for $\sum f^{\mu_i} = f^\lambda$ dimensions, i.e. all of them. So, the inclusions $V^{(i-1)} \subseteq V^{(i)} \cap \ker \theta_i$ must be equalities, giving the result.

46 Exercise
Let c be a corner of $\lambda \vdash n$. Is the map $S^\lambda \to S^{\lambda - c}$ defined by sendng e_T to 0 if $T \in \mathrm{SYT}(\lambda)$ does not have n in c, and by sending e_T to e_{T-n} otherwise an S_{n-1}-module morphism? Similarly, is the map $S^{\lambda-c} \to S^\lambda$ defined by $e_{T-c} \mapsto e_T$ an S_{n-1}-module morphism?

April 9th, 2014: Representation Ring for S_n and its Pieri Formula

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Summary Last time: showed the Specht modules $\{S^\lambda : \lambda \vdash n\}$ form a complete set of irreducible representations for S_n. Showed S^λ has basis $\{e_T : T \in \mathrm{SYT}(\lambda)\}$. Note $\sigma e_T = e_{\sigma T}$. <u>Homework:</u> determine the matrices for $S^{(2,2)}$ for s_i in this basis. We also noted the branching rules, where $S^\lambda\uparrow_{S_n}^{S_{n+1}}$ is given by the sum of S^μ where μ covers λ in Young's lattice, and similarly with the restriction (see last time). Important point: these decompositions are multiplicity free.

47 Example
$S^{(2,2)}\uparrow_{S_4}^{S_5} = S^{(3,2)} \oplus S^{(2,2,1)}$. What about $S^{(2,2)}\uparrow_{S_4}^{S_6}$? Apply the rule to each of the pieces from the S_4 to S_5 case, which gives

$$S^{(2,2)}\uparrow_{S_4}^{S_6} = S^{(4,2)} \oplus S^{(3,3)} \oplus S^{(3,2,1)} \oplus S^{(2,2,2)} \oplus S^{(2,2,1,1)}.$$

48 Theorem
What's the general rule? Let $\lambda \vdash n$. Then

$$S^\lambda\uparrow_{S_n}^{S_{n+m}} = \bigoplus_{\mu \supset \lambda, |\mu| = n+m} (S^\mu)^{a_{\lambda,\mu}}.$$

Here $a_{\lambda,\mu} = f^{\mu/\lambda}$ is the number of $\boxed{\textit{standard skew tableau}}$ *μ/λ. That is, $\mu \supset \lambda$, and $\mu - \lambda$ is filled bijectively starting at 1 with strictly increasing rows and columns.*

Definition 49. Bijectively label any $D \subset \mathbb{Z} \times \mathbb{Z}$ with $|D| = n$ using $[n]$. Let

$$a_D := \sum_{\pi \in R(D)} \pi, \qquad b_D := \sum_{\pi \in C(D)} \mathrm{sgn}(\pi)\pi, \qquad c_D := b_D \cdot a_D \in \mathbb{C}[S_n].$$

Define a $\boxed{\text{generalized Specht module}}$

$$S^D := \mathrm{Span}_{\mathbb{C}}\{\sigma c_D : \sigma \in S_n\}.$$

50 Example

Let $D = *1/2$ (the $*$ means we don't pick that box). Then S^D is spanned by c_D for $*1/2$ and $*2/1$, which decomposes as the sum of the trivial representation $(*1/2 + *2/1)$ and the sign representation $(*1/2 - *2/1)$. Homework: Where in the proof of S^λ irreducible did we use the fact that λ is a partition? (Hint: pidgeonhole argument.) Open problem: decompose S^D as a sum of S^μ with some coefficients. References: Pawlowski thesis; Reiner-Shimozono.

Definition 51. Let U, V be vector spaces over some field $\Bbbk$. Let

$$\boxed{U \otimes V} := \frac{\mathrm{Span}_{\Bbbk}\{(u,v) : u \in U, v \in V\}}{(a+b,c) = (a,c) + (b,c), (a, b+c) = (a,b) + (a,c), (\lambda a, b) = (a, \lambda b) = \lambda(a,b)}.$$

The image of (u,v) in the quotient is denoted $u \otimes v$ and is called a $\boxed{\text{simple tensor}}$. Note that if $\{b_i\}_{i\in I}$ is a basis for U and $\{c_j\}_{j\in J}$ is a basis for V, then $\{b_i \otimes c_j : i \in I, j \in J\}$ is a basis for $U \otimes V$. Hence dimension is multiplicative (in general, in the sense of cardinal arithmetic).

52 Example

Consider $\mathbb{R}^2 \otimes \mathbb{C}^3$ viewed as vector spaces over $\mathbb{Q}$. Then

$$(5,0) \otimes (17+i, 0, e^{42\pi i}) + (0,17) \otimes (i,i,i)$$

can't be turned into a single (simple) tensor. Note $\dim_{\mathbb{R}}(\mathbb{R}^2 \otimes \mathbb{C}^3) = 12$, $\dim_{\mathbb{R}}(\mathbb{R}^2 \times \mathbb{C}^3) = 8 \neq 12$.

Definition 53. If U, V are rings, define $U \otimes V$ likewise. Define multiplication as $(a \otimes b)(c \otimes d) = (ac) \otimes (bd)$.

54 Remark

If U is a G-module, V is an H-module, then $U \otimes V$ is a $G \times H$-module via

$$(g,h)(a \otimes b) := (ga) \otimes (hb).$$

(Here the tensor product is over the underlying, common field.)

55 Remark

If $G = S_n$, $H = S_m$, then what are the irreducible representations of $G \otimes H$? Answer:

$$\{S^\lambda \otimes S^\mu : \lambda \vdash n, \mu \vdash m\}.$$

Note the number of conjugacy classes is correct! (In general, a complete set of distinct irreducible representations of a product of arbitrary *finite* groups is the tensor products of their distinct irreducible representations in the same way. However, it gets very complicated for infinite groups.)

56 Notation

If V is an S_n-representation, denote its isomorphism class via $[V]$. Say

$$[W] := [U] + [V] \qquad \text{if } W = U \oplus V.$$

Definition 57. Let $\boxed{R_n} := \mathrm{Span}_{\mathbb{C}}\{[V] : V \text{ is an } S_n\text{-rep}\}$. This is spanned by $\{[S^\lambda] : \lambda \vdash n\}$. This is called the $\boxed{\text{Grothendieck group}}$ of S_n representations. (R_0 seems to be spanned by the trivial representation, with S_0 the trivial group.)

58 Example

$-42[S^{(3,2)}] + 12[S^{(5)}] \in R_5$.

Define $\boxed{R}$ as $\oplus_{n\geq 0} R_n$ as the $\boxed{\text{ring of representations}}$. Addition is formal addition. Multiplication is given by $R_n \times R_m \to R_{n+m}$ where

$$[V]\cdot[W] = [(V\otimes W)\uparrow_{S_n\times S_m}^{S_{n+m}}].$$

Homework: show

$$(V\otimes W)\uparrow_{S_n\times S_m}^{S_{n+m}} = \mathbb{C}[S_{n+m}]\otimes_{\mathbb{C}[S_n\times S_m]}(V\otimes W).$$

59 Proposition

- *R is commutative: quick from commutativity of tensor product.*
- *R is associative: uses associativity of tensor product, that induction is transitive, etc.*
- *R has a unit, $1\in\mathbb{C}$ corresponding to the trivial representation in R_0.*

60 Theorem

$$(S^\lambda\otimes S^{(m)})\uparrow_{S_n\times S_m}^{S_{n+m}} = \bigoplus S^\mu,$$

where the sum is over $\mu\supset\lambda$ such that $|\mu| = |\lambda| + m = n + m$ and where $\mu-\lambda$ is a collection of horizontal strips (i.e. no two boxes appear in the same column). We'll call this the $\boxed{\text{Pieri formula}}$.

Proof (Thanks to Brendan.) By Frobenius reciprocity,

$$(S^\lambda\otimes S^{(m)})\uparrow_{S_n\times S_m}^{S_{n+m}} = \bigoplus_{\mu\vdash n+m}(S^\mu)^{a_{\lambda,m,\mu}}$$

and

$$S^\mu\downarrow_{S_n\times S_m}^{S_{n+m}} = \bigoplus_{\lambda'\vdash n,\nu\vdash m}(S^{\lambda'}\otimes S^\nu)^{c_{\lambda',\nu,\mu}}$$

have the same coefficients $a_{\lambda,m,\mu} = c_{\lambda,(m),\mu}$. So let $V = S^\mu\downarrow_{S_n\times S_m}^{S_{n+m}}$ for $\mu\vdash n+m$. That is, $S_n\times S_m$ acts on $V = \mathrm{Span}_{\mathbb{C}}\{e_T : T\in\mathrm{SYT}(\mu)\}$. Now V may include $S^{\lambda'}\otimes S^\nu$ for $\nu\neq(m)$ or $\lambda'\neq\lambda$, but we can "filter out" these extra representations by looking at the subspace $V^{1\times S_m}$ fixed by $1_n\times S_n$, which gives us precisely

$$V^{1\times S_m} = \oplus(S^\lambda\otimes S^{(m)})^{c_{\lambda,(m),\mu}}.$$

Hence we need only decompose $V^{1\times S_n}$ and show that the non-zero multiplicites are precisely one for exactly λ with the conditions in the theorem statement. We'll decompose $V^{1\times S_n}$ by setting $z := \sum_{\pi\in 1_n\times S_m}\pi$, noting $V^{1\times S_n} = zV = \mathrm{Span}\{ze_T : T\in\mathrm{SYT}(\lambda)\}$. When does $ze_T = ze_S$? Well, certainly when S,T agree on $[1,n]$. Moreover, if $i,j\in[n+1,m]$ are in the same column of T, then $z = zt_{ij}$ and $ze_T = zt_{ij}e_T = -ze_T$, so $ze_T = 0$. This motivates...

61 Theorem

$V^{1\times S_m}$ is the span of ze_T for $T\in\mathrm{SYT}(\mu)$ where μ/λ is a horizontal strip and the values in μ/λ increase from left to right; indeed this is a basis.

We end up with

$$V^{1\times S_m} = \oplus(S^\lambda\otimes S^{(m)})$$

with the sum over $\lambda\subset\mu$, μ/λ a horizontal strip, $|\mu| = |\lambda| + m$. The multiplicity is 1 using the basis for the tensor product, giving the result.

62 Remark

This suggests a basis for symmetric polynomials, since none of our given bases obviously mimic the Pieri rule. The $\boxed{\text{Schur functions}}$ are the answer. Recall

$$\boxed{s_\lambda} := \sum_{T \in \mathrm{SSYT}(\lambda)} x^T.$$

Next time: we'll show $s_\lambda s_{(m)}$ satisfies the Pieri formula. (Recall we also used a non-intersecting lattice path definition/proof from Autumn quarter—see beginning of next lecture.)

April 11th, 2014: Pieri for Schurs; Kostka Numbers; Dual Bases; Cauchy Identity

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Summary Last time, we defined the representation ring of symmetric groups, $R = \oplus R_n$, where R_n is spanned by equivalence classes $[S^\lambda]$ is irreducible representations of S_n, $\lambda \vdash n$, with multiplication given by tensor product followed by induction, addition splitting over direct sums.

Hence

$$[S^\mu][S^\nu] = [(S^\mu \otimes S^\nu)\uparrow_{S_n \times S_m}^{S_{n+m}}] = \sum_{\lambda \vdash n+m} c^\lambda_{\mu,\nu} [S^\lambda],$$

$\mu \vdash n, \nu \vdash m$ for some $c^\lambda_{\mu,\nu} \in \mathbb{N}$. Eventual goal: show the representation ring is isomorphic to the ring of symmetric functions.

Today: Pieri formula for Schur functions and Cauchy's identity involving Schur functions.

Definition 63. Let $\boxed{\mathrm{SSYT}(\lambda)}$ be the set of semistandard Young tableaux of shape λ, meaning fillings of a partition λ with $\mathbb{P}$ such that rows weakly increase and columns strictly increase. Associate a monomial to each such object in the obvious way:

64 Example

$T = 133/25$ of shape $\lambda = (3, 2)$. The associated monomial is $x^T = x_1^1 x_2^1 x_3^2 x_5^1$.

Definition 65. We recalled the definition of the Schur functions at the end of last time. (To get comfortable with them, Sara recommends computing any 10 by hand.) Recall also the Jacobi-Trudi formula,

$$s_\lambda(x) = \det(h_{\lambda_i - i + j}(x))_{1 \le i,j \le k},$$

which we proved last quarter. (Here λ has k parts.) Note: there's also a notion of Schur polynomials where we restrict the alphabet filling the SSYT's to eg. $[n]$. We're using the infinite alphabet $\mathbb{P}$. Sara's exercise may be easier with a restricted alphabet.

66 Proposition

The s_λ are symmetric. This is immediate from the Jacobi-Trudi formula, but it's not obvious from the SSYT *definition.* Homework*: prove symmetry from the* SSYT *definition; find appropriate bijections.*

67 Example

- $s_{(m)} = h_m$
- $s_{1^m} = e_m$

using the definitions from the first day of class.

68 Theorem (Pieri Formula)

$$s_\lambda s_{(m)} = \sum s_\mu,$$

where the sum is over μ/λ which is a horizontal strip with m boxes as in the previous Pieri formula.

PROOF We'll define a bijection

$$\mathrm{SSYT}(\lambda) \times \mathrm{SSYT}((m)) \to \cup\, \mathrm{SSYT}(\mu),$$

where the union is over λ from the theorem statement; the result follows from the definition of the Schur functions. We do so by successively inserting elements into rows and "bumping" elements down. Suppose we have a row $a_1 \leq a_2 \leq \cdots \leq a_k$ and we want to insert b. If $b \geq a_k$ then add b to the end of the row. Otherwise, replace a_j by b for the smallest j such that $a_j > b$ and insert a_j into the next row.

69 Example

Starting with $T = 123/45$, let's insert 225. 2 replaces the 3 in the first row, and the 3 is then inserted in the row below, which bumps the 4, and 4 is inserted on the last line, so we have 122/33/4 after these "bumps". Inserting 25 results in 12225/33/4. Note that we added the horizontal strip $***25/**/4$.

Claim: suppose inserting a row $i_1 \leq \cdots \leq i_m$ into an SSYT T gives S; it's straightforward to check that S is an SSYT of shape $\mathrm{sh}(S)$ containing the shape of T. One can check that indeed $\mathrm{sh}(S)/\mathrm{sh}(T)$ is a sequence of m horizontal strips. (Rough outline: the "bumping path" for i_2 is always strictly to the right of the bumping path for i_1. Draw some pictures to convince yourself; better than a formal proof.)

Hence indeed we have a weight-preserving map

$$\mathrm{RSK}\colon\ \mathrm{SSYT}(\lambda) \times \mathrm{SSYT}((m)) \to \cup\, \mathrm{SSYT}(\mu).$$

Why is it reversible? The cells of $\mathrm{sh}(S)/\mathrm{sh}(T)$ must have been created from left to right, so we can "unbump" them to undo the operation.

70 Corollary

Recall $h_\mu = h_{\mu_1} h_{\mu_2} \cdots h_{\mu_k}$ by definition. What are the coefficients $h_\mu := \sum K_{\lambda,\mu} s_\lambda$?

$$\boxed{K_{\lambda,\mu}} := \#\{T \in \mathrm{SSYT}(\lambda) : x^T = x^\mu.\}$$

In particular, $K_{\lambda,\lambda} = 1$ and $K_{\lambda,\mu} = 0$ unless $\lambda \geq_D \mu$. These are called the $\boxed{\textit{Kostka numbers}}$.

PROOF Imagine repeatedly applying the Pieri rule, starting with $h_{\mu_k} \cdot 1 = s_{(\mu_k)}$; label the new cells in the first application 1, in the second application 2, etc. We get horizontal strips of μ_1 1's, μ_2 2's, etc. In this way we get an SSYT of weight μ, and indeed all such SSYT arise in this way.

For $K_{\lambda,\lambda}$, there's just one way to fill λ with λ_1 1's: all in the first row; etc.

71 Corollary

Let $\lambda \vdash n$. Then

$$\begin{aligned} s_\lambda &= \sum_{T \in \mathrm{SSYT}(\lambda)} x^T = \sum_{\alpha \vDash n} K_{\lambda,\alpha} x^\alpha \\ &= \sum_{\mu \vdash n} K_{\lambda,\mu} m_\mu \end{aligned}$$

(Recall m_μ were the monomial symmetric functions, and $\alpha \vDash n$ means α is a composition of n, which is a partition without the weakly decreasing requirement.)

Definition 72. The $\boxed{\text{Hall inner product}}$ is by definition $\langle s_\lambda s_\mu\rangle = \delta_{\lambda,\mu}$, extended sesquilinearly (i.e. linearly in the first argument, conjugate-linearly in the second).

73 Proposition

From the corollary,

$$\langle h_\mu, s_\lambda\rangle = K_{\lambda,\mu}$$

and

$$\langle m_\mu, s_\lambda\rangle = K^{-1}_{\mu,\lambda}$$

(inverse matrix). Hence

$$\langle h_\mu, m_\gamma\rangle = \sum_\lambda K_{\lambda,\mu}\langle s_\lambda, m_\gamma\rangle = \sum K_{\lambda,\mu}K^{-1}_{\gamma,\lambda} = \delta_{\mu,\gamma}.$$

Thus $\{h_\mu\}$ and $\{m_\lambda\}$ are dual bases.

74 Theorem

The $\boxed{\textit{Cauchy identity}}$ says

$$\begin{aligned}\prod_{i\geq 1}\prod_{j\geq 1}\frac{1}{1-x_iy_j} &\overset{(A)}{=} \sum_\lambda s_\lambda(x)s_\lambda(y)\\ &\overset{(B)}{=} \sum_\lambda h_\lambda(x)m_\lambda(y)\\ &\overset{(C)}{=} \sum_\lambda \frac{p_\lambda(x)p_\lambda(y)}{z_\lambda},\end{aligned}$$

where $x=(x_1,x_2,\ldots)$, $y=(y_1,y_2,\ldots)$ and $z_\lambda = (1^{m_1}2^{m_2}\cdots)(m_1!m_2!\cdots)$ for $\lambda=(1^{m_1},2^{m_2},\cdots)$ (i.e. m_1 one's, m_2 two's, etc.). The sums are over $\lambda\vdash n$ for $n\geq 0$.

Proof Today we'll prove equality A. Note

$$\prod_i\prod_j\frac{1}{1-x_iy_j} = \prod_{ij}(1+x_iy_j+(x_iy_j)^2+\cdots) = \sum(x_{i_1}y_{j_1})^{k_1}(x_{i_2}y_{j_2})^{k_2}\cdots,$$

where the sum is over all "biwords" in the alphabet of "biletters" $\{\binom{i}{j} : i\geq 1, j\geq 1\}$ written in weakly increasing lexicographic order, where the k_ℓ are multiplicities. (Not to be confused with "bywords"!)

75 Example

$\binom{1}{2}\binom{1}{2}\binom{1}{2}\binom{1}{4}\binom{2}{3}\binom{3}{7}\binom{3}{7}$ corresponds to the monomial $(x_1y_2)^3(x_1y_4)(x_2y_3)(x_3y_7)^2$.

To prove A, we'll exhibit a "bi"weight-preserving bijection between finite biwords in lexicographic order to pairs (P,Q) of semistandard Young tableaux of the same shape. Under this bijection, label i of P corresponds to x_i, and label j of Q corresponds to x_j. Start with

$$\binom{i_1}{j_1}\binom{i_2}{j_2}\cdots\binom{i_k}{j_k}.$$

In particular, $i_1\leq i_2\leq\cdots\leq i_k$. To get P, insert the letters of $j_1j_2\cdots j_k$ successively into the empty tableau using our earlier RSK insertion algorithm. Let Q' be the recording tableaux for this process, i.e. its labels indicate when the corresponding element of P was inserted. Get Q by replacing ℓ in Q' with i_ℓ.

76 Example

With the previous word, P becomes 222377/4. The recording tableau Q' was 123467/5 (denotes order of insertion). Hence $Q = 111133/2$.

P is a semistandard tableau as before. Q is certainly weakly increasing along rows. What about columns? Recall the bumping path for inserting $j_1 \leq \cdots \leq j_k$ to T; they give a horizontal strip. It follows that Q is column strict and has the same shape as P. (But we don't have $j_1 \leq \cdots \leq j_k$?) This algorithm is invertible.

April 14th, 2014: Finishing Cauchy; $R \cong \mathrm{SYM}$; Littlewood-Richardson Rule; Frobenius Characteristic Map

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Summary Last time: concluded with the Cauchy identity, in particular proving

$$\prod_{i,j\geq 1} \frac{1}{1-x_i y_j} = \sum_\lambda s_\lambda(x) s_\lambda(y).$$

We'll prove equalities (B) and (C) today.

Proof of Cauchy identity (continued). We'll start with proving (B) while making a few detours. An important generating function in general is

$$\boxed{H(t)} := \prod_{i\geq 1} \frac{1}{1-x_i t} = 1 + h_1(x)t + h_2(x)t^2 + \cdots .$$

Note

$$\begin{aligned}\prod_{i,j\geq 1} \frac{1}{1-x_i y_j} &= \prod_{j\geq 1} H(y_j) = \prod_{j\geq 1}(1 + h_1(x)y_j + h_2(x)y_j^2 + \cdots) \\ &= \sum_{n\geq 0}\sum_{\alpha\vDash n} y^\alpha (h_{\alpha_1}(x)h_{\alpha_2}(x)\cdots) = \sum_{n\geq 0}\sum_{\alpha\vDash n} h_\alpha(x) y^\alpha \\ &= \sum_{n\geq 0}\sum_{\lambda\vdash n} h_\lambda(x) m_\lambda(y),\end{aligned}$$

where as before $\alpha \vDash n$ refers to a composition of n; this proves (B).

77 Aside

Similarly, another important generating function is

$$\boxed{E(t)} := \prod_{i\geq 1}(1 + x_i t) = 1 + e_1(x)t + e_2(x)t^2 + \cdots .$$

Since $H(t)E(-t) = 1$, we find $h_n(((x) = \sum_{i=1}^n (-1)^n h_i(x) e_{n-i}(x)$.

Next we'll prove (C). First note a cute trick:

$$\begin{aligned}\log H(t) = \log \prod_{i\geq 1} \frac{1}{1-x_i t} &= \sum_{i\geq 1} -\log(1-x_i t) = \sum_{i\geq 1}\sum_{m\geq 1} \frac{(x_i t)^m}{m} \\ &= \sum_{m\geq 1} \frac{p_m(x) t^m}{m}.\end{aligned}$$

78 Aside

Another important generating function is

$$\boxed{P(t)} := \sum_{m\geq 0} p_{m+1}(x) t^m = \frac{d}{dt}\log(H(t))$$

79 Corollary
SYM $= \mathbb{Q}[e_1, \ldots] = \mathbb{Q}[h_1, \ldots] = \mathbb{Q}[p_1, \ldots]$.

For the proof of (C),

$$\begin{aligned}\log \prod_{i\geq 1}\prod_{j\geq 1} \frac{1}{1-x_iy_j} &= \log \prod_{j\geq 1} H(y_j) = \sum_{j\geq 1} \log H(y_j)\\ &= \sum_{j\geq 1}\sum_{m\geq 1} \frac{p_m(x)y_j^m}{m} = \sum_{m\geq 1}\frac{p_m(x)p_m(y)}{m},\end{aligned}$$

so (this really isn't as horrendous as it looks)

$$\begin{aligned}\prod_{i\geq 1}\prod_{j\geq 1}\frac{1}{1-x_iy_j} &= \exp\left(\sum_{m\geq 1}\frac{p_m(x)p_m(y)}{m}\right) = \sum_{k\geq 0}\frac{1}{k!}\left(\sum_{m\geq 1}\frac{p_m(x)p_m(y)}{m}\right)^k\\ &= \sum_{k\geq 0}\frac{1}{k!}\sum_{m_1+m_2+\cdots=k, m_d\in\mathbb{N}}\binom{k}{m_1, m_2, \ldots}\left(\frac{p_1(x)p_1(y)}{1}\right)^{m_1}\left(\frac{p_2(x)p_2(y)}{2}\right)^{m_2}\cdots\\ &= \sum_{m_1+m_2+\cdots=k, m_d\in\mathbb{N}}\frac{[p_1(x)p_1(y)]^{m_1}}{1^{m_1}m_1!}\frac{[p_2(x)p_2(y)]^{m_2}}{2^{m_2}m_2!}\cdots = \sum_\lambda \frac{p_\lambda(x)p_\lambda(y)}{z_\lambda},\end{aligned}$$

where $\lambda \vdash n$ for $n \geq 0$ and $z_\lambda = (1^{m_1}2^{m_2}\cdots)(m_1!m_2!\cdots)$ is the number of ways to write a fixed permutation w with cycle type $\lambda = 1^{m_1}2^{m_2}\cdots$ (i.e. m_1 one's, m_2 two's, etc.) in cycle notation so that the cycles are listed in increasing length. This is precisely (C), completing the proof.

80 Proposition
Say $\{u_\lambda\}$, $\{v_\lambda\}$ *are two bases for* SYM *such that* $\sum_\lambda s_\lambda(x)s_\lambda(y) = \sum_\lambda u_\lambda(x)v_\lambda(y)$. *Then*

$$\langle u_\lambda, v_\mu\rangle = \delta_{\lambda,\mu}$$

(using the Hall inner product), i.e. they are dual bases.

Proof Say $u_\lambda = \sum a_{\nu\lambda}s_\nu$, $v_\lambda = \sum b_{\nu\lambda}s_\nu$. Let A be the transition matrix $A = (a_{\nu\lambda})$, similarly with $B = (b_{\nu\lambda})$. Therefore

$$\langle u_\lambda, v_\mu\rangle = \sum_\nu a_{\nu\lambda}b_{\lambda\mu} = (A^TB)_{\lambda\mu}.$$

But also

$$\begin{aligned}\sum_\lambda s_\lambda(x)s_\lambda(y) &= \sum_\lambda u_\lambda(x)v_\lambda(y)\\ &= \sum_\lambda\left(\sum_\nu a_{\nu\lambda}s_\nu(x)\right)\left(\sum_\rho b_{\rho\lambda}s_\rho(y)\right)\end{aligned}$$

so

$$\sum_\lambda a_{\nu\lambda}b_{\rho\lambda} = \delta_{\nu\rho}$$

which says $AB^T = I$, so $B^TA = I$, so $A^TB = I$, so $(A^TB)_{\lambda\mu} = \delta_{\lambda\mu}$. Indeed this works in reverse.

81 Corollary
$\{p_\lambda\}, \{p_\lambda/z_\lambda\}$ *are dual bases.* $\{p_\lambda/\sqrt{z_\lambda}\}$ *is self-dual (over* $\mathbb{C}$*, say).*

82 Remark

We earlier noted $\{h_\mu\}$ and $\{m_\lambda\}$ are dual; we just found the dual of $\{p_\lambda\}$; so what's the dual of $\{e_\lambda\}$? The $\boxed{\text{forgotten basis}}$ $\boxed{\{f_\lambda\}}$. Apparently it's not used much.

83 Remark

Back to big picture: showing R is (graded-)ring isomorphic to the symmetric functions. We've got a vector space isomorphism so far:

$$\Phi\colon R = \oplus R_n \to \mathrm{SYM} = \oplus\, \mathrm{SYM}_n$$

given by $[S^\lambda] \mapsto s_\lambda$. What does $[M_\lambda]$ map to?

84 Proposition

We have

$$M^\lambda = 1\uparrow_{S_{\lambda_1}\times\cdots\times S_{\lambda_k}}^{S_n} = S^{(\lambda_1)} \otimes S^{(\lambda_2)} \otimes \cdots S^{(\lambda_k)}\uparrow_{S_{\lambda_1}\times\cdots\times S_{\lambda_k}}^{S_n},$$

so

$$[M^\lambda] = [S^{(\lambda_1)}]\cdots[S^{(\lambda_k)}].$$

Proof Induction is transitive, so

$$\begin{aligned} M^\lambda &= 1\uparrow_{S_{\lambda_1}\times\cdots\times S_{\lambda_k}}^{S_n} = S^{(\lambda_1)} \otimes S^{(\lambda_2)} \otimes \cdots S^{(\lambda_k)}\uparrow_{S_{\lambda_1}\times\cdots\times S_{\lambda_k}}^{S_n} \\ &= \left(\left(\left(S^{(\lambda_1)} \otimes S^{(\lambda_2)}\right)\uparrow_{S_{\lambda_1}\times S_{\lambda_2}}^{S_{\lambda_1+\lambda_2}} \otimes S^{(\lambda_3)}\right)\uparrow_{S_{\lambda_1+\lambda_2}\times S_{\lambda_3}}^{S_{\lambda_1+\lambda_2+\lambda_3}} \cdots \otimes S_{(\lambda_k)}\right)\uparrow_{S_{n-\lambda_k}\times S_{\lambda_k}}^{S_n}. \end{aligned}$$

85 Remark

We hope

$$\Phi([M_\lambda]) \stackrel{?}{=} h_{\lambda_1}\cdots h_{\lambda_k} = h_\lambda,$$

though we haven't shown Φ is a ring homomorphism yet, hence the question mark.

86 Proposition

Φ distributes over products of $[S^{(m)}]$'s.

Proof Both SYM and R obey the Pieri rule, with Φ translating between the two versions, so for instance $\Phi([S^\lambda][S^{(m)}]) = s_\lambda s_{(m)} = s_\lambda h_m$. Expand $\prod_i [S^{\lambda^i}]$ as a sum of $c_\nu[S^\nu]$ by repeatedly using the Pieri rule and do the same to $\prod_i s_{\lambda^i}$ resulting in a sum of $c_\nu s_\nu$: the coefficients must agree. This is precisely saying Φ distributes over $\prod_i [S^{\lambda^i}]$.

87 Lemma

- $\Phi([M^\mu]) = h_\mu$
- $\Phi([M^\lambda][M^\mu]) = \Phi([M^\lambda])\Phi([M^\mu])$
- $M^\mu = \oplus_\lambda K_{\mu,\lambda} S^\lambda$
- $M^\mu = S^\mu \oplus_{\lambda <_D \mu} K_{\mu,\lambda} S^\lambda$

Proof For the first two, use the two preceding propositions. For the third, expand h_μ in the Schur basis and apply the first one. For the fourth, recall that $K_{\mu,\mu} = 1$ and $K_{\mu,\lambda} = 0$ unless $\mu \geq_D \lambda$.

88 Theorem

Φ is a ring isomorphism.

Proof Multiplication in SYM is determined by the Pieri rule plus the Jacobi-Trudi identity, $s_\lambda = \det[h_{\lambda_i - i + j}]$, since using these we can compute $s_\lambda s_\mu$ in the Schur basis. The ring R also obeys the Pieri rule, but we haven't shown it obeys the Jacobi-Trudi identity, so we'll use another approach.

To see that Φ is a ring homomorphism, it suffices to show Φ is multiplicative on $[S^\lambda][S^\mu]$'s. By the fourth part of the lemma,

$$[M^\lambda][M^\mu] = [S^\lambda][S^\mu] + \text{lower terms},$$

where the lower terms are of the form $[S^{\lambda'}][S^{\mu'}]$ for $\lambda' \leq_D \lambda, \mu' \leq_D \mu$ with at least one strict. Suppose inductively Φ is multiplicative on these lower terms. From the second part of the lemma, Φ is multiplicative on $[M^\lambda][M^\mu]$. Apply Φ to the entire equation: the left-hand side becomes $h_\lambda h_\mu$, which we may expand as products of Schur functions using the Kostka numbers. Ignoring $[S^\lambda][S^\mu]$, this operation agrees with the result of applying Φ to the right-hand side using the inductive hypothesis and the third part of the lemma. This forces $\Phi([S^\lambda][S^\mu]) = s_\lambda s_\mu$, as desired.

89 Corollary
The representation ring R for the symmetric groups is isomorphic to SYM *as rings, via* $[S^\lambda] \mapsto s_\lambda$.

90 Corollary
$S^\mu \otimes S^\nu \uparrow_{S_{|\mu|}\times S_{|\nu|}}^{S_n} = \bigoplus_{\lambda\vdash n} c_{\mu\nu}^\lambda S^\lambda$ *and* $s_\mu s_\nu = \sum_{\lambda\vdash n} c_{\mu\nu}^\lambda s_\lambda$ *for some coefficients* $\boxed{c_{\mu\nu}^\lambda}$, *the* $\boxed{\textit{Littlewood-Richardson coefficients}}$. *(Here* $n = |\mu| + |\nu|$.*)*

91 Theorem ($\boxed{\textbf{Littlewood-Richardson Rule}}$ **)**
$c_{\mu\nu}^\lambda$ *is the number of fillings of* λ/μ *which are semistandard with content* ν *and reading words in Hebrew are lattice words. That is, reading right to left, top to bottom, count the number if 1's, 2's, etc. and ensure that at every point the number of 3's is at least as large as the number of 2's, which is at least as large as the number of 1's, etc.*

92 Example
$***11/*122/23$ is a valid filling; here $\mu = (3,1)$, $\lambda = (5,4,2)$.

Proofs: Stembridge paper; Stanley EC2, around chapter 7; Remmel-Whitney; Ravi Vakil has a geometric proof.

Definition 93 ($\boxed{\textbf{Frobenius Characteristic Map}}$ **).** For motivation, recall

$$p_\mu = \sum_\lambda \xi_\mu^\lambda m_\lambda$$

where ξ_μ^λ is $\chi_{M^\lambda}(\sigma)$ where σ has cycle type μ. By duality,

$$h_\lambda = \sum_\mu \xi_\mu^\lambda \frac{p_\mu}{z_\mu}.$$

Also recall $\Phi([M^\lambda]) = h_\lambda$.

Define the Frobenius characteristic map $\mathcal{F}\colon R \to \text{SYM}$ as follows. If χ is a character for an S_n-module V, set

$$\boxed{\mathcal{F}([V])} := \sum_{\mu\vdash n} \frac{\chi(\mu)p_\mu}{z_\mu} := \boxed{\mathrm{ch}(\chi)},$$

where $\chi(\mu)$ means the value of χ on the equivalence class with cycle type μ. From the above considerations, $\mathcal{F}([M^\lambda]) = h_\lambda = \Phi([M^\lambda])$.

94 Theorem
$s_\lambda = \sum_{\mu\vdash n} \frac{\chi^\lambda(\mu)p_\mu}{z_\mu}$, $p_\mu = \sum \chi^\lambda(\mu) s_\lambda$.

April 16th, 2014: Algebras and Coalgebras Intro

Summary Last time: finished Cauchy identities; showed $\{p_\mu\}, \{p_\mu/z_\mu\}$ are dual bases, $p_\mu = \sum_\lambda \chi^\lambda(\mu) s_\lambda$.

Also important: Murnaghan-Nakayama rule for computing entries in the character table for S^λ; see Sagan or Macdonald for details.

95 Fact
Let G be a finite group. G acts on itself by conjugation. If Ψ_G is the character of this representation and $\{\xi^\lambda\}$ is the set of irreducible characters, then

$$\langle \psi_G, \xi^\lambda \rangle = \sum_K \xi^\lambda(K)$$

where the sum is over conjugacy classes of G. (This is a good exercise.) Hence the sum on the right-hand side is in $\mathbb{N}$.

96 Open Problem
Find a combinatorial interpretation of the fact that $\sum_{\mu \vdash n} \chi^\lambda(\mu) \in \mathbb{N}$, say using the Murnaghan-Nakayama rule or using SYM.

Richard Stanley's "Positivity Problems and Conjectures in Algebraic Combinatorics", problem 12, discusses this briefly.

97 Fact
Given S_n-modules U, V, then $U \otimes_{\mathbb{k}} V$ is an S_n-module (different from our earlier operations) with ("diagonal") action

$$\sigma(u \otimes v) = \sigma(u) \otimes \sigma(v).$$

In particular, $S^\lambda \otimes S^\mu = \oplus g_{\lambda,\mu,\nu} S^\nu$, and the same sort of thing holds for any finite group. The coefficients $\boxed{g_{\lambda,\mu,\nu}} \in \mathbb{N}$ are called the $\boxed{\text{Kronecker coefficients}}$. Problem: find an efficient, combinatorial, manifestly positive rule for computing the Kronecker coefficients.

98 Example
$\chi^\lambda \chi^\mu = \sum g_{\lambda,\mu,\nu} \chi^\nu$, so we can just throw linear algebra at it, but for instance this isn't clearly positive.

99 Remark
We'll switch to algebras and coalgebras for a bit, giving some background. Sources:

- Hopf lectures by Federico Ardila.
- Darij Grinberg and Vic Reiner notes, "Hopf Algebras in Combinatorics"
- Moss Sweedler from 1969, a breakthrough book for its day making the field more accessible.

Definition 100. Let $\mathbb{k}$ be a field (though "hold yourself open" to other rings). A is an $\boxed{\text{associative } \mathbb{k}\text{-algebra}}$ if it is

(1) a $\mathbb{k}$-vector space

(2) a ring with $\mathbb{k}$-linear multiplication $A \otimes_{\mathbb{k}} A \xrightarrow{m} A$ and two-sided multiplicative identity $\boxed{1_A}$.

(3) with a linear map $\boxed{u \colon \mathbb{k} \to A}$ called the $\boxed{\text{unit}}$ with $u(1_{\mathbb{k}}) = 1_A$. (The existence and uniqueness of u is implied by (1) and (2).)

This is summarized by the following diagrams:

$$
\begin{tikzcd}
 & A \otimes A \otimes A \arrow[dl, "m\otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id}\otimes m"] & \\
A \otimes A \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\
 & A &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes_{\Bbbk} \Bbbk \arrow[d, "\mathrm{id}\otimes u"] & A \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\mathrm{id}"] & \Bbbk \otimes_{\Bbbk} A \arrow[d, "u\otimes \mathrm{id}"] \\
A \otimes A \arrow[r, "m"'] & A & A \otimes A \arrow[l, "m"]
\end{tikzcd}
$$

For instance, if we run $a \in A$ through the left square of the right diagram, we find

$$a \mapsto a \otimes 1_{\Bbbk} \mapsto a \otimes u(1_{\Bbbk}) \mapsto au(1_{\Bbbk}) = a,$$

so that $u(1_{\Bbbk})$ is a (hence, the) two-sided identity of A.

101 Example

- $\Bbbk[x]$ has $u(c) = c$
- $\mathbb{C}[S_n]$: multiplication is associative, identity is the identity permutation, so $u(c) = c[1, 2, \ldots, n]$.
- M_{22}, two by two matrices over $\mathbb{Q}_p$, say, taken as a $\mathbb{Q}$-vector space. Usual addition, multiplication, $\mathbb{Q}$-module action all works. Unit is the identity matrix, so $u(c)$ is the diagonal matrix with c's on the diagonal.
- If A, B are $\Bbbk$-algebras, then so is $A \otimes_{\Bbbk} B$ with multiplication defined via

  $$m_{A\otimes B}[(a \otimes b) \otimes (a' \otimes b')] = aa' \otimes bb'.$$

  Hence $1_{A\otimes B} = 1_A \otimes 1_B$. This is indeed associative:

  $$m_{A\otimes B} = m_A \otimes m_B \circ (1 \otimes T \otimes 1)$$

  where $T = B \otimes A \to A \otimes B$ is the "twist" operation. For practice with the above diagrams, it's good to check this explicitly by tracing through the above. The unit is $u_{A\otimes B}(c) = u_A(c) \otimes u_B(c) \in A \otimes B$—actually, this is incorrect; see the beginning of the next lecture.

102 Proposition

Why tensor products? They're a nice way to encode bilinear maps. The universal property of tensor products is the following. Let L, V, W be $\Bbbk$-vector spaces. Let $\phi \colon V \times W \to V \otimes W$ via $(v, w) \mapsto v \otimes w$. If $f \colon V \times W \to L$ is bilinear, then it factors through a unique $\Bbbk$-linear $\tilde{f} \colon V \otimes W \to L$:

$$
\begin{tikzcd}
V \times W \arrow[r, "\phi"] \arrow[dr, "f"'] & V \otimes W \arrow[d, dashed, "\exists! \tilde{f}"] \\
 & L
\end{tikzcd}
$$

Hence there is a natural bijection between bilinear $V \times W \to L$ and linear $V \otimes W \to L$.

Proof ϕ is bilinear by definition of the product. Given f bilinear, define $F(V, W)$ as the $\Bbbk$-span of (v, w) for $v \in V, w \in W$. Define $\overline{f} \colon F(V, W) \to L$ by $(v, w) \mapsto f(v, w)$. From the bilinearity of f, we see $\overline{f}(I) = 0$, where I is the submodule we quotiented by to form the tensor product, i.e. $V \otimes W := F(V, W)/I$. Hence the map descends to the quotient $\tilde{f} \colon V \otimes W \to L$ with $v \otimes w \mapsto \overline{f}(v, w) = f(v, w)$. Uniqueness comes from the fact that we've annihilated precisely the things given by bilinearity of f.

103 Remark

Weird thing: $p\colon V \otimes W \to V$ given by $v \otimes w \mapsto v$ is not well-defined. However, if $g\colon W \to \Bbbk$ is linear, then $p\colon V \otimes W \to V$ given by $v \otimes w \mapsto g(w) \cdot v$ is linear. Homework: If g is an algebra homomorphism, then p is as well.

For instance, $p_1\colon C \otimes_{\Bbbk} \Bbbk \to C$ given by $c \otimes k \mapsto k \cdot c$ is the inverse of $C \to C \otimes_{\Bbbk} \Bbbk$ given by $c \mapsto c \otimes 1_{\Bbbk}$.

Definition 104. A $\boxed{\text{coassociative coalgebra}}$ C is

(1) a $\Bbbk$-vector space

(2) with a $\Bbbk$-linear comultiplication $\Delta\colon C \to C \otimes_{\Bbbk} C$ such that

$$(\mathrm{id} \otimes \Delta) \circ \Delta(c) = (\Delta \otimes \mathrm{id}) \circ \Delta(c)$$

(3) and a $\Bbbk$-linear map called the $\boxed{\text{counit}}$

$$\boxed{\epsilon\colon C \to \Bbbk}$$

such that $p_1 \circ (\epsilon \otimes \mathrm{id}) \circ \Delta(c) = c$ and $p_2 \circ (\mathrm{id} \otimes \epsilon) \circ \Delta(c) = c$. Here $p_1\colon C \otimes \Bbbk \to C$ is given by $c \otimes k \mapsto k \cdot c$, and likewise with $p_2\colon \Bbbk \otimes C \to C$ given by $k \otimes c \mapsto k \cdot c$.

This is encoded in diagrams as before:

$$\begin{tikzcd}
 & C \otimes C \otimes C & \\
C \otimes C \arrow[ur, "\Delta \otimes \mathrm{id}"] & & C \otimes C \arrow[ul, "\mathrm{id} \otimes \Delta"'] \\
 & C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
C \otimes_{\Bbbk} \Bbbk \arrow[r, "p_1", "\sim"'] & C & \Bbbk \otimes_{\Bbbk} C \arrow[l, "p_2"', "\sim"] \\
C \otimes C \arrow[u, "\mathrm{id} \otimes \epsilon"] & C \arrow[l, "\Delta"] \arrow[r, "\Delta"'] \arrow[u, "\mathrm{id}"] & C \otimes C \arrow[u, "\epsilon \otimes \mathrm{id}"']
\end{tikzcd}$$

For instance, if we run $c \in C$ through the left square of the right diagram, we find

$$c \mapsto \Delta(c) = \sum_{i=1}^{k} a_i \otimes b_i \mapsto \sum_{i=1}^{k} a_i \otimes \epsilon(b_i) \mapsto \sum_{i=1}^{k} \epsilon(b_i) \cdot a_i = c,$$

so ϵ roughly allows us to "undo" comultiplication by applying it in the first (or second) part of the resulting tensor.

105 Example

Coalgebra examples:

- $\Bbbk[x]$ is $\Bbbk$-vector space with basis $\{1, x, x^2, \ldots\}$. Define $\Delta(x) = 1 \otimes x + x \otimes 1$, $\Delta(1) = 1 \otimes 1$, extend to be a ring homomorphism. Hence $\Delta(x^n) = (1 \otimes x + x \otimes 1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}$. What $\epsilon\colon \Bbbk[x] \to \Bbbk$ works with this counit? Trace through the diagram to see what it must be; turns out $\epsilon(1) = 1$, $\epsilon(x) = 0$.
- For $\Bbbk[x_1, x_2, \ldots, x_n]$ with $\Delta(x_i) = 1 \otimes x_i + x_i \otimes 1$; call an element with this type of comultiplication $\boxed{\text{primitive}}$.

April 18th, 2014: Skew Schur Functions and Comultiplication; Sweedler Notation; $\Bbbk$-Coalgebra Homomorphisms

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106 Remark

Correction: in $A \otimes B$, we have $u_{A\otimes B}(1) = 1_{A\otimes B} = 1_A \otimes A_B$, so $u_{A\otimes B}(c) = c(1_A \otimes 1_B)$, which is not equal to $u_A(c) \otimes u_B(c) = c^2(1_A \otimes 1_B)$.

107 Example

At the end of last lecture, we started defining comultiplication on $\Bbbk[x_1, \ldots, x_n]$ by declaring each x_i primitive. For another example, $\mathbb{C}[S_n]$ is a $\mathbb{C}$-vector space, and we can define comultiplication by $\Delta(\sigma) = \sigma \otimes \sigma$. What's the counit? Seems like we're forced to have $\epsilon(\sigma) = 1$.

108 Example

For $\mathrm{SYM}(t) = \mathbb{Q}[p_1(t), p_2(t), \ldots]$, if p_i is primitive, that means $\Delta(p_i) = 1 \otimes p_i(t) + p_i(t) \otimes 1 = \sum_{j\geq 1} 1 \otimes t_j^i + \sum_{j\geq 1} t_j^i \otimes 1$, which you can think of as $p_i(y) + p_i(x)$ where the y variables correspond to $1 \otimes t_j^i$ (i.e. $y_j^i \leftrightarrow 1 \otimes t_j^i$) and the x variables correspond to $t_j^i \otimes 1$. We have $\mathrm{SYM}(t) \otimes \mathrm{SYM}(t) \cong \mathrm{SYM}(x+y)$ under this correspondence, where $x + y := x \cup y$, whence

$$\mathrm{SYM}(t) \otimes \mathrm{SYM}(t) \cong \mathbb{Q}[p_1(x), p_1(y), p_2(x), p_2(y), \ldots].$$

Indeed, this comultiplication comes about from declaring $\Delta(f) = f(x + y)$ for $f \in \mathrm{SYM}$, which is well-defined since f is symmetric so the order in which we plug in variables is irrelevant. That is,

$$\Delta(f)|_{t^\alpha \otimes t^\beta} := f(x + y)|_{x^\alpha y^\beta}.$$

109 Example

What are $h_n(x + y)$ and $e_n(x + y)$? Let's say $x_1 < x_2 < \cdots < y_1 < y_2 < \cdots$. Then it's easy to see

$$h_n(x + y) = \sum_{i=0}^{n} h_i(x)h_{n-i}(y), \qquad e_n(x + y) = \sum_{i=0}^{n} e_i(x)e_{n-i}(y).$$

Likewise we can see

$$\Delta(h_n) = \sum_{i=0}^{n} h_i(t) \otimes h_{n-i}(t), \qquad \Delta(e_n) = \sum_{i=0}^{n} e_i(t) \otimes e_{n-i}(t).$$

Indeed, if $m \colon \mathrm{SYM}(t)\otimes\mathrm{SYM}(t) \to \mathrm{SYM}(x+y)$ is the above isomorphism, we've shown $m\Delta(h_n(t)) = h_n(x + y)$, so by linearity $m\Delta(f(t)) = f(x + y)$ for all $f \in \mathrm{SYM}$.

One can think it through to see $\Delta(s_\lambda) = \sum_{\mu\subset\lambda} s_\mu(t) \otimes s_{\lambda/\mu}(t)$ where...

Definition 110. A $\boxed{\text{skew partition}}$ $\boxed{\lambda/\mu}$ means λ, μ are partitions and $\lambda \supset \mu$. The set $\boxed{\mathrm{SSYT}(\lambda/\mu)}$ of $\boxed{\text{skew tableau}}$ of shape λ/μ consists of fillings of $\lambda - \mu$ from $\mathbb{P}$ which weakly increase along rows and strictly increase along columns. Then the $\boxed{\text{skew Schur function}}$ associated to the skew partition λ/μ is

$$\boxed{s_{\lambda/\mu}} := \sum_{T\in\mathrm{SSYT}(\lambda/\mu)} x^T.$$

Homework: show $s_{\lambda/\mu} = \det(h_{\lambda_i-\mu_j-i+j})$.

111 Theorem

$\Delta(s_\lambda) = \sum_{\mu\subset\lambda;\nu} c^\lambda_{\mu\nu} s_\mu \otimes s_\nu$ if and only if $s_{\lambda/\mu} = \sum_{\mu\subset\lambda} c^\lambda_{\mu\nu} s_\nu$ if and only if $s_\mu s_\nu = \sum c^\lambda_{\mu\nu} s_\lambda$. Hence the Littlewood-Richardson coefficients arise extremely naturally in terms of comultiplication.

Proof Assume the third equality. Use the Cauchy identity twice to see

$$\begin{aligned}\left(\prod_{i,j\geq 1}\frac{1}{1-x_iz_j}\right)\left(\prod_{i,j\geq 1}\frac{1}{1-y_iz_j}\right) &= \left(\sum_\mu s_\mu(x)s_\mu(z)\right)\left(\sum_\nu s_\nu(y)s_\nu(z)\right)\\ &= \sum_{\mu,\nu} s_\mu(x)s_\nu(y)s_\mu(z)s_\nu(z)\\ &= \sum_{\mu,\nu} s_\mu(x)s_\nu(y)\left(\sum_\lambda c^\lambda_{\mu\nu}s_\lambda(z)\right)\\ &= \sum_{\mu,\nu,\lambda} c^\lambda_{\mu\nu}s_\mu(x)s_\nu(y)s_\lambda(z).\end{aligned}$$

On the other hand, suppose $s_{\lambda/\mu} = \sum d^\lambda_{\mu\nu}s_\nu$. Then from the comment in the previous example,

$$\Delta(s_\lambda) = \sum_{\mu\subset\lambda} s_\mu(t)\otimes s_{\lambda/\mu}(t) = \sum_{\mu\subset\lambda} s_\mu(t)\otimes\left(\sum_\nu d^\lambda_{\mu\nu}s_\nu(t)\right) = \sum_{\mu\subset\lambda;\nu} d^\lambda_{\mu\nu}s_\mu(t)\otimes s_\nu(t),$$

(This is essentially the first equation in the theorem statement.) But by the remark in the previous example, $m\Delta(s_\lambda(t)) = s_\lambda(x+y)$, so if we apply m to the previous equation, we find

$$s_\lambda(x+y) = \sum_{\mu\subset\lambda;\nu} d^\lambda_{\mu\nu}s_\mu(x)s_\nu(y).$$

Finally, consider the left-hand side of the first product in this proof, the product of $1/(1-\ell_iz_j)$ where $\ell = x$ or $\ell = y$. Stated this way, we can apply the Cauchy identity to the alphabets $x+y$ and z to get

$$\begin{aligned}\text{LHS} &= \sum_\lambda s_\lambda(x+y)s_\lambda(z) = \sum_\lambda\left(\sum_{\mu,\nu} d^\lambda_{\mu\nu}s_\mu(x)s_\nu(y)\right)s_\lambda(z)\\ &= \sum_{\lambda,\mu,\nu} d^\lambda_{\mu\nu}s_\mu(x)s_\nu(y)s_\lambda(z).\end{aligned}$$

Now compare coefficients of $s_\mu(x)s_\nu(y)s_\lambda(z)$ to see that $c^\lambda_{\mu\nu} = d^\lambda_{\mu\nu}$, completing the proof.

112 Homework

Figure out the "(?)" in terms of other things that we know in

$$\prod_{i,j,k\geq 1}\frac{1}{1-x_iy_jz_k} = \sum_{\lambda,\mu,\nu}(?)s_\lambda(x)s_\mu(y)s_\nu(z).$$

113 Example

What's the counit for the above comultiplication? $\epsilon(p_i) = 0$.

114 Homework

What if we defined the e_i's to be primitive, i.e. $\Delta(e_i) = 1\otimes e_i + e_i\otimes 1$? Is the coalgebra structure really different?

115 Example

Let P be a poset. Recall $\mathrm{Int}(P)$ is the set of $\boxed{\text{intervals}}$ in P, i.e. the set of $[x,y] = \{c : x\leq c\leq y\}$. Define C as the $\Bbbk$-span of $\mathrm{Int}(P)$. For instance, $12[a,a]+6[a,b]+17[a,e]\in C$ for the poset with $a<b<c$, $a<d<e$, $b<e$.

Define the coproduct structure by "breaking chains" or "rock breaking":

$$\Delta([x,z]) := \sum_{x \leq y \leq z} [x,y] \otimes [y,z].$$

Coassociativity works perfectly well, eg.

$$(\mathrm{id} \circ \Delta) \circ \Delta([w,z]) = \sum_{w \leq x \leq y \leq z} [w,x] \otimes [x,y] \otimes [y,z].$$

What's the counit? We must have

$$(\mathrm{id} \otimes \epsilon) \circ (\Delta([x,z])) = [x,z] \otimes 1.$$

Since

$$\Delta([x,z]) = [x,x] \otimes [x,z] + [x,y] \otimes [y,z] + \cdots + [x,z] \otimes [z,z],$$

we're strongly encouraged to make $\epsilon([a,a']) = 1$ if $a = a'$ and 0 otherwise. This is called the $\boxed{\text{incidence coalgebra}}$.

There doesn't seem to be a natural algebra structure—how do you multiply two arbitrary intervals?

116 Notation ($\boxed{\textbf{Sweedler Notation}}$)

Let C be a coalgebra. We have $c = \sum_{i=0}^{n_c} a_i \otimes b_i$ for some $n_c \geq 0$ and $a_i, b_i \in C$. Let $(c) = \{0, \ldots, n_c\}$ and let $c^i_{(1)} = a_i, c^i_{(2)} = b_i$. We can write this as $c = \sum_{i \in (c)} c^i_{(1)} \otimes c^i_{(2)}$ or, making the indexes implicit,

$$\boxed{c = \sum_{(c)} c_{(1)} \otimes c_{(2)}}.$$

Note that $c^i_{(1)} \in C$ for $i \in (c)$, so we can use the same conventions to write

$$c^i_{(1)} = \sum_{(c^i_{(1)})} (c^i_{(1)})_{(1)} \otimes (c^i_{(1)})_{(2)}.$$

This is rather verbose; we can unambiguously drop the i and ignore some parentheses to get $c_{(1)} = \sum_{(c_{(1)})} c_{(1)(1)} \otimes c_{(1)(2)}$. Coassociativity then reads

$$\sum_{(c)} \sum_{(c_{(2)})} c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} = \sum_{(c)} \sum_{(c_{(1)})} c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}.$$

This is still rather verbose; we must sum over (c) to evaluate the $c_{(1)}$ or $c_{(2)}$, so we can drop the outer sum without losing information. Likewise we can drop the inner sum, so a lazy form of Sweedler notation for coassociativity reads

$$\boxed{\Delta_2(c)} := c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}.$$

Here $\Delta_2(c)$ is an $\boxed{\text{iterated comultiplication}}$, which has the obvious definition in general. Up to this point, Sweedler notation has been perfectly well-defined and unambiguous, even though it's very implicit. Some people even drop the parentheses and/or the tensor product symbols!

However, the notation becomes context-sensitive with iterated comultiplication. In particular, we set $\Delta_{(2)}(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$. Compared to $\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$, each of the symbols $(c), c_{(1)}, c_{(2)}$ have new meanings. The presence of (3) is at least a clue as to which definition of the symbols to use. More generally, we write

$$\boxed{\Delta_{n-1}}(c) := \sum_{(c)} c_{(1)} \otimes \cdots \otimes c_{(n)}.$$

Note: using $\sum_{(c)^n} c^n_{(1)} \otimes \cdots \otimes c^n_{(n)}$ would at least be unambiguous, and we could write $(c^n_{(1)})^m_{(2)}$ as $c^{nm}_{(1)(2)}$, though nothing like this seems to be in actual use.

117 Example

In this notation, $(\mathrm{id} \otimes \epsilon) \circ \Delta(c) = c \otimes 1$ becomes

$$\sum_{(c)} c_{(1)} \otimes \epsilon(c_{(2)}) = c \otimes 1 \qquad \text{or} \qquad \sum_{(c)} c_{(1)}\epsilon(c_{(2)}) = c,$$

among other possibilities.

118 Example

$\Delta_{n-1}[x, y] = \sum_{x=x_1 \leq \cdots \leq x_n = y} [x_1, x_2] \otimes [x_2, x_3] \otimes \cdots \otimes [x_{n-1}, x_n]$. $\Delta_{n-1}(\sigma) = \sigma \otimes \cdots \otimes \sigma$ (n terms) for $\sigma \in S_n$ with the previous comultiplication.

Definition 119. Let A, B be $\Bbbk$-algebras. A $\Bbbk$-linear map $f \colon A \to B$ is a $\boxed{\Bbbk\text{-algebra homomorphism}}$ $\Bbbk$-algebra homomorphism if and only if $f(aa') = f(a)f(a')$ and $f(1_A) = 1_B$. In diagrams,

$$\begin{tikzcd}
 & A \otimes A \arrow[dl, "m_A"'] \arrow[dr, "f \otimes f"] & \\
A \arrow[dr, "f"'] & & B \otimes B \arrow[dl, "m_B"] \\
 & B &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \Bbbk \arrow[dl, "u_A"'] \arrow[dd, "u_B"] \\
A \arrow[dr, "f"'] & \\
 & B
\end{tikzcd}$$

Definition 120. Let C, D be $\Bbbk$-coalgebras. A $\Bbbk$-linear map $g \colon C \to D$ is a $\boxed{\Bbbk\text{-coalgebra homomorphism}}$ $\Bbbk$-coalgebra homomorphism if and only if

$$\sum_{(c)} g(c_{(1)}) \otimes g(c_{(2)}) = \sum_{(gc)} (gc)_{(1)} \otimes (gc)_{(2)}$$

and $\epsilon_C = \epsilon_D \circ g$. In diagrams,

$$\begin{tikzcd}
 & D \otimes D & \\
D \arrow[ur, "\Delta_D"] & & C \otimes C \arrow[ul, "g \otimes g"'] \\
 & C \arrow[ul, "g"] \arrow[ur, "\Delta_C"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \Bbbk \\
D \arrow[ur, "\epsilon_D"] & \\
 & C \arrow[ul, "g"] \arrow[uu, "\epsilon_C"']
\end{tikzcd}$$

121 Example

Let C be the $\Bbbk$-span of $T = \{t_1, t_2, \ldots, t_k\}$. Give it a coalgebra structure with $\Delta_C(t_i) = t_i \otimes t_i$ and $\epsilon_C(t_i) = 1$. Let D be the incidence coalgebra for the boolean lattice B_k on T (i.e. the lattice of subsets of T). Then

$$\Delta_D([X, Z]) = \sum_{X \leq Y \leq Z} [X, Y] \otimes [Y, Z],$$

with $\epsilon_D[A, A] = 1$, $\epsilon_D[A, A'] = 0$ if $A \neq A'$.

Let's try out $g \colon C \to D$ with $t_i \mapsto [\{t_i\}, \{t_i\}]$, extended $\Bbbk$-linearly. Is this a $\Bbbk$-coalgebra homomorphism?

$$\begin{aligned}(g \otimes g)(\Delta_C(t_i)) = (g \otimes g)(t_i \otimes t_i) = g(t_i) \otimes g(t_i) &= [\{t_i\}, \{t_i\}] \otimes [\{t_i\}, \{t_i\}] \\ &= \Delta_D(g(t_i)).\end{aligned}$$

The counit also works out:

$$\epsilon_D(g(t_i)) = \epsilon_D([\{t_i\}, \{t_i\}]) = 1 = \epsilon_C(t_i).$$

Indeed, this gives an example of an injective $\Bbbk$-coalgebra homomorphism.

April 21st, 2014: Subcoalgebras, Coideals, Bialgebras

122 Remark

Today: (1) we'll discuss $\ker(f), \mathrm{im}(f)$, coideals, subcoalgebras, etc. (2) we'll introduce bialgebras.

123 Notation

Today, C, D are k-coalgebras, $B \subset C$ is a $\Bbbk$-subspace.

Definition 124. A $\Bbbk$-subspace B is a $\boxed{\text{subcoalgebra}}$ of a $\Bbbk$-coalgebra C if $\Delta(B) \subset B \otimes B$ and $\epsilon_B \colon B \to K$ is given by restricting $\epsilon_C \colon C \to k$. B is a $\boxed{\text{coideal}}$ if $\Delta(B) \subset B \otimes C + C \otimes B$ and $\epsilon(B) = 0$. Note that a subcoalgebra is a coideal iff $\epsilon(B) = 0$.

125 Example

Consider $\Bbbk[G]$ as a coalgebra using the diagonal comultiplication, where recall $\epsilon(g) = 1$. If $g \in G$, then $\Bbbk[g]$ is a subcoalgebra since $\Delta(cg) = cg \otimes g \in \Bbbk[g] \otimes \Bbbk[g]$. However, $\epsilon(\Bbbk[g]) = \Bbbk \neq 0$, so it's not quite a coideal.

What about $\Bbbk[h_n] \subset \mathrm{SYM}_{\Bbbk}$? It's a $\Bbbk$-subspace only. What do we have to add to make it a subcoalgebra? Need $\Bbbk[h_1, \ldots, h_n]$. Recall ϵ annihilates each p_n except $\epsilon(1) = 1$.

What about the $\Bbbk$-span of $\{p_n\}$? It's a coideal, but it's not a subcoalgebra, since $\Delta(p_n) = 1 \otimes p_n + p_n \otimes 1$. We can make it a subcoalgebra if we take the $\Bbbk$-span of $\{1, p_n\}$. Similarly with the $\Bbbk$-span of $\{h_\lambda\}$ where λ is a non-empty partition.

126 Proposition

Let $f \colon C \to D$ be a coalgebra homomorphism.

1. $\ker(f)$ *is a coideal;*
2. $\mathrm{im}(f)$ *is a subcoalgebra.*

Proof First, a lemma.

127 Lemma ("Useful Lemma")

Let $f \colon V \to V'$, $g \colon W \to W'$ be linear maps. Then $f \otimes g \colon V \otimes W \to V' \otimes W'$ satisfies

i) $\mathrm{im}(f \otimes g) = \mathrm{im}(f) \otimes \mathrm{im}(g)$

ii) $\ker(f \otimes g) = \ker(f) \otimes W + V \otimes \ker(g)$

Proof Homework.

Proof of (1): $c \in \ker(f)$ implies $f(c) = 0$, so $0 = \Delta_D \circ f(c) = (f \otimes f)(\Delta_C(c))$, so $\Delta_C(C) \in \ker(f \otimes f) = \ker f \otimes C + C \otimes \ker f$ from the useful lemma, which is precisely what we needed. Also, $\epsilon_C(c) = \epsilon_D(f(c)) = \epsilon_D(0) = 0$, so indeed $\epsilon_C(\ker f) = 0$.

Proof of (2): say $f(c) \in \mathrm{im}(f)$. Then $\Delta f(c) = (f \otimes f)\Delta(c) = \sum_{(c)} f(c_{(1)}) \otimes f(c_{(2)}) \in \mathrm{im}(f) \otimes \mathrm{im}(f)$, which is again what we needed.

128 Proposition

If I is a coideal of C, then C/I is a coalgebra, with $\Delta_{C/I}$ inherited from Δ_C.

Proof Our hope is

$$\begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[d, "\pi"] & C \otimes C \arrow[d, "\pi \otimes \pi"] \\ C/I \arrow[r, dashed, "\Delta_{C/I}", "\exists!"'] & C/I \otimes C/I \end{tikzcd}$$

Claim: $(\pi \otimes \pi) \circ \Delta_C$ descends to C/I if the composite annihilates I. To see this, let $i \in I$, and note $\Delta(i) \in I \otimes C + C \otimes I$, so $(\pi \otimes \pi)(\Delta(i))$ is indeed zero. $\epsilon_{C/I}(a + I) = \epsilon_C(a)$ is well-defined since $\epsilon_C(I) = 0$.

129 Theorem (Fundamental Theorem of Coalgebras)
If $f : C \to D$ is a coalgebra homomorphism, then

$$\mathrm{im}(f) \cong C/\ker(f)$$

as coalgebras.

Proof It's an isomorphism as $\Bbbk$-vector spaces. Both are coalgebras. Check that $\alpha : C/\ker(f) \to \mathrm{im}(f)$ is a morphism of coalgebras; α^{-1} exists, so check it's a morphism of coalgebras.

130 Homework
If $f : C \to D$ is an isomorphism of vector spaces and f is a coalgebra morphism, is it a coalgebra isomorphism?

131 Remark
Recall that we earlier defined a natural $\Bbbk$-algebra structure on the tensor product $A \otimes A$ of a $\Bbbk$-algebra A. Explicitly,

$$\boxed{m_{A\otimes A}} := (m_A \otimes m_A) \circ (1 \otimes T \otimes 1)$$

using the "twist" operator T given by $a \otimes a' \mapsto a' \otimes a$, and $u_{A\otimes A}(k) = k(1_A \otimes 1_A)$. Alternatively, we have

$$\boxed{u_{A\otimes A}} : \Bbbk \to \Bbbk \otimes_{\Bbbk} \Bbbk \xrightarrow{u_A \otimes u_A} A \otimes A$$

where $\Bbbk \xrightarrow{\Bbbk} \otimes_{\Bbbk}\Bbbk$ is the natural isomorphism, since then

$$u_{A\otimes A}(k) \mapsto k \otimes 1_{\Bbbk} = 1_{\Bbbk} \otimes k \mapsto u_A(k) \otimes u_A(1_{\Bbbk}) = u_A(1_{\Bbbk}) \otimes u_A(k) = k(1_A \otimes 1_A) = k(1_{A\otimes A}).$$

Likewise, given a coalgebra C, we can give $C \otimes C$ a natural coalgebra structure with

$$\boxed{\Delta_{C\otimes C}} := (\mathrm{id} \otimes T \otimes \mathrm{id}) \circ \Delta_C \otimes \Delta_C$$

and where

$$\boxed{\epsilon_{C\otimes C}} : C \otimes C \xrightarrow{\epsilon_C \otimes \epsilon_C} \Bbbk \otimes_{\Bbbk} \Bbbk \to \Bbbk.$$

Definition 132. Suppose (B, m, u) is an algebra and further suppose (B, Δ, ϵ) is a coalgebra. Then $(B, m, u, \Delta, \epsilon)$ is a $\boxed{\text{bialgebra}}$ if

(1) m and u are coalgebra homomorphisms, and

(2) Δ and ϵ are algebra homomorphisms

We're roughly saying m and Δ commute, though they don't operate on the same spaces, so that's not quite right. In diagrams, first consider

$$\begin{tikzcd}
B \otimes B \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"'] \arrow[ddrr, dashed, phantom] \arrow[drr, dashed] & B \arrow[r, "\Delta"] & B \otimes B \\
B \otimes B \otimes B \otimes B \arrow[rr, "\mathrm{id}\otimes T\otimes\mathrm{id}"] \arrow[urr, dashed] & & B \otimes B \otimes B \otimes B \arrow[u, "m\otimes m"']
\end{tikzcd}$$

The dashed lines represent $m_{B\otimes B}$ and $\Delta_{B\otimes B}$, so this is just (most of) the requirement that m is a coalgebra homomorphism or Δ is an algebra homomorphism. Similarly we require

$$\begin{tikzcd} B\otimes B \arrow[r, "m"] \arrow[d, "\epsilon\otimes\epsilon"'] \arrow[dr, dashed] & B \arrow[d, "\epsilon"] \\ \Bbbk\otimes_{\Bbbk}\Bbbk \arrow[r, "\sim"'] & \Bbbk \end{tikzcd} \qquad \begin{tikzcd} B\otimes B & B \arrow[l, "\Delta"'] \\ \Bbbk\otimes_{\Bbbk}\Bbbk \arrow[u, "u\otimes u"] & \Bbbk \arrow[l, "\sim"] \arrow[u, "u"'] \arrow[ul, dashed] \end{tikzcd}$$

where the dashed lines are $\epsilon_{B\otimes B}$ and $u_{B\otimes B}$, respectively. With the previous diagram, this makes m a coalgebra morphism and Δ an algebra morphism, respectively. Indeed, the left diagram is (most of) the requirement that $\epsilon\colon B\to\Bbbk$ is an algebra morphism, and similarly for u with the right diagram. The only remaining requirement is

$$\begin{tikzcd} \Bbbk \arrow[rr, "\sim", "\mathrm{id}"'] \arrow[dr, "u"'] & & \Bbbk \\ & B \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

which in addition to the last two diagrams says ϵ is an algebra morphism and u is a coalgebra morphism, since the unit and counit of $\Bbbk$ are id.

133 Proposition

Given $(B, m, u, \Delta, \epsilon, \Bbbk)$, B is a bialgebra if and only if either (1) or (2) holds. That is, it suffices to check that m, u are coalgebra morphisms, or that Δ, ϵ are algebra morphisms.

Proof The diagrams listed above for either set of requirements are exactly the same.

A few more equivalent conditions: (3) is

$$\begin{aligned}\Delta(gh) &= \Delta(g)\Delta(h) \\ &= (\sum_{(g)} g_{(1)}\otimes g_{(2)})(\sum_{(h)} h_{(1)}\otimes h_{(2)}) \\ &= \sum_{(g)}\sum_{(h)} g_{(1)}h_{(1)}\otimes g_{(2)}h_{(2)},\end{aligned}$$

which is also

$$\Delta(gh) = \sum_{(gh)} (gh)_{(1)}\otimes(gh)_{(2)}$$

so in the laziest notation

$$(gh)_{(1)}\otimes(gh)_{(2)} = g_{(1)}h_{(1)}\otimes g_{(2)}h_{(2)}.$$

We also have $\Delta(u(1_{\Bbbk})) = \Delta(1_B) = 1_{B\otimes B} = 1_B\otimes 1_B = u(1_{\Bbbk})\otimes u(1_{\Bbbk})$; $\epsilon(1_B) = 1_{\Bbbk}$; $\epsilon(gh) = \epsilon(g)\epsilon(h)$.

(4) is just the set of four diagrams in the definition above.

Homework: teach this to someone!

April 23rd, 2014: Bialgebra Examples; Hopf Algebras Defined

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134 Remark

Last time: did bialgebras. Today: more examples; Hopf algebras.

135 Example

$\Bbbk[x]$ is a bialgebra with the natural multiplication and with our previous comultiplication, where $\Delta(x^n) = \sum_{k=0}^n \binom{n}{k} x^k \otimes x^{n-k}$:

$$\begin{tikzcd}
x^a \otimes x^b \arrow[r, mapsto, "m"] \arrow[d, mapsto] & x^{a+b} \arrow[r, mapsto, "\Delta"] & \sum_k \binom{a+b}{k} x^k \otimes x^{a+b-k} \arrow[d, leftrightarrow, "=?"] \\
\left(\sum_i \binom{a}{i} x^i \otimes x^{a-i}\right) \otimes \left(\sum_j \binom{b}{j} x^j \otimes x^{b-j}\right) \arrow[rr, mapsto] & & \sum_k \sum \binom{a}{i}\binom{b}{j} x^{i+j} \otimes x^{a+b-i-j}
\end{tikzcd}$$

This commutes since $\binom{a+b}{k} = \sum_{i+j=k} \binom{a}{i}\binom{b}{j}$; this is the $\boxed{\text{Vandermonde convolution}}$. Also check

$$\begin{tikzcd}
1_\Bbbk \arrow[rr, leftrightarrow] \arrow[dr] & & 1_\Bbbk \\
& 1_\Bbbk \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
x^a \otimes \arrow[r, mapsto] \arrow[d, mapsto, "\epsilon\otimes\epsilon"] & x^{a+b} \arrow[d, mapsto, "\epsilon"] \\
1 \otimes 1 \arrow[r, mapsto] & \delta_{a,b,0}
\end{tikzcd}$$

136 Homework

Show $\Bbbk[x_1, \ldots, x_n]$ and SYM_k are bialgebras.

137 Example

The $\boxed{\text{unranked reduced incidence bialgebra}}$ $\boxed{\mathcal{I}}$ as a vector space is the $\Bbbk$-span of isomorphism classes of finite posets with $\widehat{0}$ and $\widehat{1}$. (Here $\widehat{0}$ is a unique minimal element; $\widehat{1}$ is a unique maximal element. Hence such a poset is given by the interval $[\widehat{0}, \widehat{1}]$.) For $I \in \mathcal{I}$, set

$$\Delta(I) := \sum_{i \in I} [\widehat{0}, i] \otimes [i, \widehat{1}]$$

and

$$\epsilon(I) := \begin{cases} 1 & I = \cdot \\ 0 & \text{otherwise} \end{cases},$$

where $\cdot$ denotes an interval of size 1. Define the algebra structure via

$$m(I \otimes J) := I \times J$$

as posets; the right-hand side is $\{(i,j) : i \in I, j \in J\}$ with $(i,j) \leq (i', j')$ iff $i \leq i', j \leq j'$. What's the identity? The singleton poset $\cdot$. To check that this defines a bialgebra, we need to check that $\Delta(I \times J)$"="$\Delta(I) \times \Delta(J)$; note that multiplication in the right is taking place in $\mathcal{I} \otimes \mathcal{I}$. We compute

$$\begin{aligned}
\Delta(I \times J) &= \sum_{(i,j) \in I \times J} [\widehat{0}_{I \times J}, (i,j)] \otimes [(i,j), \widehat{1}_{I \times J}] \\
&= \sum_{(i,j)} [\widehat{0}, i] \times [\widehat{0}, j] \otimes [i, \widehat{1}] \times [j, \widehat{1}] \\
&= m_{I \otimes I}(\Delta(I) \otimes \Delta(J)).
\end{aligned}$$

$\Delta(1_\mathcal{I}) = 1_\mathcal{I} \otimes 1_\mathcal{I} = 1_{\mathcal{I} \otimes \mathcal{I}}$ and $\epsilon(I \times J) = \epsilon(I)\epsilon(J)$ hold too. Hence $\mathcal{I}$ is indeed a bialgebra! Trying to get rid of the finite assumption is confusing/maybe not possible, so let's not.

The $\boxed{\text{reduced incidence Hopf algebra}}$ $\boxed{\mathcal{I}_{\text{ranked}}}$ as a bialgebra is the same as $\mathcal{I}$, except we only allow to ranked posets. Note that $\mathcal{I}_{\text{ranked}}$ is graded by total rank (see below for the definition of graded), that is, by the difference in rank of $\widehat{1}$ and $\widehat{0}$). Hence by Takeuchi (see below) it is a Hopf algebra. The antipode is described further in Ehrenborg's paper below.

138 Example

Let P_n be the $\Bbbk$-span of $w \in S_n$, let $\boxed{P_\infty}$ or $\boxed{\text{Perms}}$ be $\oplus_{n=0}^\infty P_n$ graded. For $u \in S_m, v \in S_n$, define $\boxed{u \sqcup\!\sqcup v} := \sum_{\text{shuffles } s \text{ of } u,v} s$, where a $\boxed{\text{shuffle}}$ is roughly an "interleaving" of the one-line notations for the two permutations, as in the example:

139 Example

Image $[4321] = [7654]$. We have

$$[312] \sqcup\!\sqcup [4321] = 3127654 + 3176254 + 3172654 + 7312654 + \cdots$$

with $35 = \binom{7}{3}$ terms.

What about comultiplication? For $w \in S_n$, use

$$\Delta(w) = \sum_{i=0}^{n} \mathrm{fl}(w|_{[i]}) \otimes \mathrm{fl}(w|_{[i+1,n]})$$

where $\boxed{\mathrm{fl}}$ denotes $\boxed{\text{flattening}}$, meaning if $a_1, \ldots, a_p \in \mathbb{R}^p$ are distinct, use $\mathrm{fl}(a_1, \ldots, a_p) := v_1 \cdots v_p \in S_p$ where $a_i < a_j \Leftrightarrow v_i < v_j$. Here $w|_{[i]} = w_1 w_2 \cdots w_i$ except $w|_{[0]}$ is the "identity" element in S_0, call it $\imath$.

140 Example

$$\Delta(312) = \imath \otimes 312 + \mathrm{fl}(3) \otimes \mathrm{fl}(12) + \mathrm{fl}(31) \otimes \mathrm{fl}(2) + 312 \otimes \imath = \imath \otimes 312 + 1 \otimes 12 + 21 \otimes 1 + 312 \otimes \imath.$$

Again we're more or less forced to pick

$$\epsilon(w) = \begin{cases} 1 & w = \imath \\ 0 & \text{otherwise} \end{cases}$$

So, we have an algebra and a coalgebra! Is it a bialgebra? Is $\Delta(u \sqcup\!\sqcup v)$"="$\Delta(u) \sqcup\!\sqcup \Delta(v)$? We compute

$$\begin{aligned} \Delta\left(\sum_{\sqcup\!\sqcup s \text{ of } u,v} s\right) &= \sum_s \sum_{(s)} \mathrm{fl}(s_{(1)}) \otimes \mathrm{fl}(s_{(2)}) \\ &= u \otimes v + \cdots \\ &= \sum a_1 \ldots a_j \otimes b_1 \ldots b_k, \qquad j + k = n + m \end{aligned}$$

On the other hand,

$$m_{P_\infty \otimes P_\infty}(\Delta(u) \otimes \Delta(v)) = m(\sum_{(u)} \sum_{(v)} u_{(1)} \otimes u_{(2)} \otimes v_{(1)} \otimes v_{(2)})$$

and

$$\Delta(u) \sqcup\!\sqcup \Delta(v) = \sum \sum (u_{(1)} \sqcup\!\sqcup v_{(1)}) \otimes (u_{(2)} \otimes v_{(2)})$$

so these two are equal, and they're equal to $\Delta(u \sqcup\!\sqcup v)$.

141 Homework

Check the other three diagrams.

142 Homework

Define fl on the relative positions of the values $1, \ldots, i$ in one-line notation rather than $w_1, \ldots, w_i$. Set $\Delta(w) = \sum w|_{[i]} \otimes \mathrm{fl}(w|_{[i+1,n]})$. How does P_∞ relate to this structure? Is it the dual? (Note: we did not forget an fl on the first factor: it's just the identity using this new definition.)

143 Remark

$\Bbbk[x]$ and P_∞ are graded; $\Bbbk[x]$ is commutative but P_∞ is not.

Definition 144. A coalgebra C is $\boxed{\text{cocommutative}}$ if

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{(c)} c_{(2)} \otimes c_{(1)}.$$

145 Example

$\Bbbk[x]$ is cocommutative; P_∞ is not. What about $\Bbbk[G]$?

	Commutative	Cocommutative	Graded
$\Bbbk[G]$	no	yes	no
$\Bbbk[x]$	yes	yes	yes
$\mathcal{I}_{\text{ranked}}$	yes	no	yes
P_∞	no	no	yes

Definition 146. A bialgebra $(B, m, u, \Delta, \epsilon, \Bbbk)$ is a $\boxed{\text{Hopf algebra}}$ if there exists a $\Bbbk$-linear function $\boxed{S}\colon H \to H$, called an $\boxed{\text{antipode}}$, such that

$$\begin{tikzcd}
& H \otimes H \arrow[rr, "S \otimes \mathrm{id}"] & & H \otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\epsilon"] & & \Bbbk \arrow[rr, "u"] & & H \\
& H \otimes H \arrow[rr, "\mathrm{id} \otimes S"] & & H \otimes H \arrow[ur, "m"'] &
\end{tikzcd}$$

or in Sweedler

$$\sum S(h_{(1)}) \cdot h_{(2)} = (u \circ \epsilon)(h) = \sum h_{(1)} \cdot S(h_{(2)}).$$

147 Remark

Recall a bialgebra axiom is that $\Bbbk \xrightarrow{u} H \xrightarrow{\epsilon} \Bbbk$ is the identity, i.e. $\epsilon \circ u = \mathrm{id}$. An antipode gives us a way to compute the opposite composition in terms of mutliplication and comultiplication, i.e.

$$u \circ \epsilon = \Delta \circ (\mathrm{id} \otimes S) \circ m = \Delta \circ (S \otimes \mathrm{id}) \circ m.$$

148 Example

For $\Bbbk[G]$, use $S(g) = g^{-1}$.

April 25th, 2014: Properties of Antipodes and Takeuchi's Formula

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Summary Last time: a bialgebra B is a Hopf algebra provided there exists a $\Bbbk$-linear function $S\colon B \to B$ such that

$$\sum_{(h)} S(h_{(1)}) h_{(2)} = u \circ \epsilon(h) = \sum_{(h)} h_{(1)} S(h_{(2)})$$

for all $h \in B$. S is called the antipode.

149 Remark

Today:

1) Properties of antipodes.

2) Hopf algebras are plentiful, i.e. Takeuchi's theorem.

150 Homework
Is every bialgebra a Hopf algebra?

Definition 151. Let C be a $\Bbbk$-coalgebra (Δ, ϵ) and A be a $\Bbbk$-alegbra (m, u). The set $\mathrm{Hom}_{\Bbbk}(C, A)$ of $\Bbbk$-linear maps from C to A can be given an algebra structure as follows, which will be denoted $\boxed{\mathrm{Hom}(C, A)}$. It is a $\Bbbk$-vector space as usual. Define the algebra multiplication through $\boxed{\text{convolution}}$

$$\boxed{(f * g)(c)} := \sum_{(c)} f(c_{(1)})g(c_{(2)}),$$

i.e. $f * g = m \circ (f \otimes g) \circ \Delta$. What is the identity of this algebra, if it has one? We find $1_{\mathrm{Hom}(C,A)} = u \circ \epsilon$:

$$\begin{aligned} f * (u \circ \epsilon)(c) &= m \circ (f \otimes (u \circ \epsilon)) \left(\sum_{(c)} c_{(1)} \otimes c_{(2)} \right) = m \left(\sum_{(c)} f(c_{(1)}) \otimes u\epsilon(c_{(2)}) \right) \\ &= m \left(\sum_{(c)} \epsilon(c_{(2)}) f(c_{(1)}) \otimes u(1_{\Bbbk}) \right) = \sum_{(c)} \epsilon(c_{(2)}) f(c_{(1)}) = f \left(\sum_{(c)} (c_{(1)} \epsilon(c_{(2)}) \right) \\ &= f(c), \end{aligned}$$

and likewise on the other side.

Warning: the multiplicative identity of $\mathrm{Hom}(C, A)$ is *not* in general the identity map even if $C = A$.

152 Notation
$\mathrm{Hom}(U, V), \mathrm{Hom}(C, D), \mathrm{Hom}(A, B), \mathrm{Hom}(C, A)$ can be ambiguous or hard to decipher: they may be $\Bbbk$-linear, coalgebra, algebra or bialgebra, or $\Bbbk$-linear or algebra maps, respectively. Such is life :(.

153 Proposition
Let B be a bialgebra. Let $\mathrm{id}\colon B \to B$. S is an antipode for B if and only if

$$S * \mathrm{id} = u \circ \epsilon = 1_{\mathrm{Hom}(B,B)} = \mathrm{id} * S.$$

Phrased this way, a Hopf algebra is precisely a bialgebra where the identity map in $\mathrm{Hom}(B, B)$ has a two-sided multiplicative inverse.

Proof Immediate from the definitions.

154 Corollary
If S exists, it's unique.

Proof If S, S' are antipodes, then $S = S * (\mathrm{id} * S') = (S * \mathrm{id}) * S' = S'$ since $S * \mathrm{id}$ and $\mathrm{id} * S'$ are both $u \circ \epsilon$, which is the identity. Really this is just the uniqueness of two-sided inverses: S and S' are inverses of id under the convolution.

155 Proposition
In any Hopf algebra H,

(1) $S(gh) = S(h)S(g)$

(2) $S \circ u = u$

(3) $\epsilon \circ S = \epsilon$

(4) $\Delta \circ S(h) = \sum_{(h)} S(h_{(2)}) \otimes S(h_{(1)}) = S \otimes S \circ T \circ \Delta(h)$

Proof (1) Consider $\Bbbk$-linear maps in $\mathrm{Hom}(H \otimes H^c, H^c)$ (where H^c indicates we're thinking of the coalgebra structure on H). We have three maps

$$\begin{aligned} m(g \otimes h) &= gh \\ N(g \otimes h) &:= S(h)S(g) \\ P(g \otimes h) &:= S(gh). \end{aligned}$$

That is, $N := m \circ (S \otimes S) \circ T$, $P := S \circ m$. Outline: a) $P * m = 1_{\mathrm{Hom}(H \otimes H, H)}$ (where this identity is $u \circ m \circ \epsilon \otimes \epsilon(g \otimes h) = \epsilon(g)\epsilon(h) \cdot 1_H$). b) $m * N = 1_{\mathrm{Hom}(H \otimes H, H)}$. c) left and right inverses agree in an associative algebra with 1, so $P = N$. For a), we compute

$$\begin{aligned} P * m(g \otimes h) &= \sum_{(g \otimes h)} P((g \otimes h)_{(1)}) m((g \otimes h)_{(2)}) \\ &= \sum_{(g)} \sum_{(h)} P(g_{(1)} \otimes h_{(1)}) m(g_{(2)} \otimes h_{(2)}) \\ &= \sum_{(g)} \sum_{(h)} S(g_{(1)} h_{(1)}) g_{(2)} h_{(2)} \\ &= \sum_{(gh)} S((gh)_{(1)})(gh)_{(2)} \\ &= S * \mathrm{id}(gh) = u \circ \epsilon(gh) = \epsilon(g)\epsilon(h) \cdot 1_H \end{aligned}$$

where the second equality is true because

$$\begin{aligned} \Delta_{H \otimes H}(g \otimes h) &= \mathrm{id} \circ T \circ \mathrm{id} \circ (\Delta \otimes \Delta)(g \otimes h) \\ &= \mathrm{id} \circ T \circ \mathrm{id} \sum g_{(1)} \otimes g_{(2)} \otimes h_{(1)} \otimes h_{(2)} \\ &= \sum g_{(1)} \otimes h_{(2)} \otimes g_{(2)} \otimes h_{(2)} \end{aligned}$$

and the fourth equality is true because

$$\begin{aligned} \Delta(gh) &= \Delta(g) \cdot \Delta(h) \\ &= \sum g_{(1)} \otimes h_{(1)} \otimes g_{(2)} \otimes h_{(2)} \\ &= \sum_{(gh)} (gh)_{(1)} \otimes (gh)_{(2)} \end{aligned}$$

For b), we compute

$$\begin{aligned} m * N(g \otimes h) &= \sum_{g \otimes h} m((g \otimes h)_{(1)}) N((g \otimes h)_{(2)}) \\ &= \sum_{(g)} \sum_{(h)} m(g_{(1)} \otimes h_{(1)}) N(g_{(2)} \otimes h_{(2)}) \\ &= \sum_{(g)} \sum_{(h)} g_{(1)}(h_{(1)} S(h_{(2)})) S(g_{(2)}) \\ &= \sum_{(g)} g_{(1)}(\epsilon(h) 1_H) S(g_{(2)}) \\ &= \epsilon(g)\epsilon(h) \cdot 1_H \end{aligned}$$

(2) Similar / homework.

(3) Similar / homework.

(4) Similar / homework.

156 Homework

If H is commutative or cocommutative, then $S \circ S = \mathrm{id}$.

157 Proposition

S cannot have odd order > 1.

Proof Suppose $S^{2k+1} = \mathrm{id}$. Then

$$gh = S^{2k+1}(gh) = S^{2k}(S(h)S(g)) = S^{2k+1}(h)S^{2k+1}(g) = hg$$

so H is commutative, hence S has order 1 or 2, a contradiction.

158 Theorem (Taft 1971)

There are Hopf algebras with antipode of order $2, 4, 6, \ldots, \infty$.

Definition 159. A $\boxed{\text{graded } \Bbbk\text{-bialgebra}}$ $B = \oplus_{n \geq 0} B_n$ is a bialgebra where $B_i B_j \subset B_{i+j}$ and $\Delta(B_n) \subset \oplus_{i+j=n} B_i \otimes B_j$. We say B is $\boxed{\text{connected}}$ if $B_0 \cong \Bbbk$ and this respects the bialgebra structure in the sense that $u \circ \epsilon = \mathrm{id}_{B_0}$. (The direct sum is as $\Bbbk$-vector spaces.)

160 Remark

$\epsilon(B_n) = 0$ for all $n > 0$ for a graded $\Bbbk$-bialgebra, which can be proved by chasing around the counit and coproduct diagrams. Hence H is connected iff $u \circ \epsilon = \mathrm{id}_B$, i.e. iff $u = \epsilon^{-1}$.

The terminology probably comes from topology, where a space is connected if and only if its 0th homotopy group is $\mathbb{Z}$.

161 Theorem ($\boxed{\textbf{Takeuchi's Formula}}$)

A graded connected bialgebra is a Hopf algebra with antipode

$$S = \sum_{n \geq 0} (-1)^n \cdot m^{(n-1)} \circ f^{\otimes n} \circ \Delta^{(n-1)}$$

where $m^0 := \mathrm{id}$, $\Delta^0 := \mathrm{id}$, $m^{-1} := u$, $\Delta^{-1} := \epsilon$, $f := \mathrm{id} - u \circ \epsilon \in \mathrm{Hom}(H^c, H)$, *so* $f(h) = h - \epsilon(h)1_H$.

162 Remark

What does f do to the graded pieces B_n for $n \geq 0$? For $c \in B_0$, we have $f(c) = c - \epsilon(c)1_H = 0$, and $f(B_n) = B_n$ is the identity for $n > 0$. If k is large enough compared to m, we see

$$f^{\otimes k} \Delta^{(k-1)}(B_m) \in \sum_{i_1 + \cdots + i_k = m} B_{i_1} \otimes \cdots \otimes B_{i_k}.$$

If $k > m$, then some $i_j = 0$, so f kills it. Hence the sum is well-defined.

Proof First note

$$\begin{aligned} m^{(n-1)} f^{\otimes n} \Delta^{(n-1)}(h) &= \sum_{(h)} f(h_{(1)}) f(h_{(2)}) \cdots f(h_{(n)}) \\ &= f^{*n}(h). \end{aligned}$$

Hence $S = \sum_{n \geq 0} (-1)^n f^{*n}$, so

$$\begin{aligned} S * \mathrm{id} &= \left(\sum_{n \geq 0} (-1)^n f^{*n} \right) * (f + u \circ \epsilon) \\ &= \sum_{n \geq 0} (-1)^n f^{*(n+1)} + \sum_{n \geq 1} (-1)^n f^{*n} \\ &= f^{*0} = u \circ \epsilon, \end{aligned}$$

and similarly on the other side.

163 Homework

Look at what this formula implies for $\Bbbk[x]$–what is the antipode of x? Also do it for the graded bialgebra we called Perms or P_∞. Also do it for (ranked) posets $\mathcal{I}_{\text{ranked}}$.

April 28th, 2014: Homological Properties of Hopf Algebras: projdim, injdim, gldim, GKdim, ...

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164 Remark

Today is James Zhang's first guest lecture. He'll discuss homological properties of Hopf algebras.

165 Example

Some more examples of Hopf algebras:

- G a group and $k[G]$ the group algebra where k is a field. We have $\Delta g = g \otimes g$ for all $g \in G$, and $S(g) = g^{-1}$. Since this is cocommutative, $S^2 = \mathrm{id}$, which is of course also immediate.
- Algebraic geometry: let G be an algebraic group, $\mathcal{O}(G)$ the regular functions on G; commutative implies $S^2 = \mathrm{id}$.
- Lie theory: if L is a Lie algebra, let $U(L)$ be the universal enveloping algebra, namely the free algebra $\Bbbk\langle L\rangle$ modulo $xy - yx - [x, y]$ for all $x, y \in L$. This is a Hopf algebra with a coproduct given by
$$\Delta(x) = x \otimes 1 + 1 \otimes x$$
for all $x \in L$. Note, for instance, that $\Delta(x^2) = (\Delta x)^2 = x^2 \otimes 1 + 2x \otimes x + 1 \otimes x^2$. Note that $S(x) = -x$ for all $x \in L$. This is again cocommutative, and again we can see directly that $S^2 = \mathrm{id}$.
- $S^2 \neq \mathrm{id}$ sometimes: for "quantum groups", $\mathcal{O}_g(G)$ where G is a semisimple Lie group or $U_q(L)$ where L is a semisimple Lie algebra and $q \in \Bbbk^x$, the order of S depends on q^2. It turns out that $\mathcal{O}_1(G) = \mathcal{O}(G)$ but $U_1(L)$ is not generally $U(L)$. For instance, if $q = -1$, then $S^2 = \mathrm{id}$ (for both?).
- $\mathbb{F}_p[x]/(x^p)$ for p prime has two possible Hopf structures. First comes from $\Delta x = x \otimes 1 + 1 \otimes x$, and second uses $\Delta x = x \otimes 1 + 1 \otimes x + x \otimes x$. Again $S^2 = \mathrm{id}$ in either case.

166 Remark

The homological properties of Hopf algebras are connected to many areas of math: noncommutative algebra, algebraic geometry, invariant theory, string theory. Our goal for this lecture and the next will be to find common features of Hopf algebras, to find hidden invariants, and to see some open questions ~~which will be our homework~~.

167 Notation

H will be a Hopf algebra over $\Bbbk$ with multiplication m, unit μ, comultiplication Δ, counit ϵ, antipode S.

168 Open Problem

If H is Noetherian (as a ring), is S bijective? Open question in general, known in some special cases: graded connected case, finite dimensional case, "p_i", "semi-prime".

Definition 169. Let M be a left H-module. The $\boxed{\text{projective dimension}}$ of M is defined to be

$$\boxed{\operatorname{projdim} M} := \min\{n : 0 \to p^n \to p^{n-1} \to \cdots \to p^0 \to M \to 0\}$$

where the minimum is over projective resolutions of M, i.e. the p^i's are projective modules and the sequence is exact. This may well be ∞. Likewise, the $\boxed{\text{injective dimension}}$ of M is defined to be

$$\boxed{\operatorname{injdim} M} := \min\{n : 0 \to M \to I^0 \to \cdots \to I^n \to 0\}$$

where the minimum is over injective resolutions of M, i.e. the I^i are injective modules and the sequence is exact. Similarly, the $\boxed{\text{global dimension}}$ of H is

$$\boxed{\operatorname{gldim} H} := \max\{\operatorname{projdim} M : M \in H\text{-mod}\}.$$

(You can use injdim instead of projdim.) We also define the $\boxed{\text{Gelfand-Kirillov dimension}}$ or $\boxed{\text{GK dimension}}$ of H to be

$$\boxed{\operatorname{GKdim} H} := \limsup_{n\to\infty} \frac{\log(\dim V^n)}{\log n}$$

where $V \ni 1$ is any finite dimensional $\boxed{\text{generating space}}$ of H, meaning $H = \cup_{n\geq 0} V^n$. This turns out to be independent of V, so long as it exists.

170 Proposition

Assume H is finitely generated over k. Then V exists. In this case,

$$\operatorname{GKdim} H = 0 \Leftrightarrow \dim_{\Bbbk} H < \infty.$$

171 Example

$k[x_1, \ldots, x_n]$ has GKdim of n.

172 Open Problem

If $\operatorname{GKdim} H < \infty$, is $\operatorname{GKdim} H \in \mathbb{N}$?

173 Example

$H = U(L)$ implies $\operatorname{GKdim} H = \operatorname{gldim} H = \dim_k L$.

174 Fact

If A is a commutative finitely generated k-algebra, then $\operatorname{GKdim} A$ is the Krull dimension of A.

175 Open Problem

Is $\operatorname{GKdim} H \geq \operatorname{gldim} H$ if $\operatorname{gldim} < \infty$?

176 Open Problem

Suppose H is Noetherian. Is it true that $\operatorname{GKdim} H \geq \operatorname{Kdim} H$, where Kdim is the Krull dimension for a not necessarily commutative ring?

Definition 177. A Hopf algebra H is $\boxed{\text{Frobenius}}$ if H is finite dimension and the left H-module of the dual vector space H^* is isomorphic to H as a left H-module. (You might call this "left Frobenius".) (See the aside following this lecture for a discussion of H^* in the finite dimensional case.)

178 Homework

(1) There is one Hopf algebra structure on $\Bbbk[x_1, x_2]$ where $\operatorname{char} \Bbbk = 0$ up to isomorphism. (Assume the standard algebra structure, and hunt for coalgebra/bialgebra/Hopf algebra structures.)

(2) There exist two Hopf algebra structures on $\Bbbk[x_1, x_2, x_3]$ up to isomorphism where $\operatorname{char} \Bbbk = 0$.

179 Remark

Homological properties in $\boxed{GK0}$, meaning Hopf algebras with GKdim $= 0$. (H finite implies $\operatorname{GKdim} H = 0$, as noted above.)

180 Theorem (Larson-Sweedler)

A finite dimensional Hopf algebra H is Frobenius.

181 Lemma

Being Frobenius is equivalent to saying $\operatorname{injdim} H = 0$*, where* H *is viewed as either a left or right* H*-module. This is the same as saying* ${}_H H$ *is injective. (*H *finite dimensional?)*

Definition 182. (a) H is $\boxed{\text{Gorenstein}}$ if $\operatorname{injdim} {}_H H = \operatorname{injdim} H_H < \infty$.

(b) H is $\boxed{\text{regular}}$ if $\operatorname{gldim} H < \infty$.

(c) H is $\boxed{\text{semisimple}}$ if $H = \oplus M_{n_i}(D_i)$ is a direct sum of matrix rings over division rings. (This is in analogy with Artin-Wedderburn.)

183 Theorem (Larson-Radford)

Suppose $\operatorname{char} k = 0$ *and* H *is finite dimensional over* $\Bbbk$*. Then the following are equivalent:*

(a) H *is semisimple.*

(b) H *is regular of dimension* (gldim) 0*.*

(c) H *is cosemisimple. (This means* H^* *is semisimple, where* H^* *is the dual Hopf algebra; see below.)*

(d) H *is* $S^2 = \mathrm{id}$*.*

Proof If H is semisimple, every module is both projective and injective, so the global dimension is immediately 0. The rest are harder.

184 Example

This fails if $\operatorname{char} \Bbbk = p > 0$. Take $H = \mathbb{F}_p[x]/(x^p)$; this has non-trivial nilradical, hence is not semisimple. However, $S^2 = \mathrm{id}$ since it's cocommutative.

185 Remark

Is there a more general homological theorem that applies when $S^2 \neq \mathrm{id}$?

186 Theorem (Maschke)

Let G *be a finite group. Take* $H = \Bbbk[G]$ *the group ring of* G *viewed as a Hopf algebra. Then* H *is semisimple if and only if* $\operatorname{char} k \nmid \dim H$*.*

187 Open Problem

Suppose H is Noetherian and $\operatorname{injdim} H_H \leq 1$. Is $\operatorname{GKdim} H \leq 1$?

Aside: Dual Algebras, Coalgebras, Bialgebras, and Hopf Algebras

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Definition 188. Let (A, m, u) be a finite dimensional associative $\Bbbk$-algebra. There is a natural finite dimensional coassociative $\Bbbk$-coalgebra structure, which we'll call (A^*, m^*, u^*) by a minor abuse of notation. More precisely, the comultiplication and counit are given by

$$A^* \xrightarrow{m^*} (A \otimes A)^* \xrightarrow{\sim} A^* \otimes A^*, \qquad A^* \xrightarrow{u^*} \Bbbk^* \xrightarrow{\sim} \Bbbk,$$

where $*$ indicates the $\Bbbk$-dual functor.

Proof Apply the $-* := \operatorname{Hom}_\Bbbk(-, \Bbbk)$ functor to the diagrams defining the algebra structure of A. We see a problem immediately:

$$\begin{tikzcd}
 & A \otimes A \otimes A \arrow[dl, "m \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id} \otimes m"] & \\
A \otimes A \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\
 & A &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (A \otimes A \otimes A)^* & \\
(A \otimes A)^* \arrow[ur, "(m \otimes \mathrm{id})^*"] & & (A \otimes A)^* \arrow[ul, "(\mathrm{id} \otimes m)^*"'] \\
 & A^* \arrow[ul, "m^*"] \arrow[ur, "m^*"'] &
\end{tikzcd}$$

We roughly need to distribute the *'s over the tensor products. For that, first consider the natural map $U^* \times V^* \to (U \otimes V)^*$ given by sending $(f : U \to \Bbbk, g : V \to \Bbbk)$ to the map $u \otimes v \mapsto f(u)g(v)$. This map is $\Bbbk$-bilinear, so factors through $U^* \times V^* \to U^* \otimes V^*$. Indeed, this induced map is injective: $f \otimes g$ is sent to $u \otimes v \mapsto f(u)g(v)$, and if this latter map is always zero, then $f \equiv 0$ or $g \equiv 0$, so $f \otimes g = 0$. Assuming U and V are finite dimensional vector spaces, the dimensions of either side of $U^* \otimes V^* \to (U \otimes V)^*$ are the same, so we have an isomorphism.

Indeed, this argument shows $(U \otimes V)^*$ with the above map from $U^* \times V^*$ satisfies the same universal property as $U^* \otimes V^*$. For instance, we get a map

$$\begin{tikzcd}
A^* \arrow[r, "m^*"] \arrow[dr, dashed, "\exists!", "\overline{m^*}"'] & (A \otimes A)^* \arrow[d] \\
 & A^* \otimes A^*
\end{tikzcd}$$

Using this property many times, we find $\overline{m^*} : A^* \to A^* \otimes A^*$ satisfies the diagram for coassociativity. Likewise $u^* : A^* \to \Bbbk^*$ has an associated map $\overline{u^*} : A^* \to \Bbbk$ using the natural isomorphism $\Bbbk \cong \Bbbk^*$, and one may check with $\overline{m^*}$ it satisfies the diagram for the counit. For convenience, we drop the overlines.

Definition 189. Let (B, Δ, ϵ) be a finite dimensional coassociative $\Bbbk$-coalgebra. There is a natural finite dimensional associative $\Bbbk$-algebra structure we'll call $(B^*, \Delta^*, \epsilon^*)$. More precisely, the multiplication and unit are given by

$$B^* \otimes B^* \xrightarrow{\sim} (B \otimes B)^* \xrightarrow{\Delta^*} B^*, \qquad \Bbbk \xrightarrow{\sim} \Bbbk^* \xrightarrow{\epsilon^*} B^*.$$

Proof The proof is the same as in the previous case, with the same tacit isomorphisms applied to distribute duals over tensor products.

190 Proposition

Let $(B, m, u, \Delta, \epsilon)$ be a finite dimensional $\Bbbk$-bialgebra. The two structures previously defined give a finite dimensional $\Bbbk$-bialgebra $(B^*, \Delta^*, \epsilon^*, m^*, u^*)$.

Proof Treat the compatibility diagrams as before.

191 Proposition

Let $(H, m, u, \Delta, \epsilon, S)$ be a finite dimensional Hopf algebra over $\Bbbk$. The bialgebra previously defined is a finite dimensional Hopf algebra with antipode S^*. This is the $\boxed{\textit{dual Hopf algebra}}$

Proof In this case, $S^* : H^* \to H^*$ requires no implicit isomorphisms to distribute, so we can literally use the dual.

192 Remark

There is a more general construction that works for infinite dimensional algebras, though it's more involved. Note the coalgebra maps can at least be defined even without the finite dimensional hypothesis.

April 30th, 2014: (Left) Integrals; from $GK0$ to $GK1$ and Beyond

193 Remark
James Zhang, second guest lecture.

194 Remark
We'll tacitly assume frequently that $GK0$ is equivalent to finite dimensional. Technically $GK0$ just means GKdim is 0, but as mentioned last time if H is a finitely generated k-algebra, they are equivalent. (Warning: there may be missing assumptions or other minor errors in this lecture. Double-check with the literature to be sure.)

195 Remark
We'll compare the following theorems from last time to infinite versions today.

196 Theorem (Larson-Sweedler)
Every $GK0$ (i.e. finite dimensional) Hopf algebra is Gorenstein of $\operatorname{injdim} 0$.

197 Theorem (Larson-Radford)
Suppose $\operatorname{char} k = 0$ *and* $\operatorname{GKdim} H = 0$. *The following are equivalent:*

(a) H *is semisimple, i.e.* $H = \oplus M_{n_i \times n_i}(D_i)$ *for division rings* D_i.

(b) H *is regular of* $\operatorname{gldim} 0$.

(c) H *is cosemisimple.*

(d) $S^2 = \mathrm{id}$.

Definition 198. A $\boxed{\text{left integral}}$ element in a Hopf algebra H is some $t \in H$ such that

$$ht = \epsilon(h)t, \qquad \forall h \in H.$$

That is, t is an eigenvector of the "trivial character" ϵ. We use $\boxed{\int^\ell}$ for the space of left integrals in H. Note that 0 is trivially left integral, though frequently we don't want to consider it. Of course, this space is a $\Bbbk$-vector space. "'integral' is the most important invariant in the study of $GK0$ Hopf algebras."

199 Lemma
If H has $GK0$, then $\int^\ell$ is one-dimensional, i.e. there is a unique (up to scalar) non-trivial left integral.

200 Example
Let $H = k[G]$ for G a finite group. Then $\int^\ell$ is spanned by $\sum_{g \in G} g$.

201 Notation
At least in the $GK0$ case, we have several abuses of notation:

(i) $\int^\ell$ can refer to an arbitrary non-trivial left integral.

(ii) $\int^\ell$ may also refer to itself as an $H/\operatorname{Ann}_R(\int^\ell)$ module.

(iii) $\int^\ell$ may also refer to the map $H \to H/\operatorname{Ann}_R(\int^\ell)$.

202 Theorem (Maschke Restated)
H is semisimple $\Leftrightarrow \epsilon(\int^\ell) \neq 0$.

203 Remark

We have some general structure:

$\dim/\Bbbk$	GKdim	geometry	algebraic group
$<\infty$	0	points	finite group
∞	1	curve	1-dim alg. group
∞	2	surface	2-dim alg. group
$\vdots$	$\vdots$		
∞	∞		

Definition 204. If M, N are left H-modules, then we define $\boxed{\operatorname{Ext}^1_H(M,N)}$ first as a set: it consists of the short exact sequences $0 \to N \to E \to M \to 0$ up to isomorphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

We can give it an abelian group structure using the $\boxed{\text{Baer sum}}$ construction, which we do not discuss. (It is not clear to me how to give it either an H-module or $\Bbbk$-module structure.) Of course, the usual construction of taking a projective resolution $P^* \to M$, applying $\operatorname{Hom}(-,N)$, and taking homology works.

More generally, we define $\operatorname{Ext}^i_H(M,N)$ as the set of exact sequences $0 \to N \to E_1 \to \cdots \to E_i \to M \to 0$ mod a similar equivalence relation. (The Wikipedia article "Ext Functor" has a few details.)

205 Remark

Standing assumption: H is a finitely generated Noetherian $\Bbbk$-algebra.

206 Theorem

If $\operatorname{GKdim} H = 1$, *then* H *is Gorenstein of* $\operatorname{injdim} H = 1$. *Moreover,* $\operatorname{Ext}^0_H(\Bbbk, H) = \operatorname{Hom}_H(\Bbbk, H) = 0$ *and* $\operatorname{Ext}^1_H(\Bbbk, H)$ *is one-dimensional.*

The analogous question even for $\operatorname{GKdim} H = 2$ *is open.*

Definition 207. The $\boxed{\text{homological integral}}$ of H (f.g. Noetherian) is the H-module

$$\boxed{\int_H^\ell} := \operatorname{Ext}_H^{\operatorname{injdim} H}(\Bbbk, H)_H.$$

208 Remark

If $\operatorname{GKdim} H = 0$, then $\int^\ell \cong \operatorname{Hom}_H(\Bbbk, H)$ via $f \mapsto f(1)$. If H is infinite dimensional over $\Bbbk$, then there does not exist a classical integral of H, so this is a generalization of our previous definition.

209 Corollary

If $\operatorname{GKdim} H = 1$, *then* S *is bijective.*

Proof Complicated, though the key step is the theorem above.

210 Open Problem

If $\operatorname{GKdim} H = 1$, is S^2 of finite order?

211 Example

Fix $q \neq 0$. Consider $k\langle g^{\pm 1}, x\rangle/(gx = qxg)$. Use $\Delta g = g \otimes g$, $\Delta x = x \otimes 1 + g \otimes x$. $\operatorname{GKdim} H = 2$. S^2 has infinite order if $q \notin \sqrt{1}$. In this example, gldim is 2.

For another example (Taft), let $q^n = 1$ and consider

$$k\langle g, x\rangle/(g^n = 1, x^n = 0, gx = qxg).$$

In this example, gldim $= \infty$ since $S^2 \neq \mathrm{id}$.

212 Conjecture

If $\operatorname{GKdim} H = 1$, then H is a finitely generated module over $Z(H)$, the center of H.

213 Theorem (Wu-Zhang)

Suppose $\operatorname{GKdim} H = 1$. *If* $\operatorname{gldim} H < \infty$, *then* $\operatorname{gldim} H = 1$, H *is isomorphic to a finite direct sum of prime rings of* $\operatorname{GKdim} 1$, *and the conclusion of the conjecture holds.*

214 Remark

This is something like the (b) implies (a) part of the Larson-Radford theorem above. The converse is not true: if H is a finite direct sum of prime rings of $\operatorname{GKdim} 1$ (even over $\mathbb{C}$), the global dimension could be infinite. Hence roughly (a) implies (b) does not hold.

215 Theorem (Wu-Zhang)

Suppose conjecture 1 holds. If $S^2 = \mathrm{id}$ *and* $\operatorname{char} \Bbbk = 0$, *then* $\operatorname{gldim} H = 1$.

216 Remark

Together with the previous theorem, roughly (d) $\Rightarrow$ (b) $\Rightarrow$ (a), but the reverse implications do not hold. (The next step, $GK2$, is unclear.)

217 Example

(a) Suppose H has $GK0$. Then $H \otimes \Bbbk[x]$ and $H \otimes \Bbbk[g^{\pm 1}]$ are $GK1$. This allows us to make many $GK0$ Hopf algebras. (The same trick works more generally.)

(b) If $q^n = 1$, take $k\langle g, x\rangle/(g^n = 1, gx = qxg)$ and use the same coproduct as Taft's example above. This is in $GK1$, has $\operatorname{gldim} 1$, and is prime.

218 Conjecture

If $\operatorname{GKdim} H < \infty$, $S^2 = \mathrm{id}$, and $\operatorname{char} k = 0$, then $\operatorname{gldim} H < \infty$.

May 2nd, 2014: Dual Hopf Algebras and QSYM

Summary Sara is lecturing again today. Last time: graded connected bialgebras are Hopf algebras using

$$S(h) = \sum_{n \geq 0} (-1)^n m^{\otimes n-1} f^{\otimes n} \Delta^{n-1}$$

with $f = \mathrm{id} - u \circ \epsilon$.

Today: SYM to QSYM and dual Hopf algebras.

219 Remark

Recall $\mathrm{SYM} = \oplus_{n \geq 0} \mathrm{SYM}_n$, where SYM_n is the $\Bbbk$-span of e_λ for $\lambda \vdash n$. The coproduct satisfies

$$\Delta(e_n) = \sum_{i=0}^{n} e_i \otimes e_{n-i}.$$

We noted this gave rise to

$$e_n(x+y) = \sum_{i=0}^{n} e_i(x)e_{n-i}(y)$$

where $x+y := \{x_1, y_1, x_2, y_2, \ldots\}$ is the union of alphabets x and y.

220 Example

Take the $\boxed{\text{Vandermonde determinant}}$ $\boxed{V(x_1,\ldots,x_n)}$ $:= \prod_{1\le i<j\le n}(x_i - x_j)$. This is antisymmetric, but V^2 is symmetric. We see

$$\sum_{i_1<\cdots<i_n} V^2(x_{i_1},\ldots,x_{i_n}) = V^2[e_n],$$

where the notation on the right is a $\boxed{\text{plethsym}}$ $V^2[e_n]$, meaning you feed in the e_n's as arguments to the symmetric function V^2.

What is the antipode, using Takeuchi's formula?

$$S(e_k) = \sum_{n\ge 0}(-1)^n \sum_{i_1+\cdots+i_n=k, i_j\ge 1} e_{i_1}\cdots e_{i_n}$$

221 Example

$$\begin{aligned}
S(e_0) &= 1\\
S(e_1) &= -e_1\\
S(e_2) &= e_1e_1 - e_2 = x_1^2 + x_1x_2 + \cdots = h_2\\
S(e_3) &= -e_1^3 + e_1e_2 + e_2e_1 - e_3 = -h_3
\end{aligned}$$

One would guess $S(e_k) = (-1)^k h_k$. Using the diagram for the antipode, we find

$$\begin{tikzcd}
 & \sum e_i\otimes e_{n-i} \arrow[r, mapsto, "?"] & (-1)^i h_i \otimes e_{n-i} \arrow[dr, mapsto] & \\
e_n \arrow[ur, mapsto] \arrow[r, mapsto, "\epsilon"] & 0 \arrow[rr, mapsto, "u"] & & \sum(-1)h_i e_{n-i} = 0
\end{tikzcd}$$

(unless $n=0$), and we can do the same going around the other way. We've shown this identity using the Cauchy identity, since $H(-t)E(t) = 1 = E(-t)H(t)$, which (should) give an inductive proof of our guess.

222 Example

What about $S(s_\lambda)$? We have $S(s_{(n)}) = (-1)^n S_{(1^n)}$, $S(s_{(1^n)}) = (-1)^n s_{(n)}$. More generally, $S(s_\lambda) = (-1)^{|\lambda|} S_{\lambda'}$ where λ' is the conjugate of λ.

Proof Use Jacobi-Trudi:

$$s_{\lambda/\mu} = \det(h_{\lambda_i-\mu_j+i-j}) = \det(e_{\lambda'_i-\mu'_j+i-j}).$$

Definition 223 ($\boxed{\textbf{Duality}}$ and Hopf algebras). Suppose V is a vector space over $\Bbbk$. $V^* = \mathrm{Hom}_{\Bbbk}(V,\Bbbk)$ is the space of $\boxed{\text{linear functionals}}$. A linear map $\phi\colon V\to W$ induces a map $\phi^*\colon W^*\to V^*$ as usual. We have a bilinear form $V^*\times V\to\Bbbk$ where $(f,v)\mapsto f(v)$. Note $(\phi^*(g),v) = (g,\phi(v))$.

If C is a coalgebra, $C^* = \mathrm{Hom}(C, \Bbbk)$ is an algebra with multiplication

$$fg(v) = \sum f(v_{(1)})g(v_{(2)}).$$

and unit ϵ^* (where we identify $\Bbbk^*$ and $\Bbbk$).

If A is a finite dimensional algebra, then $(A \otimes A)^* \cong A^* \otimes A^*$, and we can define a coalgebra structure on A^* as

$$(\Delta(f), a \otimes b) = f(ab) = (f, ab)$$

Definition 224. The identity $(A \otimes A)^* = A^* \otimes A^*$ fails in infinite dimensions. As a partial solution, suppose $H = \oplus H_n$ is a graded Hopf algebra of "$\boxed{\text{finite type}}$", meaning each H_n is finite dimensional. Then let

$$H^\circ = \bigoplus_{n \geq 0} H_n^*$$

and use multiplication and coproduct as above on each piece. This gives the $\boxed{\text{dual Hopf algebra}}$ in this case.

225 Remark

If $\{h_i\}$ is a basis for H respecting the graded structure, and $\{f_i\}$ is the dual basis of H° (i.e. union of dual bases for H_n^*'s), so $(f_i, h_j) = \delta_{ij}$, then what are the structure constants? $h_i h_j = \sum c_{ij}^k h_k$ satisfies

$$\Delta_{H^\circ}(f_k) = \sum c_{ij}^k f_i \otimes f_j.$$

The counit in H° kills all but H_0^*.

226 Theorem

SYM *is self-dual.*

Proof $\{s_\lambda\}$ form an orthonormal basis. $\Delta_{\mathrm{SYM}}(s_\lambda^*) = \sum c_{\mu\nu}^\lambda s_\mu^* \otimes s_\nu^*$.

227 Remark

Recall

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T = \mathrm{ch}(V^\lambda),$$

where $\dim(V^\lambda) = f^\lambda = \#\,\mathrm{SYT}(\lambda)$. This suggests we might want to partition $\mathrm{SSYT}(\lambda)$ into pieces induced by $\mathrm{SYT}(\lambda)$.

228 Example

$1117/227/3 \in \mathrm{SSYT}((4,3,1)) \mapsto 1237/456/8 \in \mathrm{SYT}((4,3,1))$. Indeed, $1, 2, 13, 122/14, 17, 22/306 \in \mathrm{SSYT}((4,3,1))$ maps to the same thing. Call this procedure "straightening".

Hence

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} \sum_{S \in \mathrm{SSYT}(\lambda), \mathrm{strait}(S) = T} x^S.$$

The inner sum for fixed T is called F_T, the fundamental quasisymmetric functions, discussed more formally below.

229 Example

F_T for $T = 12/34$ of shape $(2,2)$ gives $x_1x_2x_3x_4 + x_1^2x_2^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2 + \cdots$.

Definition 230. If $f \in \Bbbk[[x_1, x_2, \ldots]]$ is of bounded degree, it is a $\boxed{\text{quasisymmetric function}}$ provided

$$f|_{x_1^{\alpha_1} \cdots x_k^{\alpha_k}} = f|_{x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}}$$

for all $1 \le i_1 < \cdots i_k$ and $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a composition (i.e. $\alpha_j \ge 1$).

$\boxed{\text{QSYM}}$ is the set of all quasisymmetric functions. It comes with a grading $\text{QSYM} = \oplus_{n\ge 0} \text{QSYM}_n$ where QSYM_n consists of the homogeneous quasisymmetric functions of degree n. Note $\text{QSYM}_0 = \Bbbk$. QSYM is a graded connected algebra. The multiplication is just multiplication of power series. The comultiplication $\Delta(f)$ is determined by $f(x \cup y)$, just as for SYM above. This gives a bialgebra structure, so it must be a Hopf algebra.

It comes with a (homogeneous) basis of $\boxed{\text{monomial quasisymmetric functions}}$

$$\boxed{M_\alpha} := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} + \text{ all shifts}$$

where $\alpha \vDash n$. Hence $\dim(\text{QSYM}_n)$ is the number of compositions of n, which is 2^{n-1} by taking subsets of $[n-1]$.

231 Homework

Describe multiplication and coproduct on M_α's, and the antipode too!

232 Remark

Note $F_S = F_T$ is possible. For instance, this holds if $S = 1237/456/8$ and $T = 1237/4568$, which requires a moment's thought. However, F_T is not equal to F_U for $U = 12378/456$. For instance, $x_1^5 x_2^3 \in F_U$ but not in F_T. When are they equal, and when are they not?

Definition 233. Given $T \in \text{SYT}(\lambda)$, define $\boxed{\text{Des}(T)}$, the $\boxed{\text{descent set of } T}$, as the set of j where $j+1$ is lower than j in T (i.e. strictly south, weakly west, in English notation). Note that if $\lambda \vdash n$, $\text{Des}(\lambda) \subset [n-1]$. There is a natural way to associate a composition of n with $\text{Des}(\lambda)$, defined by example below.

234 Example

$\text{Des}(1237/456/8) = \{3, 7\}$, $\text{Des}(1237/4568) = \{3, 7\}$, and $\text{Des}(12378/456) = \{3\}$. Write out $**3|***7|*$ for the first, giving associated composition $3 + 4 + 1$. Similarly we get $3 + 4 + 1$ and $3 + 5$ for the second and third.

Definition 235. The $\boxed{\text{fundamental quasisymmetric function}}$ for $\alpha \vDash n$ is defined as follows. Let $\text{set}(\alpha)$ mean the set corresponding to the composition α. Define

$$F_\alpha := F_{\text{Des}(T)} = \sum_{1 \le i_1 \le \cdots \le i_n} x_{i_1} \cdots x_{i_n}$$

where we require strict inequality $i_j < i_{j+1}$ to hold for $j \in \text{set}(\alpha)$. The previous F_T for $T \in \text{SYT}(\lambda)$ are related to these by letting α be the composition associated to $\text{Des}(T)$.

(Note: this was copied from lecture, but I can't make it make sense. To be updated.) (Reply: We have strict inequality in the case where $j \in \text{set}(\alpha)$.)

236 Theorem (Gessel)

$s_\lambda = \sum_{T \in \text{SYT}(\lambda)} F_{\text{Des}(T)}$.

237 Remark

Expand $s_\mu[s_\lambda] = \sum d_{\mu\lambda}^\nu s_\nu$; this is super hard and the coefficients are unknown except in the most basic of cases.

May 5th, 2014: Rock Breaking, Markov Chains, and the Hopf Squared Map

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238 Remark

Eric and Chris are presenting today on "Hopf algebras and Markov chains: two examples and theory" by Diaconis, Pang, and Ram.

Definition 239. We start with the $\boxed{\text{Hopf squared map}}$ of a bialgebra H,

$$\boxed{\psi^2}\colon H \to H := m\Delta$$

240 Example

$\psi^2(e_n) = \sum_{i=0}^n e_i e_{n-i}$ for SYM. More generally, $\Delta(e_\lambda) = \sum_{\alpha\subset\lambda} e_\alpha \otimes e_{\lambda-\alpha}$ where the α are weak compositions which are subsets of the partition λ in the obvious sense.

241 Example

ψ^2 on $k\langle x_1, \ldots, x_n\rangle$ with coproduct given by x_i primitive: then

$$\Delta(x_{i_1}\cdots x_{i_k}) = \sum_{S\subset\{1,\ldots,k\}} \prod_{j\in S} x_{i_j} \otimes \prod_{j\in S^c} x_{i_j},$$

so that for instance

$$\begin{aligned} m\Delta(x_1x_2x_3) &= \underline{x_1x_2x_3} + x_1\underline{x_2x_3} + x_2\underline{x_1x_3} + x_3\underline{x_1x_2} \\ &\quad + x_1x_2\underline{x_3} + x_1x_3\underline{x_2} + x_2x_3\underline{x_1} + x_1x_2x_3 \\ &= 4x_1x_2x_3 + x_2x_1x_3 + x_3x_1x_2 + x_1x_3x_2 \end{aligned}$$

(The underlined terms come from the S^c term.)

Definition 242 ($\boxed{\textbf{Markov Chains}}$). Let Ω denote a finite "state space". A sequence $\{x_k\}_{k\geq 0}$ is a Markov chain if

$$\mathbb{P}(x_i = y : x_{i-1} = y') = \mathbb{P}(x_i = y : x_0 = y').$$

In words, the probability that you go to a new state only depends on where you are, not when you get there.

243 Example

From the previous example, divide by 2^3 so the coefficients sum to 1. You can construct a Markov chain by giving the conditional probabilities above in a matrix with y vs. y'. For instance, we can use the result of $m\Delta(x_1x_2x_3)$ to fill in such a matrix, where the row corresponding to $x_1x_2x_3$ is includes $1/2, 1/8, 1/8, 1/8, 0, 0, 0, 0$.

244 Remark

Suppose $f = \sum a_i f_i$ where the $f_i\colon \Omega \to \mathbb{R}$ are eigenfunctions (of the transition matrix) with eigenvalues β_i. We can compute the expected value of f after k steps starting at x_0 as

$$\begin{aligned} \mathbb{E}_{x_0}(f(x_k)) &= \sum_i a_i \mathbb{E}_{x_0}(f_i(x_k)) = \sum_i a_i \mathbb{E}_{x_0}(\beta_i f(x_{k-1})) \\ &= \sum_i a_i \beta_i^k \mathbb{E}_{x_0}(f(x_0)). \end{aligned}$$

Definition 245 ($\boxed{\textbf{Rock breaking process}}$). Let (n) be an object of mass n, which can be broken into two parts, (j) on the left and $(n-j)$ on the right. Let the probability that we go from (n) to (j) and $(n-j)$ be $\binom{n}{j}/2^n$. At the next step, (j) breaks up into $(j_1), (j-j_1)$ with probability $\binom{j}{j_1}/2^j$ and $(n-j)$ breaks up into $(j_2), (n-j-j_2)$ with probability $\binom{n-j}{j_2}/2^{n-j}$. Note it's possible for $j=0$, in which case we just get (n).

This is defined on partitions by breaking each part independently and interpreting the result as a partition. Note that 1^n is the final state. For instance, one could ask what the expected time for reaching this state is from some initial state.

246 Remark

Our goal is to find a basis for SYM_n in which the (scaled) comultiplication map $\frac{1}{2^n}\psi^2\colon \mathrm{SYM}_n \to \mathrm{SYM}_n$ corresponds to the transition matrix of the rock breaking process defined above. In the $n=2$ case, our naive basis is e_{1^2}, e_2, and $\psi^2(e_{1^2}) = 4e_{1^2}$, $\psi^2(e_2) = e_{1^2} + 2e_2$. $\psi^2/2^2$ in this basis is

$$\begin{pmatrix} 1 & 0 \\ 1/4 & 1/2 \end{pmatrix}$$

The second row doesn't add up to 1, so this isn't a transition matrix. Since we understand our present basis well, let's just rescale it by some $\phi\colon \{e_{1^2}, e_2\} \to \mathbb{R}_{>0}$ where $\boxed{\hat{e}_\lambda} := e_\lambda/\phi(e_\lambda)$. Start with $\phi(e_{1^2}) = 1 = \phi(e_1)^2$. Then

$$\begin{aligned}\psi^2(\hat{e}_2) &= \phi(e_2)^{-1}(e_{1^2} + 2e_2) = \phi(e_2)^{-1}e_{1^2} + 2\phi(e_2)^{-1}e_2 \\ &= \phi(e_2)^{-1}\hat{e}_{1^2} + 2\hat{e}_2.\end{aligned}$$

The matrix is now

$$\begin{pmatrix} 1 & 0 \\ \phi(e_2)^{-1}/4 & 1/2 \end{pmatrix}$$

forcing $\phi(e_2) = 1/2$.

What about the $n=3$ case? We know $\hat{e}_{1^3}, \hat{e}_{12}$, and we can solve for the remaining element.

247 Theorem

Let H be a Hopf algebra over $\mathbb{R}$ that is either a polynomial algebra or a cocommutative free associative algebra. Also suppose we have a basis B such that $\Delta(c)$ has non-negative coefficients, and given any basis element c of degree greater than 1, then $\boxed{\overline{\Delta}}c)PrimitivelyTwistedCoproduct := \Delta(c) - 1\otimes c - c\otimes 1 \neq 0$.

A result of the paper is that there is a new basis $\hat{B}$ which is a rescaled version of the original such that $\psi^2/2^n\colon \hat{B}_n \to \hat{B}_n$ has as matrix a transition function for some Markov chain.

248 Proposition

For any generator, we have the right eigenvector

$$e(x) = \sum_{n\geq 1} (-1)^{a-1}/am^{[a]}\overline{\Delta}^{[a]}(x)$$

249 Homework

If H is commutative, check ψ^a is an algebra homomorphism. If H is cocommutative, check that ψ^a is a coalgebra homomorphism. (To be clear, $\overline{\Delta}^{[a]} = (\mathrm{id}\otimes\cdots\otimes\overline{\Delta})\circ\overline{\Delta}^{[n-1]}$.

250 Example (Rock-breaking example)

Use $\hat{e_i} = i!e_i$ so $\phi(e_i) = 1/i!$. Now

$$\begin{aligned}\psi^2(\hat{e}_n) &= n!m\left(\sum_{i=0}^n e_i e_{n-i}\right) \\ &= \sum_{i=0}^n \binom{n}{i}\hat{e}_i\hat{e}_{n-i}\end{aligned}$$

In particular, this gives part of the transition matrix for the rock-breaking Markov process.

251 Proposition

If μ, λ are partitions of n, then the right eigenfunction associated to μ evaluated at λ is

$$f_\mu(\lambda) = \frac{1}{\prod_i \mu_i!}\sum\prod_j \frac{\lambda_j!}{a_1(\mu^j)!\cdots a_{\lambda_j}(\mu^j)!},$$

where the sum is over all sets $\{\mu^j\}$ such that μ^j is a partition of λ_j, $\coprod \mu^j = \mu$, and $a_i(\mu^j)$ is the number of parts of size i in μ^j.

252 Example

Say $\mu = 1^{n-2}2$. For any λ, $f_\mu(\lambda) = \sum_j \binom{\lambda_j}{2}$. This has associated eigenvalue $1/2$ and is non-zero on all λ except the trivial partition 1^n.

For instance, if $x_0 = (n)$, we find

$$\begin{aligned}\mathbb{P}(x_k \neq 1^n) = \mathbb{P}(f_\mu(x_k) \geq 1) \leq \mathbb{E}(f_\mu(x_k)) = 1/2^k f_\mu(x_0)\\ = \binom{n}{2}/2^k.\end{aligned}$$

(The inequality uses Markov's inequality.)

Aside: Hopf Algebras for $\Bbbk[x, y, z]$ and $\mathrm{Hom}(A, A)$ Invariants

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253 Proposition

Let A, A' be algebras, C', C coalgebras, all over $\Bbbk$. As before, endow $\mathrm{Hom}_\Bbbk(C, A)$ with an algebra structure. An algebra morphism $A \to A'$ induces an algebra morphism $\mathrm{Hom}_\Bbbk(C, A) \to \mathrm{Hom}_\Bbbk(C, A')$, a coalgebra morphism $C' \to C$ induces an algebra morphism $\mathrm{Hom}_\Bbbk(C, A) \to \mathrm{Hom}_\Bbbk(C', A)$, and these operations are both functorial (the former covariant, the latter contravariant).

In particular, if A and B are bialgebras and $\phi\colon A \to B$ is an isomorphism of bialgebras, then we have

$$\begin{tikzcd} \mathrm{Hom}(A,A) \arrow[r, "\phi\circ-"] & \mathrm{Hom}(A,B) \\ \mathrm{Hom}(B,A) \arrow[u, "-\circ\phi"] \arrow[r, "\phi\circ-"] & \mathrm{Hom}(B,B) \arrow[u, "-\circ\phi"'] \end{tikzcd}$$

and every arrow is an isomorphism. Indeed, the composite $\phi_*\colon \mathrm{Hom}(A, A) \to \mathrm{Hom}(B, B)$ sends id_A to id_B. If A, B are also Hopf algebras with antipodes S_A, S_B, then $\phi_* S_A = S_B$.

254 Proposition

Suppose $\phi\colon A \to B$ is an isomorphism of Hopf algebras with antipodes S_A, S_B. ϕ_* induces an isomorphism

$$\Bbbk[S_A, \mathrm{id}_A] \xrightarrow{\phi_*} \Bbbk[S_B, \mathrm{id}_B]$$

which acts via $S_A \mapsto S_B, \mathrm{id}_A \mapsto \mathrm{id}_B$. (For instance, the left-hand side is isomorphic to a quotient of $\Bbbk[x, x^{-1}]$.)

255 Example

Consider $A = \Bbbk[x_1, x_2, \ldots]$, given the usual Hopf algebra structure where each x_i is primitive. Let $P_n := \mathrm{Span}_\Bbbk\{\prod_{j=1}^n x_{i_j}\}$. If S denotes the antipode, then

$$S(q) = (-1)^n q \qquad \text{if } q \in P_n.$$

Hence $S \otimes S = \mathrm{id} \otimes \mathrm{id}$ for all $q \otimes q \in A \otimes A$, and if $\mathrm{char}\, \Bbbk = 2$, then $S^2 - \mathrm{id}^2 = 0 \in \mathrm{Hom}(A, A)$ trivially.

If $\mathrm{char}\, \Bbbk \neq 2$, then for instance

$$(S \otimes S)(1 \otimes x_1) = 1 \otimes (-x_1) \neq 1 \otimes x_1 = (\mathrm{id} \otimes \mathrm{id})(1 \otimes x_1),$$

whence $S^2(x_1) = -2x_1$ while $\mathrm{id}^2(x_1) = 2x_1$. In this case, $S^2 - \mathrm{id}^2 \neq 0 \in \mathrm{Hom}(A, A)$.

256 Example

Consider $B = \Bbbk[e_1, e_2, e_3]$, given a Hopf algebra structure via $\Delta(e_k) = \sum_{i=0}^{k} e_i \otimes e_{k-i}$ ($e_0 = 1$). The antipode satisfies

$$\begin{aligned} S(e_1) &= -e_1 & S^2(e_1) &= -2e_1 \\ S(e_2) &= e_1^2 - e_2 & S^2(e_2) &= 3e_1^2 - 2e_2 \\ S(e_3) &= 2e_1e_2 - e_1^3 - e_3 & S^2(e_3) &= 6e_1e_2 - 4e_1^3 - 2e_3. \end{aligned}$$

(Here as usual S^2 denotes $S \cdot S \in \operatorname{Hom}(B, B)$.)

In particular, $(S^2 - \mathrm{id}^2)(e_2) = (3e_1^2 - 2e_2) - (e_1^2 + 2e_2) = 2e_1^2 - 4e_2$. This is zero iff $\operatorname{char} \Bbbk = 2$.

257 Remark

If A (with three variables) and B are isomorphic and $\operatorname{char} \Bbbk = 2$, then $S_B^2 - \mathrm{id}_B^2 = 0 \in \operatorname{Hom}(B, B)$.

Proof If $\phi\colon A \to B$ were an isomorphism, the induced map $\phi_*\colon \Bbbk[S_A, \mathrm{id}_A] \to \Bbbk[S_B, \mathrm{id}_B]$ would send $S_A^2 - \mathrm{id}_A^2 = 0$ to $S_B^2 - \mathrm{id}_B^2$.

258 Remark

Identifying B with the subalgebra of symmetric polynomials in three variables generated by e_1, e_2, e_3, we may alternatively use power sum symmetric polynomials p_1, p_2, p_3 as algebraic basis. Recall these are primitive with repect to the comultiplication above. From Newton's Identities, it follows that

$$\begin{aligned} e_1 &= p_1 \\ e_2 &= \frac{1}{2}(p_1^2 - p_2) \\ e_3 &= \frac{1}{6}(p_1^3 - 3p_1p_2 + 2p_3) \end{aligned}$$

Hence if $\operatorname{char} \Bbbk \neq 2, 3$, we may identify x_i with p_i and use these formulas to construct a bialgebra, hence Hopf algebra, isomorphism from A to B.

259 Remark

Some haphazard thoughts: a Hopf algebra A has a "primitive subspace" spanned by primitive elements, where the antipode is particularly well-behaved. There are relations between $S, \mathrm{id}, 1 \in \operatorname{Hom}(A, A)$ on this subspace. Powers of this subspace are also interesting. There are still relations between $S, \mathrm{id}, 1$ on the powers, but I wasn't able to find a nice general pattern. Maps between Hopf algebras send the primitive subspace of the source into the primitive subspace of the target. Perhaps other notions besides "primitive" have similar properties and induce similar "filtrations" which in good situations are non-trivial. The two Hopf algebras above both have at least one non-zero primitive element, which seems important.

Addendum from Darij Grinberg: a commutative cocommutative graded connected Hopf algebra in characteristic 0 is the symmetric algebra of its space of primitive elements, which follows, for instance, from the Milnor-Moore theorem. There is more freedom in positive characteristic, for instance using $\Bbbk[x]/(x^p)$ for $\operatorname{char} \Bbbk = p$.

May 7th, 2014: Chromatic Polynomials and applications of Möbius inversion

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260 Remark

Kolya and Connor are lecturing today on Stanley's "Symmetric Function Generalization of the Chromatic Polynomial of a Graph".

Definition 261. Let G be a graph with vertices V, edges E. A $\boxed{\text{coloring}}$ of G is a function $k\colon V \to \mathbb{P}$; a $\boxed{\text{proper coloring}}$ is a coloring such that for all $ab \in E$, $k(a) \neq k(b)$, i.e. no vertexes connected by an edge have the same color. $\boxed{\chi_G(n)}$ is the number of colorings $V \to [n]$.

262 Example

(1) Take a (connected) tree G with d vertices, so $d-1$ edges. Claim: $\chi_G(n) = n(n-1)^{d-1}$. One can see this by deletion-contraction (below) and induction. Alternatively, simply color some vertex one of n colors, so each adjacent vertex can have $(n-1)$ colors, and we will always have $(n-1)$ colors for the next vertex since we have no cycles.

(2) Take G with four vertexes, with one being a "central" vertex connected to each of the others, and no other edges. Values of $\chi_G(n)$ for $n = 1, 2, 3, \ldots$ are $0, 2, 24, \ldots$.

263 Proposition
Given a graph G, for any edge e in G,

$$\chi_G = \chi_{G-e} - \chi_{G/e}$$

where G/e means G with the edge e "contracted" (i.e. its two vertices identified) and $G-e$ means G with the edge e removed.

Proof Easy ennumerative exercise.

Definition 264. Given a graph $G = (V, E)$, define the $\boxed{\text{chromatic polynomial}}$

$$\boxed{X_G} := \sum_{\kappa} x_{\kappa(v_1)} \cdots x_{\kappa(v_n)}$$

where the sum is over proper colorings of G and the product is over all vertices. This is symmetric and homogeneous of degree $\#V$.

265 Example
Let G be a "bow tie" graph, where the crossing point is a vertex. Then

$$X_G = 4m_{(2,2,1)} + 24m_{(2,1,1,1)} + 120m_{(1,1,1,1,1)}$$

(using the monomial symmetric functions).

266 Example
Let G be a square with one diagonal edge added in. Add a vertex with a single edge to one of the box's vertices which wasn't used when adding the diagonal edge. Then X_G is the same as for the previous example, so $G \mapsto X_G$ is not in general injective.

Definition 267. A $\boxed{\text{stable partition}}$ of $V(G)$, the vertex set of G, is a set partition of $V(G)$ where no two vertices in the partition are connected by an edge. We set $\boxed{a_\lambda}$ as the number of stable partitions of $V(G)$ of "type" $\lambda \vdash n$ (where $n = \#V$). Now $X_G = \sum_{\lambda \vdash n} a_\lambda \tilde{m}_\lambda$ where $\tilde{m}_\lambda = m_\lambda r_1! r_2! \cdots$ if $\lambda = (r_1, r_2, \ldots)$.

Definition 268. Let $\boxed{\mathcal{G}}$ be the k-span of $[G]$ where G is a finite simple graph. Define multiplication by cartesian product. Define a coproduct as follows:

$$\Delta([G]) = \sum_{V_1 \coprod V_2 = V(G)} G|_{V_1} \otimes G|_{V_2},$$

where $G|_{V_1}$ refers to the induced subgraph of G with respect to V_1. That is, remove all other vertices not in V_1.

269 Example

If G is a triangle with an edge removed, then $\Delta(G)$ is $1\otimes[G]+2(\cdot\otimes -)+[G]\otimes 1+2(-\otimes\cdot)+(\cdot\cdot\otimes\cdot)$.

270 Proposition

The map $\mathcal{G}\to \mathrm{SYM}$ given by $G\mapsto X_G$ is a Hopf algebra morphism. Indeed, $\mathcal{G}$ is graded by number of vertices and connected, so is a Hopf algebra.

271 Theorem (Humpert-Martin)

Let M be the $\boxed{\text{graphic matroid}}$ *of G. (It is the set of subsets F of edges such that if v and v' are connected in that subset, possibly by a sequence of several edges, then vv' is in that subset.) Then the antipode is*

$$S[G]=\sum_F(-1)^{|V|-|F|}acyc(G/F)G_{V,F}$$

where acyc refers to the number of $\boxed{\text{acyclic orientations}}$ *and the sum is over F in the graphic matroid.*

272 Theorem (Reiner)

$\mathcal{G}$ is self-dual via $[G]\mapsto\sum[H]^*$ where the sum is over all H where $V(H)=V(G)$ and $E(H)\cap E(G)=\varnothing$. For instance, the triangle with an edge removed maps to the sum of three disjoint points's dual plus $-\coprod\cdot$'s dual.

273 Theorem

$$X_G=\sum_{S\subset E}(-1)^{\#S}p_{\lambda(S)}$$

where $\lambda(S)$ is a partition whose parts are the sizes of components of G_S, and the p's are power symmetric functions.

Proof Let K_S be the set of colorings which are monochromatic on components of G_S. Set $x^\alpha = x_{\alpha(v_1)}\cdots x_{\alpha(v_n)}$. Then

$$\sum_{\alpha\in K_S}x^\alpha=p_{\lambda(S)}$$

since we get a factor of p_m for each component of size m. The right-hand side of the formula in the theorem is

$$\sum_{S\subset E}(-1)^{\#S}\sum_{\alpha\in K_S}x^\alpha=\sum_{\beta\colon V\to\mathbb{P}}x^\beta\sum_{S\subset E_\beta}(-1)^{\#S}$$

Here, we have reversed the order of the sum with $E_\beta:=\{(u,v)\in E:\beta(u)=\beta(v)\}$. This equality oocurs because β is monochromatic on the components of G_S if and only if all edges in S are between vertices of the same color in β. Hence,

$$\sum_{S\subset E}(-1)^{\#S}p_{\lambda(S)}=\sum_{\beta\colon V\to\mathbb{P}}x^\beta\delta_{E_\beta=\varnothing}$$

However, $E_\beta=\varnothing$ if and only if β is a proper coloring. Hence, we conclude that

$$X_G=\sum_{S\subset E}(-1)^{\#S}p_{\lambda(S)}.$$

Definition 274. Let $\mathcal{I}$ be our interval bialgebra as before on some poset P, where the coproduct of an interval was given by breaking it up into two pieces in all possible ways. Let $\mathrm{Hom}(\mathcal{I},\Bbbk)$ be an algebra using convolution (pre- and post-composing with $\Delta\colon\mathcal{I}\to\mathcal{I}\otimes\mathcal{I}$ and $m\colon\Bbbk\otimes\Bbbk\to\Bbbk$, using $-\otimes\mathrm{id}$ in between), which in particular has identity

$$1([s,t])=\delta_{s=t}$$

The $\boxed{\text{zeta function}}$ is $Z([s,t])=\delta_{s\le t}$. The $\boxed{\text{Möbius function}}$ is the inverse of the zeta function in this algebra,

$$\mu Z=Z\mu=1.$$

275 Proposition

$\mu(s,s) = 1$ and

$$\mu([s,t]) = -\sum_{s \le x \le t} \mu([s,x]).$$

276 Proposition

For any $f, g\colon P \to \Bbbk$, we have

$$g(t) = \sum_{s \le t} f(s) \qquad \Leftrightarrow \qquad f(t) = \sum_{s \le t} g(s)\mu(s,t).$$

This is called $\boxed{\textit{Möbius inversion}}$.

Definition 277. Define the $\boxed{\text{lattice of contractions}}$ L_G as the set of connected (set) partitions of the vertices of a graph G ordered by refinement; it is a sublattice of the partition lattice. The type is as usual the obvious associated integer partition of $\#V(G)$.

278 Example

For the square with four vertices (and four edges), the elements of L_G with three blocks are

$$12|3|4 \qquad 13|2|4 \qquad 1|3|24 \qquad 1|2|34$$

279 Theorem

We have $X_G = \sum_{\pi \in L_G} \mu([0,\pi]) p_{type(\pi)}$, where $\mu([0,\pi])$ is the Möbius function on L_G.

Proof Use Möbius inversion. For $\sigma \in L_G$, let

$$Y_\sigma = \sum_{\alpha \in J_\sigma} x^\alpha$$

where $\alpha \in J_\sigma$ iff α is a coloring where the blocks of σ are precisely the monochromatic connected pieces of α. Now

$$p_{\text{type}(\pi)} = \sum_\alpha x^\alpha = \sum_{\sigma \ge \pi} Y_\sigma$$

where the α are monochromatic on blocks of π. One can see this by conditioning on what the monochromatic connected blocks of α are. By Möbius inversion,

$$Y_\sigma = \sum_{\sigma \ge \pi} \mu(\pi,\sigma) p_{\text{type}(\sigma)}.$$

Now notice $Y_{\hat{0}} = \sum_\alpha x^\alpha$ where the α's are monochromatic connected pieces are $1|2|\cdots|n$. But this is just equivalent to being a proper coloring, so

$$X_G = \sum_{\pi \in L_G} \mu(\hat{0},\pi) p_{\text{type}(\pi)}$$

280 Example (Forests)

Let G be a forest. We show $X_G = \sum_{\lambda \vdash d} e_\lambda b_\lambda p_\lambda$ where $e_\lambda = (-1)^{d-|\lambda|} = \mu(0,\pi)$ for any $\text{type}(\pi) = \lambda$ and b_λ is the number of connected partitions of type λ. First, observe that L_G is a Boolean algebra, by the poset isomorphism where each connected partition is sent to the edges its blocks contain. It follows that we have the Mobius function $\mu(0,\pi) = (-1)^{d-|\lambda|}$ for any $\text{type}(\pi) = \lambda$. The result then follows from the preceding theorem.

281 Conjecture
$X_G \neq X_H$ for two non-isomorphic trees G, H.

282 Remark
$X_G = X_H$ iff $b_\lambda(G) = b_\lambda(H)$.

Definition 283. Consider some labeling γ on the edges of a graph G. Call $S \subset E$ a $\boxed{\text{broken circuit}}$ if it is a cycle with its largest edge removed, with respect to the labeling γ. Furthermore, let B_G denote the set of all subsets of E which do not contain a broken circuit.

284 Theorem
We have $X_G = \sum_{S \in B_G} (-1)^{\#S} p_{\lambda(S)}$

Proof From other results, we have that

$$\mu(0, \pi) = \sum_S (-1)^{\#S}$$

where the sum is over all $S \in B_G$ such that G_S has the blocks of π as its connected components. The result then follows from using the preceding theorem.

May 9th, 2014: Reduced Incidence Coalgebras; Section Coefficients; Matroids and Hereditary Classes; Many Examples

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285 Remark
Neil is presenting on "Coalgebras and Bialgebras in Combinatorics" by Joni and Rota. No particular big result; just a lengthy collection of interesting examples and observations.

Definition 286. Recall that if P is a poset, then we can form the $\boxed{\text{incidence coalgebra of } P}$, denoted $\boxed{\mathcal{C}(P)}$, as follows. It is the free $\Bbbk$-vector space spanned by the segments $[x, y]$ of P, with comultiplication defined by breaking up the segments,

$$\Delta[x, y] = \sum [x, z] \otimes [z, y].$$

(For this sum to be finite, P must be locally finite, i.e. P must have finite intervals.)

Definition 287. An equivalence relation $\sim$ on intervals of a poset P is $\boxed{\text{order-compatible}}$ if

$$\{[x, y] - [u, v] : [x, y] \sim [u, v]\}$$

is a coideal. Given such a relation, we define the $\boxed{\text{reduced incidence coalgebra}}$ as the quotient coalgebra $\mathcal{C}(P)/\sim$, abbreviated as RIC.

288 Example
The $\boxed{\text{standard RIC}}$ is the RIC defined by the equivalence relation $[x, y] \sim [u, v]$ iff $[x, y] \cong [u, v]$ as posets.

Definition 289. A collection of $\boxed{\text{section coefficients}}$ $\boxed{(i|j, k)}$ on a set G is a map

$$(i, j, k) \mapsto (i|j, k) \in \mathbb{Z}_{\geq 0}$$

such that

(1) For all i, $\#\{(j, k) : (i|j, k) \neq 0\} < \infty$.

(2) $\sum_k (i|j,k)(k|p,q) = \sum_s (i|s,q)(s|j,p)$; this quantity is called $\boxed{(i|j,p,q)}$, a $\boxed{\text{multisection coefficient}}$.

Intuitively, we "cut i into pieces j and k". Given a coalgebra with basis x_i, we can take $(i|j,k)$ to be the coefficient of $x_j \otimes x_k$ in $\Delta(x_i)$, so long as those coefficients are in $\mathbb{Z}_{\geq 0}$.

Definition 290. If G is a semigroup, $\boxed{\text{bisection coefficients}}$ additionally require

$$\text{(3)} \quad (i+j|p,q) = \sum (i|p_1,q_1)(j|p_2,q_2),$$

where the sum is over $p_1 + p_2 = p$ and $q_1 + q_2 = q$.

291 Example

The binomial coefficients are bisection coefficients: $(n|j,k) = n!/(j!k!)$ if $j+k=n$ and 0 otherwise. The bisection condition is Vandermonde's convolution identity,

$$\binom{i+j}{p} = \sum_{p_1+p_2=p} \binom{i}{p_1}\binom{j}{p_2}.$$

292 Example

We define a system of section coefficients on the incidence coalgebra using

$$([x_1,x_2]|[y_1,y_2],[z_1,z_2]) = \begin{cases} 1 & \text{if } x_1 = y_1, x_2 = z_2, y_2 = z_1 \\ 0 & \text{otherwise} \end{cases}$$

293 Proposition

If there is a map $\epsilon\colon G \to \Bbbk$ which satisfies

$$\sum_j (i|j,k)\epsilon(j) = \delta_{i,k} \qquad \text{and} \qquad \sum_k (i|j,k)\epsilon(k) = \delta_{i,j},$$

then we can form a coalgebra from the free $\Bbbk$-vector space spanned by elements of G with coproduct $\Delta x_i = \sum (i|j,k) x_j \otimes x_k$. This is cocommutative if and only if $(i|j,k) = (i|k,j)$. If we use bisection coefficients, we get a bialgebra structure using $x_i x_j = x_{i+j}$ as our multiplication.

294 Example

Let $\mathcal{B}$ be the boolean lattice on $\mathbb{P}$, and consider $\mathcal{C}(\mathcal{B})$.

(1) The $\boxed{\text{boolean coalgebra}}$ on $\mathcal{B}$ uses

$$\Delta A = \sum_{A_1 \coprod A_2 = A} A_1 \otimes A_2.$$

This arises as a quotient of $\mathcal{C}(\mathcal{B})$ using $[A,B] \sim [C,D]$ if $B - A = D - C$.

(2) The $\boxed{\text{binomial coalgebra}}$ is given as follows. Let $s > 0$. Define $\mathcal{B}_s$ using the $\Bbbk$-vector space $\Bbbk[x_1,\ldots,x_s]$ and comultiplication given by

$$\Delta(x_1^{n_1}\cdots x_s^{n_s}) = \sum \binom{n_1}{m_1}\cdots\binom{n_s}{m_s} x_1^{m_1}\cdots x_s^{m_s} \otimes x_1^{n-m_1}\cdots x_s^{n-m_s},$$

where the sum is over $(m_1,\ldots,m_s) \leq (n_1,\ldots,n_s)$.

This arises as the quotient of $\mathcal{C}(\mathcal{B})$ by the equivalence relation where $[A,B] \sim [C,D]$ if for all $k = 1,\ldots,s$, the number of $i \in B - A$ where $i \equiv_s k$ is the same as the number of $j \in D - C$ where $j \equiv_s k$.

For instance, if $s = 1$, we have $[A,B] \sim [C,D]$ iff $|B-A| = |D-C|$. If $s = 2$, we have $[A,B] \sim [C,D]$ iff the number of even (respectively, odd) integers in $B - A$ is the same as the number in $D - C$.

295 Example
Some coalgebras arise as duals of formal power series.

(1) The $\boxed{\text{divided powers coalgebra}}$ $\mathcal{D}$ is the standard RIC of $\mathcal{C}(\mathbb{Z}_{\geq 0})$ (standard order). One can check that, using a coproduct on $k[x]$ given by

$$\Delta x^n = \sum_{k=0}^{n} x^k \otimes x^{n-k},$$

$\mathcal{D}^* \cong k[[x]]$ as algebras.

(2) The $\boxed{\text{Dirichlet coalgebra}}$ D is defined as follows. Order $\mathbb{P}$ by divisibility. Use $[i,j] \sim [k,l]$ iff $j/i = l/k$. Set $D = \mathcal{C}(\mathbb{P})/\sim$. An isomorphic description is to take the $\Bbbk$-span of formal symbols $\{n^x : n \in \mathbb{Z}_{\geq 0}\}$ using

$$\Delta n^x = \sum_{pq=n} p^x \otimes q^x.$$

This suggests an algebra structure given by $n^x m^x = (nm)^x$, which unfortunately does not give a bialgebra, but does give the identity

$$\Delta(n^x m^x) = \Delta(n^x)\Delta(m^x)$$

whenever $(n,m) = 1$.

In this case, D^* is isomorphic as an algebra to the algebra of formal Dirichlet series via $f \mapsto \sum_n f(n^s)/n^s$.

Definition 296. Let S be a set. A $\boxed{\text{matroid}}$ $M(S)$ is a "closure relation" on S, meaning a map associating a subset A of S to its "closure" $\overline{A}$, such that the following holds:

1. $A \subset \overline{A}$; $\overline{\overline{A}} = \overline{A}$; $A \subset B$ implies $\overline{A} \subset \overline{B}$; and
2. If $A \subset S$ and $p, q \in S$, and $p \in \overline{A \cup \{q\}}$, then $p \in \overline{A}$ implies $q \in \overline{A \cup \{p\}}$.

Definition 297. Let $A, B \in M(S)$ with $A \subset B$. A $\boxed{\text{segment}}$ $M(A, B; S)$ is the matroid defined on $B - A$ as follows. If $C \subset B - A$, then $\tilde{C} := \overline{C \cup A} - A$.

Isomorphism classes of matroids are called $\boxed{\text{types}}$; lattices of closed sets are called $\boxed{\text{geometric lattices}}$ (i.e. starting from a matroid, form a lattice by inclusion using the closure of the union).

Definition 298. Given a geometric lattice L, let S be the $\boxed{\text{atoms}}$ of L, that is, the elements covering $\widehat{0}$. For $A \subset S$, let $\overline{A} := \{p \in S : p \leq \sup A\}$, where $\sup A$ is the join of the atoms. The result is the $\boxed{\text{combinatorial geometry}}$ of L.

Definition 299. A $\boxed{\text{hereditary class}}$ H is a family of types of combinatorial geometries such that

(1) If $\alpha, \beta \in H$, then $\alpha + \beta \in H$.

(2) If $[M(S)] \in H$, $A, B \subset M(S)$, then $[M(A, B; S)] \in H$.

(3) If $\alpha \in H$ and $\alpha = \alpha_1 + \alpha_2$, then $\alpha_1, \alpha_2 \in H$.

300 Proposition
Given a hereditary class of matroids, we can form section coefficients $(\alpha|\beta, \gamma)$ as follows. If $[M(S)] = \alpha$, let the section coefficients be the number of closed sets A of such that $\beta = [M(\varnothing, A; S)]$ and $\gamma = [M(A, S; S)]$.

May 12th, 2014—Two Homomorphisms from $\mathcal{I}_{\text{ranked}}$ to QSYM; Möbius Functions on Posets

Summary Hailun is presenting "On Posets and Hopf Algebras" by Ehrenborg. Previous work included Schmidt's "Antipodes and Incidence Coalgebra" from 1987 and Gessel's "Multipartite P-partitions and inner products of skew Schur functions" from 1984. The main results presented today are:

(1) There is a Hopf algebra homomorphism from the reduced incidence Hopf algebra of posets to QSYM.

(2) There is a "twisted" version of the Hopf algebra homomorphism from (1), and a connection to the Zeta polynomial of a poset.

301 Remark

Let $\mathcal{I}_{\text{ranked}}$ be the reduced incidence Hopf algebra defined above. The bialgebra structure and grading have already been given explicitly. The antipode by definition (using Sweedler notation and $\cdot$ for multiplication) satisfies

$$\epsilon(P) \cdot 1 = \sum_{(P)} S(P_{(1)}) \cdot P_{(2)}.$$

This can be thought of as a recursive formula for defining the antipode, made precise in the following recursive version of Takeuchi's formula:

302 Lemma

Suppose $B = \oplus_{n \geq 0} B_n$ is a graded bialgebra (so $B_0 = \Bbbk$, $B_i \cdot B_j \subset B_{i+j}$, and $\Delta(B_n) \subset \oplus_{i+j=n} B_i \otimes B_j$). Then B is a Hopf algebra with antipode recursively defined by $S(1) = 1$ and, for $x \in B_n$ with $n \geq 1$,

$$S(x) = -\sum_{i=1}^{m} S(y_i) \cdot z_i,$$

where

$$\Delta(x) = x \otimes 1 + \sum_{i=1}^{m} y_i \otimes z_i$$

and $\deg y_i < \deg x$.

For convenience, we'll denote $\mathcal{I}_{\text{ranked}}$ by $\boxed{\mathcal{H}}$ today.

303 Remark

Further recall the $\boxed{\text{Möbius function}}$ for a poset P with finite intervals: $\mu \colon P \times P \to \mathbb{Z}$ is given by

$$\mu(x, y) = \begin{cases} -\sum_{x \leq z < y} \mu(x, z) & x < y \\ 1 & x = y \end{cases}$$

(We aren't particularly interested in $\mu(x, y)$ for $x > y$; set it to 0 in that case if you wish.) Write $\boxed{\mu(P)} := \mu(\widehat{0}, \widehat{1})$, assuming $\widehat{0}, \widehat{1}$ exist in P.

Likewise recall the $\boxed{\text{zeta function}}$ on $\mathcal{I}$, given by $Z(P) = 1$ for all P, extended linearly. As before, Z and μ are inverses in the $\text{Hom}(\mathcal{H}, \Bbbk)$ algebra.

In fact, one can check $Z(S(P)) = \mu(P)$, so the antipode is a sort of generalization of the Möbius function.

Definition 304 (QSYM recap). QSYM is a graded bialgebra (a Hopf algebra) as usual; for instance, in the monomial basis $\{M_a\}$ indexed by compositions, the coproduct is "rock breaking". Fundamental basis elements F_a satisfy

$$F_a = \sum_{b \le a} M_b.$$

For convenience, we'll use

$$\boxed{\widetilde{M_a}} := (-1)^{m-k} \sum_{b \le a} M_b = (-1)^{m-k} F_a,$$

where a is a composition of m into k parts.

305 Proposition

The antipode S for QSYM *is given in the monomial basis by*

$$S(M_a) = (-1)^{\ell(a)} \sum_{b \le a} M_{b^*},$$

where if $b = (b_1, \dots, b_k)$, then $b^* := (b_k, \dots, b_1)$, and the sum is over refinements b of the composition a. ($\ell(a)$ denotes the number of parts of a.)

Proof Use the recursive version of the antipode formula from the previous lemma and induct. See Proposition 3.4 in the paper for more details.

306 Remark

The first main result of the paper is the following theorem. It associates to each ranked finite poset P with $\widehat{0}$ and $\widehat{1}$ a corresponding quasisymemtric function, and the resulting map respects the bialgebra structures involved.

Here $\boxed{\rho}(x)$ for $x \in P$ is the rank of P, normalized so $\rho(\widehat{0}) = 0$. We write

$$\rho(x, y) := \rho(y) - \rho(x),$$

which is the rank of the interval $[x, y]$. Similarly $\boxed{\rho(P)} := \rho(\widehat{0}, \widehat{1})$ is the rank of the poset P. (Recall that $\mathcal{H}$ is graded by poset rank.)

307 Theorem

There is a Hopf algebra homomorphism $F \colon \mathcal{H} \to$ QSYM defined as follows:

$$F(P) = \sum M_{(\rho(x_0, x_1), \dots, \rho(x_{n-1}, x_n))},$$

where the sum is over chains $\widehat{0} = x_0 < x_1 < \cdots < x_n = \widehat{1}$.

Proof Clearly $F(1) = 1$, and it's easy to see F commutes with the counit. We must show F respects multiplication and comultiplication.

Note that QSYM is the inverse limit of QSYM_n (where QSYM_n is the same as QSYM but we only use n variables). We have projection maps $\mathrm{QSYM} \to \mathrm{QSYM}_n$, which just set the unused variables to 0. Let F_n be the composite $\mathcal{H} \xrightarrow{F} \mathrm{QSYM} \to \mathrm{QSYM}_n$.

For multiplication, it then suffices to show $F_n \colon \mathcal{H} \to \mathrm{QSYM}_n$ is an algebra homomorphism. One may show that

$$F_n(P; t_1, \dots, t_n) = \sum t_1^{\rho(x_0, x_1)} \cdots t_n^{\rho(x_{n-1}, x_n)},$$

where the sum is over chains as above. Letting $\kappa_i \colon \mathcal{H} \to \Bbbk[t_i]$ via $\kappa_i(P) = t_i^{\rho(P)}$, we see the κ_i are multiplicative, and moreover we have

$$F_n = \mu^n \circ (\kappa_1 \otimes \cdots \otimes \kappa_n) \circ \Delta^n,$$

and each map in the composite is multiplicative.

Finally, we show F is a coalgebra homomorphism. We compute as follows (this isn't nearly as bad as it looks):

$$\begin{aligned}
(F \otimes F) \circ \Delta(P) &= \sum_{\widehat{0} \leq x \leq \widehat{1}} F([\widehat{0}, x]) \otimes F([x, \widehat{1}]) \\
&= \sum_{\widehat{0} \leq x \leq \widehat{1}} \left(\sum_{\widehat{0} = y_0 < \cdots < y_k = x} M_{(\rho(y_0, y_1), \ldots, \rho(y_{k-1}, y_k))} \right) \\
&\qquad \otimes \left(\sum_{x = z_0 < \cdots < z_n = \widehat{1}} M_{(\rho(z_0, z_1), \ldots, \rho(z_{k-1}, z_k))} \right) \\
&= \sum_{\widehat{1} = x_0 < \cdots < x_n = \widehat{0}} \sum_{k=0}^{n} M_{(\rho(x_0, x_1), \ldots, \rho(x_{k-1}, x_k))} \\
&\qquad \otimes M_{(\rho(x_k, x_{k+1}), \ldots, \rho(x_{n-1}, x_n))} \\
&= \sum \Delta(M_{(\rho(x_0, x_1), \ldots, \rho(x_{n-1}, x_n))}) \\
&= \Delta(F(P)).
\end{aligned}$$

Definition 308. Let P be a finite ranked poset with $\widehat{0}, \widehat{1}$. Pick $S \subset [\rho(P)-1]$, and define the $\boxed{\text{rank-selected poset}}$ $\boxed{P(S)}$ by

$$P(S) := \{x \in P : \rho(x) \in S\} \cup \{\widehat{0}, \widehat{1}\}.$$

Further, let $\boxed{\alpha(S)}$ denote the number of maximal chains in $P(S)$.

309 Lemma

Let P be a ranked poset of rank m. Then

$$F(P) = \sum_{S \subset [m-1]} \alpha(S) M_{\omega(S)},$$

where $\omega(S)$ is the composition of m corresponding to S via the "stars and bars" bijection (see below).

Definition 310. Let $\Omega \colon \mathrm{QSYM} \to \mathrm{QSYM}$ be the involution given by

$$\Omega(M_a) = \widetilde{M_a}.$$

Let $\widetilde{F} \colon \mathcal{H} \to \mathrm{QSYM}$ be the sign-twisted version of F given by

$$\widetilde{F}(P) = \sum (-1)^{\rho(P)} \mu(x_0, x_1) \cdots \mu(x_{n-1}, x_n) M_{\rho(x_0, x_1) \cdots \rho(x_{n-1}, x_n)},$$

where the sum is over chains as before.

311 Proposition

The following diagram commutes:

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "F"] \arrow[dr, "\widetilde{F}"'] & \mathrm{QSYM} \arrow[d, "\Omega"] \\
& \mathrm{QSYM}
\end{tikzcd}$$

Proof One main ingredient is Philip Hall's formula for the Möbius function,

$$\mu(a,b) = \sum_{a=y_0<\cdots<y_n=b} (-1)^n.$$

For further details, see Proposition 5.1 of Ehrenborg.

312 Corollary
Malvenuto and Reutenauer showed Ω is a Hopf algebra homomorphism, so $\widetilde{F}$ is as well.

Definition 313. Call the vector $(1,1,\ldots,1,0,0,\ldots)$ with x initial one's and zeros everywhere after 1_x. Note that

$$M_a(1_x) = \binom{x}{\ell(a)},$$

which can be interpreted as a polynomial identity. Hence we may define a $\Bbbk$-linear function $\Psi\colon \mathrm{QSYM} \to \Bbbk[x]$ by "evaluating at 1_x", or more formally by sending M_a to $\binom{x}{\ell(a)}$.

Definition 314. The $\boxed{\text{Zeta polynomial}}$ of a finite poset with $\widehat{0}$ and $\widehat{1}$ is defined to evaluate to the number of multichains from $\widehat{0}$ to $\widehat{1}$ (where a "multichain" is a chain as above but where $\leq$ is used instead of $<$).

315 Proposition
The Zeta polynomial for a poset P as above is given by $\Psi \circ F(P)$.

Proof See proof of Corollary 6.2 in Ehrenborg.

May 14th, 2014: Characters; Combinatorial Hopf Algebras; (QSYM, ζ_Q) as Terminal Object; Even and Odd Subalgebras; Dehn–Sommerville Relations

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Summary Jose is lecturing today on "Combinatorial Hopf algebras and generalized Dehn–Sommerville relations" by Aguiar, N. Bergeron, and Sottile. Today's outline:

(1) Character group

(2) Combinatorial Hopf algebras

(3) (QSYM, ζ) as terminal object

(4) Dehn–Sommerville relations

316 Notation
$(H, m, u, \Delta, \epsilon, S)$ is a graded, connected Hopf algebra throughout this lecture, and our Hopf algebras are generally graded connected today.

Definition 317. A $\boxed{\text{character}}$ of H is a linear map $\zeta\colon H \to \Bbbk$ which is an algebra map, i.e. $\zeta(ab) = \zeta(a)\zeta(b)$ and $\zeta(1) = 1$. We can multiply linear functionals $\phi, \psi\colon H \to \Bbbk$ using convolution:

$$\phi\psi\colon H \xrightarrow{\Delta} H \otimes H \xrightarrow{\phi\otimes\psi} \Bbbk \otimes \Bbbk \xrightarrow{m} \Bbbk.$$

Let $\boxed{\chi(H)}$ be the set of characters on H. The convolution gives it a group structure with identity ϵ and inverse $\zeta^{-1} = \zeta \circ S$.

If H is cocommutative, then $\chi(H)$ is commutative. If $\alpha\colon H' \to H$ is a Hopf algebra morphism, there is an induced morphism $\alpha^*\colon \chi(H) \to \chi(H')$ given by $\zeta \mapsto \zeta \circ \alpha$. Indeed, this is a contravariant functor from the category of Hopf algebras to the category of groups.

Definition 318. Let $\boxed{\overline{(\cdot)}}\colon H \to H$ be the involution defined by $h \mapsto \overline{h} := (-1)^n h$ for $h \in H_n$. This is linear, so induces an automorphism $\chi(H) \to \chi(H)$, which we'll denote $\zeta \mapsto \overline{\zeta}$. Call a character ζ $\boxed{\text{even}}$ iff $\overline{\zeta} = \zeta$, and call it $\boxed{\text{odd}}$ iff $\overline{\zeta} = \zeta^{-1}$.

Definition 319. Let $\boxed{\chi_+(H)}$ be the set of even characters in $\chi(H)$, which is a subgroup in general, and let $\boxed{\chi_-(H)}$ be the set of odd characters, which is a subgroup when H is cocommutative.

A character ζ comes with two other induced characters, namely

$$\begin{aligned} \boxed{\nu} &:= \overline{\zeta}^{-1}\zeta && \text{(odd)} \\ \boxed{\chi} &:= \overline{\zeta}\zeta && \text{(even).} \end{aligned}$$

320 Theorem

Every character ζ can be written uniquely as a product of an even and odd character,

$$\zeta = \zeta_+\zeta_-$$

Definition 321. A $\boxed{\text{combinatorial Hopf algebra}}$ (CHA) is a pair (H, ζ) where H is a Hopf algebra and ζ is a character. We define the category of CHA's using these objects and the following morphisms. $\alpha\colon (H', \zeta') \to (H, \zeta)$ is a morphism when $\alpha\colon H' \to H$ is a graded Hopf algebra morphism and α is compatible with the characters in the sense that

$$\begin{tikzcd} H' \arrow[rr, "\alpha"] \arrow[dr, "\zeta'"'] & & H \arrow[dl, "\zeta"] \\ & \Bbbk & \end{tikzcd}$$

Definition 322. Given a CHA (H, ζ), call ζ the $\boxed{\text{zeta character}}$, call ζ^{-1} the $\boxed{\text{Möbius character}}$, call $\chi = \overline{\zeta}\zeta$ the $\boxed{\text{Euler character}}$, and call $\nu = \overline{\zeta^{-1}}\zeta$ the $\boxed{\text{odd character}}$. (Minor note: $\overline{\zeta^{-1}} = \overline{\zeta}^{-1}$.)

Some motivation for this notation: let R be the Hopf algebra of posets from last time. Then $\zeta(P) := 1$, the Möbius character is $\zeta^{-1}(P) = \mu([0_p, 1_p])$, we can define a simplicial complex from a poset whose Euler characteristic is given by $\chi(P)$, and the odd character is related to the Dehn–Sommerville relations.

Definition 323. The (Hopf) algebra of $\boxed{\text{Non-commutative symmetric functions}}$ $\boxed{\text{NSYM}}$ (also written $\boxed{\mathcal{H}}$) is the free algebra over non-commutative variables $H_1, H_2, \ldots$ graded by $H_i \in \mathcal{H}_i$. Use the "rock-breaking" coproduct

$$\Delta(H_n) := \sum_{i=0}^{n} H_i \otimes H_{n-i}.$$

Here we write H_0 for $1_{\mathcal{H}}$ for convenience. There is a map NSYM $\to$ SYM given by $H_n \mapsto h_n$; the kernel is the commutator subalgebra, so essentially SYM is just making NSYM commutative.

324 Remark

Let H be a Hopf algebra. Recall that $H^* = \oplus_{n=0}^{\infty} H_n^*$ (so long as the graded pieces are each finite dimensional). There is a map

$$\text{NSYM} \to (\text{QSYM})^*$$

given by $H_i \mapsto M^*_{(i)}, H_\alpha \mapsto M^*_\alpha$ where α is a composition and $H_\alpha := H_{\alpha_1} \cdots H_{\alpha_n}$. Indeed, this is an isomorphism!

Definition 325. Define a character $\zeta_Q\colon \text{QSYM} \to \Bbbk$ by

$$M_\alpha \mapsto \begin{cases} 1 & \text{if } \alpha = (n) \\ 0 & \text{otherwise} \end{cases}$$

which is given by evaluating at $(1, 0, 0, \ldots)$.

326 Theorem

(QSYM, ζ_Q) is a terminal object in the category of CHA's. That is, for every CHA (H, ζ), there is a unique morphism $\psi\colon (H, \zeta) \to (\text{QSYM}, \zeta_Q)$. Furthermore, given $h \in H_n$,

$$\psi(h) = \sum_{\alpha \vDash n} \zeta_\alpha(h) M_\alpha$$

where ζ_α is defined as follows. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$, so k is the number of parts of the composition.

$$\begin{tikzcd}
H \arrow[r, "\Delta^{(k-1)}"] \arrow[ddrr, dashed, "\zeta_\alpha"'] & H^{\otimes k} \arrow[r, two heads] & H_{\alpha_1} \otimes \cdots \otimes H_{\alpha_k} \arrow[d, "\zeta^{\otimes k}"] \\
 & & \Bbbk^{\otimes k} \arrow[d, "m^{(k-1)}"] \\
 & & \Bbbk
\end{tikzcd}$$

This composite is a (graded) linear functional, but is not in general a character since it is not in general an algebra homomorphism.

Proof Some definitions first. Let $\zeta_n := \zeta|_{H_n} \in \mathcal{H}_n^*$. Define $\phi\colon$ NSYM $\to \mathcal{H}^*$ given by $H_n \mapsto \zeta_n$. One can check that using the formula for ψ above, $\psi = \phi^*$; implicitly we use the fact that $\mathcal{H}_n^{**} \cong \mathcal{H}_n$ since each graded piece is finite dimensional. Since ϕ is an algebra map, $\phi^* = \psi$ is a coalgebra map. One can also see this gives uniqueness, using the fact that NSYM is a free algebra: any other ψ' can be dualized to give ϕ with the same action on each H_n, hence the same action on all of NSYM; dualize again to get $\psi = \psi'$. The only remaining step is showing ψ is an algebra map.

Indeed, at this point we've shown that (QSYM, ζ_Q) is a terminal object in the category of coalgebras (with characters, which are only assumed to be linear functionals in this case). Running $a \otimes b$ through the top row of the following diagram gives $\psi(a)\psi(b)$:

$$\begin{tikzcd}
H^{\otimes 2} \arrow[r, "\psi^{\otimes 2}"] \arrow[dr, "\zeta^{\otimes 2}"'] & \text{QSYM}^{\otimes 2} \arrow[r, "m"] \arrow[d, "\zeta_Q^2"] & \text{QSYM} \arrow[dl, "\zeta_Q"] \\
 & \Bbbk &
\end{tikzcd}$$

Similary, running $a \otimes b$ through the top row of the next diagram gives $\psi(ab)$:

$$\begin{tikzcd}
H^{\otimes 2} \arrow[r, "m"] \arrow[dr, "\zeta^{\otimes 2}"'] & H \arrow[r, "\psi"] \arrow[d, "\zeta"] & \text{QSYM} \arrow[dl, "\zeta_Q"] \\
 & \Bbbk &
\end{tikzcd}$$

Since (QSYM, ζ_Q) is the terminal object for these coalgebras, the top rows $H^{\otimes 2} \to \text{QSYM}$ must in fact agree, so $\psi(a)\psi(b) = \psi(ab)$.

327 Remark

If you're only interested in cocommutative Hopf algebras, you can replace QSYM with SYM in the above.

Definition 328 (Even and odd subalgebras). Let ϕ, ψ be two characters for H. Where are they equal? Let $\boxed{S(\phi, \psi)}$ be the maximal graded subcoalgebra such that $\psi = \phi$ on $S(\phi, \psi)$. (The arbitrary union of graded subcoalgebras is a subcoalgebra, in analogy to the arbitrary intersection of graded subalgebras being a subalgebra.) Define $\boxed{I(\phi, \psi)}$ be the ideal in H^* generated by $\phi_n - \psi_n$ (again, $\phi_n := \phi|_{H_n}$).

329 Theorem

(i) $S(\phi, \psi) = \{h \in H : f(h) = 0$ *for all* $f \in I(\phi, \psi).\}$

(ii) $I(\phi, \psi)$ *is a graded Hopf ideal (i.e. an ideal, a coideal, and graded).*

(iii) $S(\phi, \psi)$ *is a Hopf subalgebra (i.e. a subalgebra and subcoalgebra).*

(iv) $S(\phi, \psi) = \ker((\mathrm{id} \otimes (\phi - \psi) \otimes \mathrm{id}) \circ \Delta^{(2)})$.

Definition 330. Let $\boxed{S_+(H, \zeta)} := S(\overline{\zeta}, \zeta)$ be the $\boxed{\text{even subalgebra}}$. Roughly, this is the largest sub-Hopf-algebra where ζ is even. Similarly define $\boxed{S_-(H, \zeta)} := S(\overline{\zeta}, \zeta^{-1})$, called the $\boxed{\text{odd subalgebra}}$.

331 Proposition

A quick computation shows $S(\overline{\zeta}, \zeta) = S(\nu, \epsilon)$ *is the even subalgebra, and* $S(\overline{\zeta}, \zeta^{-1}) = S(\chi, \epsilon)$ *is the odd subalgebra. Hence*

$$\begin{aligned} S_-(H, \zeta) &= \ker((\mathrm{id} \otimes (\overline{\zeta} - \zeta^{-1}) \otimes \mathrm{id}) \circ \Delta^{(2)}) \\ &= \ker((\mathrm{id} \otimes (\chi - \epsilon) \otimes \mathrm{id}) \circ \Delta^{(2)}) \end{aligned}$$

Definition 332. Consider QSYM. Let $h = \sum_{\gamma \vDash n} f_\gamma M_\gamma$ be a quasisymmetric function. If we use the first kernel from the previous proposition, we find $h \in S_-(H, \zeta)$ if and only if for all $\alpha = (a_1, \ldots, a_k)$,

$$(-1)^{a_i} f_\alpha = \sum_{\beta \vDash a_i} (-1)^{\ell(\beta)} f_{\alpha 0_i \beta},$$

where $\alpha 0_i \beta := (a_1, \ldots, a_{i-1}, \beta, a_{i+1}, \ldots, a_k)$. Using the second kernel, the condition is instead

$$\sum_{j=0}^{a_i} (-1)^j f_{(a_1, a_2, \ldots, a_{i-1}, j, a_i - j, a_{i+1}, \ldots)} = 0.$$

These two equations are the (generalized) $\boxed{\text{Dehn-Sommerville relations}}$.

333 Example

P is Eulerian if and only if $\mu([x, y]) = (-1)^{\ell(x) - \ell(y)}$.

May 16th, 2014: The Centralizer Theorem; Schur-Weyl Duality; Partition Algebras

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Summary Brendan will be presenting on "Partition algebras" by Halverson-Ram.

334 Theorem (Centralizer Theorem)

Let A be a finite dimensional algebra over an algebraically closed field $\Bbbk$. Suppose M is a semisimple A-module, so $A = \sum_{\lambda \in \widehat{M}} m_\lambda A^\lambda$ for some simple A-modules A^λ and some index set $\widehat{M}$. Set $Z := \mathrm{End}_A(M)$, the A-linear maps $M \to M$, which is a subset of $\mathrm{End}(M)$, the $\Bbbk$-linear maps $M \to M$. Then

(1) The simple Z-modules are indexed by $\widehat{M}$.

(2) As an (A, Z)-bimodule, $M \cong \oplus_{\lambda \in \widehat{M}} A^\lambda \otimes_k Z^\lambda$ for simple Z-modules Z^λ (note: multiplicity free).

(3) $\mathrm{End}_Z(M) \cong A$.

A remark on the name: we have $\phi\colon A \to \mathrm{End}(M)$, and $Z = \{f \in \mathrm{End}(M) : fg = gf, \forall g \in \mathrm{im}\,\phi\}$, so Z is the centralizer of the image of ϕ.

335 Example

Take $V = \mathbb{C}^n$. Then $V^{\otimes k}$ is an S_k-module:

$$\sigma(v_1 \otimes \cdots \otimes v_k) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

Let $A = \mathbb{C}[S_k]$, $M = V^{\otimes k}$. By Maschke's theorem, the hypotheses of the centralizer theorem are satisfied. What is $Z = \mathrm{End}_{S_k}(V^{\otimes k}, V^{\otimes k})$? That is, which linear $T\colon V^{\otimes k} \to V^{\otimes k}$ satisfy $T(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}) = \sigma T(v_1 \otimes \cdots \otimes v_n)$.

The answer is (maybe–there may be an error here) maps of the form $v_1 \otimes \cdots \otimes v_k \mapsto gv_1 \otimes \cdots \otimes gv_k$ for a fixed $g \in \mathrm{GL}(V)$. This is the standard action of $\mathrm{GL}(V)$ on $V^{\otimes k}$ using the diagonal action. So, as an $(S_k, \mathrm{GL}(V))$-bimodule,

$$V^{\otimes k} \cong \bigoplus_{\lambda \in \widehat{M}} S^\lambda \otimes E^\lambda$$

for Specht modules S^λ and simple $\mathrm{GL}(V)$-modules E^λ. Indeed, $\widehat{M}$ is the set of partitions of k with length $\leq \dim V$. This is called $\boxed{\text{Schur-Weyl duality}}$.

336 Example

Consider $V^{\otimes 2} \cong S^2(V) \oplus \Lambda^2(V)$ (for char $\Bbbk \neq 2$), using the decomposition

$$v \otimes w = \frac{1}{2}[(v \otimes w + w \otimes v) + (v \otimes w - w \otimes v)].$$

One can check $S^2(V)$ and $\Lambda^2(V)$ are irreducible $\mathrm{GL}(V)$-modules. As $(S_2, \mathrm{GL}(V))$-modules,

$$V^{\otimes 2} \cong (\mathrm{triv} \otimes S^2(V)) \oplus (\mathrm{sgn} \otimes \Lambda^2(V)).$$

Letting $M = V^{\otimes k}$, $n = \dim V$, we have

$$\mathrm{GL}(V) \supset O(V) \supset S_n$$

where S_n permutes basis elements, and

$$S_k \subset \mathbb{C}B_k(n) \subset \mathbb{C}A_k(n) \subset \mathbb{C}A_{k+1/2}(n)$$

where S_k permutes tensor factors, $\mathbb{C}B_k(n)$ is the Brauer algebra, $\mathbb{C}A_k(n)$ and $\mathbb{C}A_{k+1/2}(n)$ are "partition algebras".

Definition 337. The $\boxed{\text{partition algebra}}$ $\boxed{\mathbb{C}A_k(n)}$ has as $\mathbb{C}$-basis the set partitions of

$$\{1, 2, \ldots, k, 1', 2', \ldots, k'\}.$$

For instance, one element is $\{13, 244'5', 56', 2'3', 6, 1'\}$. One can draw this by labeling vertexes $1, \ldots, n$ in the top row, labeling vertexes $1', \ldots, n'$ in the bottom row, and drawing a path between elements in the same blocks. (There are many such representations in general.) See the paper for many pretty pictures. We "stack" two partitions π, ρ resulting in $\pi * \rho$ by stacking the resulting diagrams vertically, deleting and components that don't go entirely from "top to bottom". We define multiplication of π and ρ as

$$\pi\rho := n^c(\pi * \rho)$$

where c is the number of components we deleted in $\pi * \rho$.

Definition 338. We define an action of $\mathbb{C}A_k(n)$ on $V^{\otimes k}$ as follows. Suppose $v_1, \ldots, v_n$ is a basis of V with $\dim V = n$. Now k-tuples $I \in [n]^k$ index a basis for $V^{\otimes k}$,

$$v_I := v_{i_1} \otimes \cdots \otimes v_{i_k} \qquad \text{if } I = (i_1, \ldots, i_k).$$

View (I, J) as a function $\{1, 2, \ldots, k, 1', 2', \ldots, k'\} \to [n]$ where I corresponds to the unprimed elements and J to the primed elements. For π a partition, define

$$(\pi)_{IJ} = \begin{cases} 1 & \text{if } (I, J) \text{ is constant on blocks of } \pi \\ 0 & \text{else} \end{cases}$$

Then $\pi \colon V^{\otimes k} \to V^{\otimes k}$ via

$$\pi v_I = \sum_J (\pi)_{IJ} v_J.$$

339 Example

Let $\pi = \{1', 122', 33'\}$, $k = 3$, $n = 2$. Then $\pi v_{112} = v_{112} + v_{212}$. Now consider applying the transposition $(2, 1) \in S_2$; this gives $\pi v_{221} = v_{221} + v_{121}$. So, the $\mathbb{C}A_k(n)$-action and the action of S_n commute.

340 Theorem

The above-defined map $\mathbb{C}A_k(n) \to \operatorname{End}(V^{\otimes k})$ has image $\operatorname{End}_{S_n}(V^{\otimes k})$. We can decompose $V^{\otimes k}$ as a $(\mathbb{C}A_k(n), S_n)$-bimodule

$$V^{\otimes k} \cong \bigoplus_{\lambda \in \widehat{A}_k(n)} A_k^\lambda(n) \otimes S^\lambda$$

where the $A_k^\lambda(n)$ are simple modules and the S^λ are Specht modules. As S_n-modules,

$$V^{\otimes k} \cong \bigoplus_{\lambda \in \widehat{A}_k(n)} \dim(A_k^\lambda(n)) S^\lambda.$$

A relatively easy computation gives (as S_n-modules)

$$V^{\otimes k} \cong (\operatorname{Ind}_{S_{n-1}}^{S_n} \operatorname{Res}_{S_{n-1}}^{S_n})^k 1,$$

which is analogous to the Pieri formula.

May 19th, 2014: Internal Product on SYM; P-partitions; Factorizations with Prescribed Descent Sets

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Summary Jair is presenting "Multipartite P-partitions and inner products of skew Schur functions" by Gessel, 1984.

341 Remark

Recall Λ (the symmetric functions in countably many variables $t = t_1, t_2, \ldots$) is a bialgebra using the usual multiplication and the following comultiplication. First view $\Lambda \otimes \Lambda$ as Λ on the alphabet $x \cup y = x_1, y_1, x_2, y_2, \ldots$. Given $f(t)$, we define $\Delta(f(t)) = f(x \cup y)$; since f is symmetric, the order in which we substitute variables is irrelevant, so this is well-defined. An alternate coproduct Δ' is given by $\Delta' f(t) = f(xy)$, where we interpret the right-hand side as being over $xy := \{x_i y_j : x_i \in x, y_j \in y\}$.

Recall that, for the usual coproduct, p_i is primitive, so $\Delta p_i = 1 \otimes p_i + p_i \otimes 1$. For this alternate coproduct, we have

$$\begin{aligned} \Delta' p_i &= \Delta'(t_1^i + t_2^i + \cdots) = \Delta'((x_1 y_1)^i + (x_1 y^2)^i + (x_2 y_1)^i + \cdots \\ &= x_1^i (y_1^i + y_2^i + \cdots) + x_2^i (\cdots) + \cdots = p_i(x) p_i(y). \end{aligned}$$

Hence $\Delta' p_i = p_i(x) \otimes p_i(y)$, so p_i is grouplike under Δ'.

Fact: $(\Lambda, m, \Delta', u, \epsilon)$ is a bialgebra.

Definition 342. We have a multiplication on Λ^* induced by Δ'. We also have an isomorphism

$$\begin{aligned} \Lambda &\to \Lambda^* \\ f &\mapsto \langle f, - \rangle \end{aligned}$$

using the Hall inner product. This induces a multiplication $*$ on Λ as the unique solution to

$$\langle f * g, h \rangle = \langle f \otimes g, \Delta' h \rangle.$$

Consider this operation on the p_i's. Recall that $\{p_\lambda\}$ and $\{p_\lambda / z_\lambda\}$ are dual bases, where if $\lambda = 1^{m_1} 2^{m_2} \cdots$, then $z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$. Hence

$$\begin{aligned} \langle p_\mu * p_\nu, p_\lambda \rangle &= \langle p_\mu \otimes p_\nu, p_\lambda \otimes p_\lambda \rangle \\ &= \langle p_\mu, p_\lambda \rangle \langle p_\nu, p_\lambda \rangle \\ &= \delta_{\mu\lambda} z_\lambda \delta_{\nu\lambda} z_\lambda. \end{aligned}$$

Thus $p_\mu * p_\nu = 0$ if $\mu \neq \nu$, and otherwise

$$\langle p_\mu * p_\mu, p_\lambda \rangle = z_\lambda^2 \delta_{\mu\lambda} = \langle z_\mu p_\mu, p_\lambda \rangle,$$

so that $p_\mu * p_\nu = \delta_{\mu\nu} z_\mu p_\mu$. Call the $*$ operation the $\boxed{\text{internal product}}$.

343 Proposition

Fact: $*$ is equivalent to tensoring S_n-representations, in the following sense. Recall the Frobenius characteristic map

$$\operatorname{ch} \chi = \sum_\mu \frac{\chi(\mu)}{z_\mu} p_\mu,$$

where χ is a class function on S_n and the sum is over conjugacy classes.

Claim:

$$\operatorname{ch} \chi_1 \otimes \chi_2 = \operatorname{ch} \chi_1 * \operatorname{ch} \chi_2$$

and

$$\operatorname{ch}(V_1 \otimes V_2) = \operatorname{ch} V_1 * \operatorname{ch} V_2$$

for S_n-modules V_1, V_2. (Compare to our earlier definition,

$$\operatorname{ch}(V_1 \otimes V_2) {\uparrow}_{S_n \times S_m}^{S_{n+m}} = (\operatorname{ch} V_1)(\operatorname{ch} V_2).)$$

Proof (of claim). Recall $s_\lambda = \sum \frac{\chi^\lambda(\mu)}{z_\mu} p_\mu = \operatorname{ch} \chi^\lambda$, where the χ^λ are the irreducible characters. Hence

$$\begin{aligned} s_\lambda * s_\nu &= \sum_{\mu_1, \mu_2} \frac{\chi^\lambda(\mu_1) \chi^\nu(\mu_2)}{z_{\mu_1} z_{\mu_2}} p_{\mu_1} * p_{\mu_2} \\ &= \sum_\mu \frac{\chi^\lambda(\mu) \chi^\nu(\mu)}{z_\mu} p_\mu \\ &= \operatorname{ch} \chi^\lambda \otimes \chi^\nu. \end{aligned}$$

Definition 344. If π is a permutation on n letters, let $D(\pi)$ denote the usual descent set of π, which is a subset $[n-1]$. There is a standard "stars and bars" bijection between n-compositions and subsets of $[n-1]$, so we may consider $D(\pi)$ as an n-composition.

If $\alpha = (\alpha_1, \alpha_2, \ldots)$ is a composition, let $\boxed{R(\alpha)}$ be the corresponding $\boxed{\text{ribbon}}$ λ/μ, defined as in the following example:

345 Example

$R(3214)$ is $- - - * * * */ - - - */ - - * */ * **$ where the $*$'s indicate boxes taken and the $-$'s indicate boxes skipped, and the /'s indicate new rows; this is in English notation. Note the number of $*$'s, read bottom-to-top, is 3214. Further, 3214 corresponds via stars and bars to

$$* \ * \ * \,|\, * \, * \,|\, * \,|\, * \ * \ * *$$

A function $f\colon [10] \to \mathbb{P}$ is "compatible" with this if

$$f(1) \leq f(2) \leq f(3) > f(4) \leq f(5) > f(6) > f(7) \leq f(8) \leq f(9) \leq f(10).$$

Filling $R(3214)$ subject to these inequalities corresponds precisely to filling $R(3214)$ weakly increasing along rows and strictly increasing down columns. This property characterizes $R(3214)$. Moreover, the pattern of $*$'s form a "ribbon" in an obvious way.

346 Theorem

Let $\pi \in S_n$ and suppose $D_1, \ldots, D_k \subset [n-1]$. The ("absurd fact") number of factorizations $\pi = \pi_1 \cdots \pi_k$ where $D(\pi_i) = D_i$ for all i is

$$\langle s_{R(D(\pi))}, s_{R(D_1)} * \cdots * s_{R(D_k)} \rangle.$$

PROOF Lengthy; we'll just hit a few high points for the rest of the lecture.

Definition 347. Let P be a poset $P = ([n], \leq_P)$. A $\boxed{\text{partition}}$ P-partition is a function

$$f\colon [n] \to \mathbb{P}$$

which is

i. weakly increasing, so $i \leq_p j$ implies $f(i) \leq f(j)$;

ii. "squishes nicely", so that if $i < j$ (in $[n]$) with $f(i) = f(j)$, then $i <_p j$.

Equivalently, if $i <_p j$ and $i > j$, then $f(i) < f(j)$.

348 Example

Let P have vertices $1, 2, 3, 4$ with covering relations $1 < 2, 1 < 3, 3 < 4$. Then a P-partition must satisfy $f(1) \leq f(2), f(1) \leq f(4), f(4) < f(3)$.

Definition 349. Let $A(P) = \{\text{P-partitions}\}$ and let Γ_P be the generating function

$$\Gamma_P := \sum_{f \in A(p)} \prod_{i=1}^{n} x_{f(i)} \in \mathrm{QSYM}\,.$$

If $\pi \in S_n$, form a poset $\pi(1) < \pi(2) < \pi(3) < \cdots$. Then $\Gamma_\pi = F_{D(\pi)}$, the fundamental quasisymmetric function associated with the descents of π.

350 Remark

How do these P-partitions connect to counting factorizations? The idea is

$$\Delta'(F_{D(\pi)}) = \sum_{\tau\sigma = \pi \in S_n} F_{D(\tau)}(x) F_{D(\sigma)}(y).$$

351 Example

Given skew shape λ/μ, take labeled poset $P_{\lambda/\mu}$ by $- - * * */ * * * /*$ which maps to the poset with covering relations $2 < 1, 2 < 3, 3 < 4, 5 < 4, 5 < 6, 6 < 7$. Then $\Gamma_{P_{\lambda/\mu}} = s_{\lambda/\mu}$.

352 Conjecture

Only the skew shapes give posets with symmetric associated quasisymmetric functions; see Stanley.

May 21, 2014: (Special) Double Posets, E-Partitions, Pictures, Yamanouchi Words, and a Littlewood-Richardson Rule

Summary Today, Michael is lecturing on "A Self Paired Hopf Algebra on Double Posets and a Littlewood-Richardson Rule" by Malvenuto and Reutenauer. Outline:

(1) $\mathbb{Z}D$ is a graded Hopf algebra

(2) $\mathbb{Z}DS$ is a special case

(3) Connection to L-R rule

Definition 353. A $\boxed{\text{double poset}}$ is a triple $(E, <_1, <_2)$ where E is finite, $<_1$ and $<_2$ are two partial orders on E. (Today, E will typically refer to a double poset.) Define a morphism between double posets as one which is order-preserving for both partial orders, so we have a notion of isomorphism. Today we'll tacitly take double posets only up to isomorphism, i.e. we'll implicitly use isomorphism classes of double posets.

Let $\boxed{D}$ be be the set of double posets, and $\mathbb{Z}D$ the formal $\mathbb{Z}$-linear combinations of double posets.

354 Example

Given a poset, label the vertices giving a total order; use the original order as $<_1$, and use the labeling order for $<_2$.

Definition 355. Let $E, F \in D$. Define EF to be the poset $(E \cup F, <_1, <_2)$ where $<_1$ compares pairs of elements of E or of F using their first orders and pairs consisting of one element of E and one element of F are incomparable. $<_2$ is the same using the second orders, except $e <_2 f$ for any $e \in E$, $f \in F$.

Definition 356. Suppose $(E, <)$ is a poset. An $\boxed{\text{inferior order ideal}}$ I is a subset of E which is closed under $<$, i.e. if $y \in I$ and $x < y$, then $x \in I$. Likewise a $\boxed{\text{superior order ideal}}$ is closed under $>$. Define (I, S) to be a $\boxed{\text{decomposition}}$ of E if I is an inferior order ideal, S is a superior order ideal, and E is the disjoint union of I and S. (Hence I and S are complements.)

Definition 357. Let $(E, <_1, <_2)$ be a double poset. Define a decomposition of E to be a decomposition of $(E, <_2)$, but endow I, S with partial orders $<_1$ and $<_2$ by restriction, yielding double posets.

Definition 358. We define a comultiplication $\Delta\colon \mathbb{Z}D \to \mathbb{Z}D \otimes \mathbb{Z}D$ by breaking into decompositions:

$$\Delta(E, <_1, <_2) = \sum_{(I,S) \text{ of } (E,<_1,<_2)} (I, <_1, <_2) \otimes (S, <_1, <_2).$$

359 Theorem

$\mathbb{Z}D$ is a graded Hopf algebra over $\mathbb{Z}$ (graded by size of E). (This is different from our usual assumption that Hopf algebras are over fields, but it works out.) See the paper for a terminal morphism to QSYM.

Definition 360. Let $E \in D$. If $<_2$ is a total order, then E is called a $\boxed{\text{special double poset}}$. This is just a labeled poset. Let $\boxed{\mathbb{Z}DS}$ be the subset of $\mathbb{Z}D$ spanned by special posets.

Definition 361. A $\boxed{\text{linear extension}}$ of a special double poset E is a total order on E that extends $<_1$. Given a linear extension, write the elements of E in increasing order, $e_1 <_1 e_2 <_1 e_3 <_1 \cdots$. Now let $\omega\colon E \to [\#E]$ be the labelling given by $<_2$. Define $\sigma \in S_{\#E}$ by $\sigma(i) = \omega(e_i)$. For instance, with the "Y"-shaped poset labeled 1 at the bottom, 2 at the branch, and 3 and 4 at the tips, extend the poset to a linear order by taking $3 > 4$. Then σ in one-line notation is 1243.

362 Theorem

$\mathbb{Z}DS$ is a sub-bialgebra of $\mathbb{Z}D$, and there is a bialgebra homomorphism

$$\mathbb{Z}DS \to \mathbb{Z}S$$

which sends a special double poset to the sum of all its linear extensions.

Definition 363. A $\boxed{\text{Yamanouchi word}}$ is as usual: it's a word in $\mathbb{P}$ such that when read left to right the number of 1's always weekly exceeds the number of 2's, which always weekly exceeds the number of 3's, etc. For instance, $w = 11122132$ is Yamanouchi. It has $\boxed{\text{weight}}$ $1^4 2^3 3^1$. The $\boxed{\text{complement}}$ of a word in $\mathbb{P}$ is given by replacing i by $k - i + 1$, where k is the maximum element of $\mathbb{P}$ appearing in the word. For instance, the complement of w is 33322312.

Definition 364. If $E \in D$, a map $\chi \colon E \to X$ (where X is a totally ordered set) is an $\boxed{\text{partition}}$$E$-partition if

(1) $e <_1 e' \Rightarrow \chi(e) \leq \chi(e')$, and

(2) $e <_1 e'$ and $e \geq_2 e'$ together imply $\chi(e) < \chi(e')$.

(Compare to the previous lecture's P-partitions. There $<_2$ comes from the standard ordering on $[n]$, though this is a little more general.)

Definition 365. Given a special double poset $(E, <_1, <_2)$ with labeling ω (i.e. ω encodes the total order $<_2$ as above) and a word $w = a_1 \cdots a_n$ on a totally ordered alphabet A, we say w $\boxed{\text{fits into}}$ E if the map $e \mapsto a_{w(e)}$ is an E-partition.

366 Example

Consider the case when the word is the one-line notation of a permutation in S_n. The map to consider takes e to $\tau\omega(e)$. Since τ is bijective, the second condition for an E-partition is vacuously satisfied, so we just need the map $(E, <_1) \to [n]$ to be weakly increasing.

Definition 367. Given a partition ν of n, we can construct a special double poset as follows. Let $\boxed{E_\nu}$ be the Ferrers diagram of ν. For concreteness, use French notation and consider $E_\nu \subset \mathbb{N} \otimes \mathbb{N}$. Let $<_1$ be the partial order induced by the component-wise order on $\mathbb{N} \times \mathbb{N}$. Let $<_2$ be the partial order induced by $(x, y) <_2 (x', y')$ if and only if $y > y'$ or $y = y'$ and $x < x'$. Hence E_ν is a double poset.

Definition 368. A $\boxed{\text{picture}}$ between two double posets E, F is a bijection $\phi \colon E \to F$ such that (1) $e <_1 e' \Rightarrow \phi(e) <_2 \phi(e')$ and (2) $f <_1 f' \Rightarrow \phi^{-1}(f) \leq_2 \phi^{-1}(f')$. (The origin of the term "picture" is unclear; originally due to Zelevinsky in 1981.)

Given a double poset E, define $\boxed{\widetilde{E}}$ to be E but using the opposite posets for both $<_1$ and $<_2$.

369 Theorem

Let E be a special double poset. Suppose ν is a partition. Then the number of pictures from E to E_ν is

i. the number of Yamanouchi words of weight ν whose complement fits into E;

ii. the number of Yamanouchi words whose mirror image fits into $\widetilde{E}$

Note: Zelevinsky says (ii) gives the Littlewood-Richardson coefficients. We only have one partition ν here, so we need two more: these can be encoded in the special double poset E, though it seems the translation is not entirely immediate.

Proof See paper. We'll just discuss one proposition it relies on.

370 Proposition

Let T be a standard Young tableau and let w be the associated Yamanouchi word. If u is the complement of w, then the reading word of T is equal to the inverse of the standard permutation of u.

371 Example

Consider the tableau $T = 3/25/1467$. The associated Yamanouchi word is $w = 1231211$ where we read off the row number of $1, 2, 3, \ldots$. Hence $u = 3213233$. The reading word is obtained by just reading normally (in French): 3251467. The standardization of u is 4215367, whose inverse is indeed 3251467. The standarization is constructed in blocks as $--1----, -21-3--, 4215367$.

Aside: Affine Group Schemes and Commutative Hopf Algebras

Summary This explains and formalizes the statement "commutative Hopf algebras are equivalent to affine group schemes".

Definition 372. An $\boxed{\text{affine group scheme over } \Bbbk}$ is an affine $\Bbbk$-scheme G together with (scheme) morphisms $M \colon G \times G \to G$ (multiplication), $I \colon G \to G$ (identity), and $U \colon \operatorname{spec} \Bbbk \to G$ (unit) satisfying the usual group axioms in diagrammatic form. The associativity and unit axioms are:

$$\begin{tikzcd}
G \times G \times G \arrow[r, "(M,\mathrm{id})"] \arrow[d, "(\mathrm{id},M)"'] & G \times G \arrow[d, "M"] \\
G \times G \arrow[r, "M"'] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{spec} \Bbbk \times G \arrow[d, "U \times \mathrm{id}"'] & G \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\mathrm{id}"] & G \times \operatorname{spec} \Bbbk \arrow[d, "\mathrm{id} \times U"] \\
G \times G \arrow[r, "M"] & G & G \times G \arrow[l, "M"']
\end{tikzcd}$$

The inverse axiom is

$$\begin{tikzcd}
 & G \times G \arrow[dr, "M"] & \\
G \arrow[ur, "{\mathrm{id}, I}"] \arrow[r] \arrow[dr, "{I, \mathrm{id}}"'] & \operatorname{spec} \Bbbk \arrow[r, "U"] & G \\
 & G \times G \arrow[ur, "M"'] &
\end{tikzcd}$$

(Recall that a $\Bbbk$-scheme by definition comes with a map $G \to \operatorname{spec} \Bbbk$, which endows the rings of sections $\Gamma(G, U)$ as U varies over open subsets of G with (compatible) $\Bbbk$-module structures. In particular, the global sections $\boxed{\Bbbk[G]}$ are a $\Bbbk$-vector space. Recall further that $\operatorname{spec} \Bbbk$ is the final object in the category of $\Bbbk$-schemes, which explains its appearance in the inverse map and axiom: in the category of sets, $\operatorname{spec} \Bbbk$ would be replaced by a singleton, and we would have literally the definition of a group. Of course, rings of sections of schemes are by assumption commutative, so $\Bbbk[G]$ is a commutative ring.

Also, given affine $\Bbbk$-schemes $\operatorname{spec} A$ and $\operatorname{spec} B$, we have $\operatorname{spec} A \times \operatorname{spec} B = \operatorname{spec}(A \otimes_{\Bbbk} B)$. In particular, $\operatorname{spec} \Bbbk \times \operatorname{spec} A = \operatorname{spec}(\Bbbk \otimes_{\Bbbk} A) = \operatorname{spec} A$.)

373 Proposition

The category $\mathcal{G}$ of affine group schemes over $\Bbbk$ is (anti-)equivalent to the category $\mathcal{H}$ of commutative Hopf algebras over $\Bbbk$. More precisely, the global sections functor $\mathcal{G} \to \mathcal{H}$ is well-defined and part of an (anti-)equivalence of categories, the other half of which is the spec *functor $\mathcal{H} \to \mathcal{G}$, which is also well-defined.*

Proof Given an affine group scheme G, let $H := \Bbbk[G]$ be its global sections. Let $\Delta\colon H \to H \otimes_{\Bbbk} H$ be the map of global sections corresponding to $M\colon G \times G \to G$, let $\epsilon\colon H \to \Bbbk$ correspond to $U\colon \operatorname{spec} \Bbbk \to G$, and let $S\colon H \to H$ correspond to $I\colon G \to G$.

For instance, if G is a variety, $H = \Bbbk[G]$ consists of algebraic functions $G \to \Bbbk$, and the global sections functor corresponds to taking pullbacks: $I\colon G \to G$ is in particular a map of sets and the induced map $S\colon H \to H$ is just given by $f\colon G \to \Bbbk$ maps to $f \circ I\colon G \to \Bbbk$.

In any case, apply the (contravariant) global sections functor to the associativity and unit diagrams (together with the comment above about products in G corresponding to tensor products over $\Bbbk$) to get

$$\begin{tikzcd}
H \otimes_{\Bbbk} H \otimes_{\Bbbk} H & H \otimes_{\Bbbk} H \arrow[l, "\Delta\otimes\mathrm{id}"'] \\
H \otimes_{\Bbbk} H \arrow[u, "\mathrm{id}\otimes\Delta"] & H \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes_{\Bbbk} H \arrow[r, "\sim"] & H & H \otimes_{\Bbbk} k \arrow[l, "\sim"'] \\
H \otimes_{\Bbbk} H \arrow[u, "\epsilon\otimes\mathrm{id}"] & H \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[u, "\mathrm{id}"'] & H \otimes_{\Bbbk} H \arrow[u, "\mathrm{id}\otimes\epsilon"']
\end{tikzcd}$$

Of course, these are just the coproduct diagrams. Δ and ϵ are also algebra morphisms, and H is in particular a $\Bbbk$-algebra with multiplication $m\colon H \otimes_{\Bbbk} H \to H$ and unit $u\colon \Bbbk \to H$, so we have a bialgebra structure $(H, m, u, \Delta, \epsilon)$. There is one wrinkle: the inverse axiom's diagram isn't quite the same as the Hopf algebra diagram. We claim the map of global sections corresponding to $G \overset{\mathrm{id},I}{\to} G \times G$ is precisely the composite

$$H \otimes_{\Bbbk} H \overset{S\otimes\mathrm{id}}{\to} H \otimes_{\Bbbk} H \overset{m}{\to} H,$$

in which case the corresponding diagram is

$$\begin{tikzcd}
& H \otimes_{\Bbbk} H \arrow[dl, "m"'] & & H \otimes_{\Bbbk} H \arrow[ll, "S\otimes\mathrm{id}"'] & \\
H & & \Bbbk \arrow[ll, "u"'] & & H \arrow[ll, "\epsilon"'] \arrow[ul, "\Delta"'] \arrow[dl, "\Delta"] \\
& H \otimes_{\Bbbk} H \arrow[ul, "m"] & & H \otimes_{\Bbbk} H \arrow[ll, "\mathrm{id}\otimes S"'] &
\end{tikzcd}$$

which is exactly the antipode diagram. (The claim is an easy exercise. It is essentially the same as proving $\operatorname{spec} A \times \operatorname{spec} B = \operatorname{spec}(A \otimes_{\Bbbk} B)$.)

The rest of the proof is tedious and trivial: maps of affine group schemes over $\Bbbk$ indeed induce maps of their corresponding Hopf algebras; this is functorial; the spec functor indeed yields affine group schemes over $\Bbbk$ where the comultiplication corresponds to the group multiplication, the counit corresponds to the group unit, and the antipode corresponds to the group inverse; this correspondence is also functorial; and these functors give an (anti-)equivalence of categories, since they are contravariant; it is not an (anti-)isomorphism of categories because affine schemes are merely isomorphic to spec of their global sections.

May 23rd, 2014: Open Problem Day

Summary ¡Today will be open problem day! Monday there is no class, Wednesday will be Monty, Friday will be Sara.

374 Remark

Sadly, these notes have been redacted. They were distributed in an email to the class, though.

May 28th, 2014: Hecke Algebra of S_n; Kazhdan-Lusztig Polynomials; Cells

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Summary Monty McGovern is lecturing on the Kazhdan-Lusztig polynomials today.

375 Remark

Recall the standard presentation for the symmetric group expressing it as a Coxeter group, $S_n = \langle s_1, \ldots, s_{n-1} : s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1\rangle$ where m_{ij} is 2 if $|i-j| \geq 2$ and 3 if $|i-j| = 1$ for $i \neq j$.

Definition 376. The Hecke algebra $\boxed{H}$ attached to S_n is the following structure. It is an algebra over $\mathbb{Z}[q, q^{-1}]$ generated by $T_1, \ldots, T_{n-1}$ subject to relations

$$(T_i + 1)(T_i - q) = 0, \qquad (T_i T_j)^{m_{ij}} = 1 \quad (i \neq j)$$

with m_{ij} as above. Then we may set $T_{w_i} = T_{s_{i_1}} \cdots T_{s_{i_k}}$ if $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression for w.

Claim: T_w is independent of the choice of decomposition, and the T_w form a free basis for H over $\mathbb{Z}[q, q^{-1}]$.

377 Remark

We will produce a (Kazhdan-Lusztig) basis of H, also indexed by S_n, with triangular change of basis matrix to T_w, such that we can find quotients of certain left ideals spanned by the C_w (the new basis vectors) which are irreducible H-modules.

Definition 378. Recall the Bruhat order on S_n, in the following guise. Given a reduced decomposition $w = s_{i_1} \cdots s_{i_k}$, the elements $v < w$ in Bruhat order are precisely the subexpressions of this reduced word (i.e. "cross off" some of the factors s_{i_j}).

Definition 379. Define an involution ι on H as follows. Unusually, ι does not fix the base ring of the algebra H, but instead $\iota q = q^{-1}$. Also use $\iota T_w = T_{w^{-1}}^{-1}$.

Proof These operations preserve the defining operations.

We will find a new basis of H indexed by $W := S_n$ consisting of elements fixed by ι. Since $T_s^{-1} = q^{-1} T_s - (1 - q^{-1})$ is asymmetric, we need to enlarge our scalars to find ι-fixed elements. In particular, use $\mathbb{Z}[q^{1/2}, q^{-1/2}]$.

Definition 380. Let $\boxed{C_s} := q^{-1/2}(T_s - q)$. One can check this is fixed by ι. Naively, we might try using $C_w = C_{s_{i_1}} \cdots C_{s_{i_k}}$ for $w = s_{i_1} \cdots s_{i_k}$, but this is not independent of reduced decomposition. However, we will take this as a "first approximation" and tweak it.

381 Notation

Let $\boxed{\epsilon(w)} := (-1)^{\ell(w)}$, which is 1 if w is even and -1 if w is odd. Let $\boxed{q_w} := q^{\ell(w)}$ where $\ell(w)$ is the length of a reduced decomposition for w.

382 Theorem (Kazhdan-Lusztig, Inv. Math 53 (1979), 105-184)

For each $w \in S_n$, there is a unique $\boxed{C_w} \in H$ such that

(a) $\iota C_w = C_w$

(b) $C_w = \epsilon_w q_w^{1/2} \sum_{x \le w} \epsilon_x q_x^{-1} \overline{P_{x,w}} T_x$ *where* $P_{w,w} = 1$, $P_{x,w} \in \mathbb{N}[q]$ *has degree* $\le 1/2(\ell(w) - \ell(x) - 1)$ *if* $x < w$, *and* $\overline{P_{x,w}} = \iota P_{x,w}$. *Finally,* $P_{x,z} = 0$ *if* $x \not\le z$.

Requirement (b) is a refinement of the statement above that the change of basis matrix between $\{C_w\}$ *and* $\{T_x\}$ *is triangular. The* $\boxed{P_{x,w}}$ *are the* $\boxed{\text{Kazhdan-Lusztig polynomials}}$.

383 Proposition

Their proof of existence yields an inductive formula for the Kazhdan-Lusztig polynomials,

$$P_{x,z} = q^{1-c} P_{sx,sw} + q^c P_{x,sw} - \sum_{\substack{z<sw \\ sz<z}} \mu(z, sw) q_z^{-1/2} q_w^{1/2} P_{x,z}$$

where: s *is a transposition chosen so that* $sw < w$; $c = 0$ *if* $x < sx$ *and* $c = 1$ *if* $x > sx$; $\mu(z, sw) :=$ *coefficient of* $q^{1/2(\ell(sw)-\ell(z)-1)}$ *in* $P_{z,sw}$. *(One may argue that indeed the* $P_{x,w}$ *involve only integral powers of* q, *hence they are honest polynomials in* q.*)*

384 Corollary

All constant terms of non-zero $P_{x,w}$ *are 1. Moreover,* $P_{x,w} = 0$ *iff* $x \not\le w$.

(Indeed, a result of Polo says that every polynomial in $\mathbb{N}[x]$ *with constant term 1 is a Kazhdan-Lusztig polynomial for some Coxeter group.)*

385 Proposition

Using the Kazhdan-Lusztig basis (it is indeed a basis from the triangular relationship (b) above), we can deduce an "almost triangular" formula for the left action of H *on a* C_w *(there is a similar formula for the right action). In particular,*

$$T_s C_w = \begin{cases} -C_w & sw < w \\ qC_w + q^{1/2} C_{sw} + q^{1/2} \sum_{\substack{z<w \\ sz<z}} \mu(z, w) C_z & sw > w \end{cases}$$

Roughly, much of the complicated information that goes into constructing the polynomials gets washed out by this action.

Definition 386. Write $\boxed{x \le_L w}$ if the left ideal HC_w contains an element involving C_x. Similarly we define $\boxed{x \le_R w}$ if the right ideal $C_w H$ contains an element involving C_x. Likewise define $\boxed{x \le_{LR} w}$ if $HC_w H$ (the set of sums of products $hC_w h'$) contains a term involving C_x.

387 Proposition

We can decompose H *into left ideals spanned by* C_w. *It is clear that*

$$\{\mathbb{Z}[q^{1/2}, q^{-1/2}] C_x : x \le_L w\}$$

is a left ideal of H, *called the* $\boxed{\text{left cone}}$ *over* w. *Similarly we have the* $\boxed{\text{right cone}}$ *and the* $\boxed{\text{two-sided cone}}$.

Definition 388. To make these cones irreducible (as H-modules), we pass to the corresponding "cells", as follows. Define $\boxed{x <_L w}$ if $x \le_L w$ but $w \not\le_L x$ (this is essentially fixing up antisymmetry). Then the span of all C_x with $x \le_L w$ modulo the span of all C_x with $x <_L w$ is a quotient by the construction of left ideals of H, so is a left module for H, which at least has a better chance of being irreducible. We call this quotient or the set of x with $x \sim_L w$ (meaning $x \le_L w$ and $w \le_L x$) the $\boxed{\text{left cell of } w}$. We similarly define $\boxed{\text{right cells}}$ and $\boxed{\text{two-sided cells}}$ or $\boxed{\text{double cells}}$, giving equivalence relations $\sim_R, \sim_{LR}$.

These relations have "many, many, many applications to representation theory of left cells".

389 Proposition

For $W = S_n$, the left cells are irreducible H-modules, and likewise for right and two-sided cells. Moreover, the left, right, and two-sided cells may be computed by RSK, $w \mapsto (T_I(w), T_R(w))$ where T_I is the insertion tableau, T_R is the recording tableau. In particular, w is in the same left cell as v (i.e. $w \sim_L v$) iff $T_I(w) = T_I(v)$, w is in the same right cell as v iff $T_R(w) = T_R(v)$, and w and v are in the same double cell iff $T_I(w) = T_I(v), T_R(w) = T_R(v)$.

390 Remark

This implies that left, right, and two-sided cells as modules are irreducible. (Note that two-sided cells are bimodules, so they use both left and right actions.) There are generalizations to other groups, for instance the hyperoctahedral group, but these are no longer irreducible in general.

More information on this comes from Chapter 7 of Humphreys "Reflection Groups and Coxeter Groups", 1990.

May 30th, 2014—Immaculate Basis for QSYM; (Standard) Immaculate Tableaux; Representations of 0-Hecke Algebras

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Summary Sara is lecturing again today. She will be presenting on "Indecomposable Modules for the Dual Immaculate Basis of QSYM", by Berg, N. Bergeron, Saliola, Serreno, Zabrocki (BBSSZ).

391 Remark

The following will be assumed as known background on QSYM.

Recall QSYM is defined to be the $\mathbb{Q}$-span of all

$$M_\alpha := \sum_{i_1 < \cdots < i_m} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots$$

where $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{P}^m$ is a (strong) composition of n, written $\alpha \vDash n$. Compositions are ordered by refinement (where refining a composition decreases it in this order). We define the fundamental quasisymmetric functions in terms of the monomial quasisymmetric functions as

$$\begin{aligned} F_\alpha &:= \sum_{\beta \leq \alpha} M_\beta \\ &= \sum_{i_1 \leq \cdots \leq i_j} x_{i_1} \cdots x_{i_n} \end{aligned}$$

where if $j \in \omega(\alpha)$, we require $i_j < i_{j+1}$. Here $\omega(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \cdots, \alpha_1 + \cdots + \alpha_{m-1}\}$. (We don't need to include α_m, since $\alpha \vDash n$ implies it.)

We can multiply fundamental quasisymmetric functions using

$$F_\alpha F_\beta = F_{D(u)} F_{D(v)} = \sum_{w \in u \sqcup\!\sqcup v} F_{D(w)},$$

where $u \in S_{|\alpha|}$, $v \in S_{|\beta|}$, $D(u) = \alpha$, $D(v) = \beta$, and $u \sqcup\!\sqcup v$ is the shuffle product of u and v. The coproduct is

$$\Delta(F_\alpha) = \sum F_\gamma \otimes F_\beta$$

where the sum is over compositions (γ, β) such that the ribbon corresponding to α can be cut into ribbons corresponding to γ and β.

The antipode is given by

$$S(F_\alpha) = (-1)^{|\alpha|} F_{\omega(\alpha)}.$$

Also recall Perms is the Hopf algebra of permutations, defined to have basis P_w for $w \in \cup_n S_n$. It has the "shuffle product" and the "cut coproduct".

SYM $:= \mathbb{Q}[h_1, h_2, \ldots]$ and NSYM $:= \mathbb{Q}\langle H_1, H_2, \ldots \rangle$ with $\Delta(H_n) := \sum_{i+j=n} H_i \otimes H_j$.

392 Theorem (Ditters Conjecture, proved by Hazewinkel)

QSYM *is freely generated as a polynomial ring. They're indexed by compositions which are certain Lyndon words.*

393 Open Problem

How do you write Hazewinkel's generators of QSYM using $\{M_\alpha\}$ or $\{F_\alpha\}$?

Definition 394. BBSSZ previously defined the $\boxed{\text{immaculate basis}}$ $\boxed{\mathfrak{S}_\alpha}$ for NSYM (see BBSSZ's "The immaculate basis of the non-commutative symmetric functions"). Note that NSYM $\twoheadrightarrow$ SYM $\hookrightarrow$ QSYM naturally. Moreover, Perm maps surjectively onto QSYM via $p_w \mapsto F_{D(w)}$. This suggests NSYM may map into Perm. Indeed,

$$\begin{tikzcd}
\text{NSYM} \arrow[r, hook] \arrow[d, two heads] \arrow[dr, leftrightarrow, "\text{dual}"'] & \text{Perm} \arrow[d, two heads] \\
\text{SYM} \arrow[r, hook] & \text{QSYM}
\end{tikzcd}$$

via

$$\begin{tikzcd}
H_\alpha \arrow[r, mapsto] \arrow[d, mapsto] & \sum_{D(w)=\omega(\alpha)} P_w \arrow[d, mapsto] \\
h_\alpha \arrow[r, mapsto] & h_\alpha
\end{tikzcd}$$

Hence there are some $K_{\alpha,\beta}$ such that

$$H_\beta = \sum K_{\alpha,\beta} \mathfrak{S}_\alpha,$$

since the $\mathfrak{S}_\alpha$ are a basis. In fact, $\boxed{K_{\alpha,\beta}}$ is the number of "immaculate tableaux" (defined below) of shape α content β. We define $\boxed{\mathfrak{S}_a^*}$ as the dual immaculate basis element.

Definition 395. Let $\alpha, \beta \vDash n$ (to be clear, strong compositions). An $\boxed{\text{immaculate tableau}}$ of shape α and content β is a filling of the diagram of α with β_1 one's, β_2 two's, etc., such that

(i) each row is weakly increasing

(ii) the first column is strictly increasing from top to bottom

396 Example

Consider 112/35/4445. (Since α is just a composition, the row lengths do not have to be weakly monotonic.) Here $\alpha = (3, 2, 4)$ and $\beta = (2, 1, 1, 3, 2)$.

Let $\boxed{D_{\alpha,\beta}}$ be defined to be the number of immaculate tableaux of shape α, content β. An immaculate tableau T of content 1^n is $\boxed{\text{standard}}$ (and is highly restricted). For instance, $T = 1235/4/67$.

The $\boxed{\text{descent set}}$ of an immaculate tableau T is the set of all i such that $i+1$ is in a row strictly below i. For T above, $D(T) = \{3, 5\}$.

397 Proposition
$\mathfrak{S}^*_\alpha = \sum_T F_{D(T)}$, *where the sum is over standard immaculate tableaux of shape* α.

398 Theorem (BBSSZ-2013)
$\mathfrak{S}^*_\alpha$ *is the Frobenius character of an indecomposable representation of* $H_n(0)$, *the* 0*-Hecke algebra.*

399 Remark
Recall $H_n(q)$, the Hecke algebra of S_n, was defined last time, with generators T_i for $1 \le i \le n-1$ with certain relations, or as the span of T_w for $w \in S_n$ where

$$T_{s_i} T_w = \begin{cases} T_{s_i,w} & s_i w > w \\ qT_{s_i w} + (q-1)T_w & s_i w < w \end{cases}$$

At $q = 1$, we just get the group algebra of S_n. At $q = 0$, we get the relations $T_i^2 = -T_i$, and $T_i T_j = T_j T_i$. (There are several slightly different conventions; for instance, we used a different one on the first day.) Another presentation: $H_n(0) = \langle \pi_i : 1 \le i < n \rangle$ subject to $\pi_i^2 = \pi_i$ (use $\pi_i \leftrightarrow 1 + T_i$).

400 Theorem (Norton, 1979)
The irreducible representations of $H_n(0)$ *are indexed by compositions* α. *Each irreducible* L_α *is one-dimensional and spanned by* $v_\alpha \in L_\alpha$ *such that*

$$\pi_i(v_\alpha) = \begin{cases} 0 & i \in \omega(\alpha) \\ v_\alpha & \text{otherwise} \end{cases}$$

401 Remark
Let $\boxed{G_n}$:= the Grothendieck group of finite dimensional representations of $H_n(0)$ up to isomorphism with addition given by exact sequences. In particular, it is the span of $[A]$ where A is a finite dimensional representation of $H_n(0)$, modded out by $[B] = [A] + [C]$ when

$$0 \to A \to B \to C \to 0$$

is an exact sequence of $H_n(0)$-modules. Let $\boxed{G} := \oplus G_n$

402 Remark
$[N][M] = \operatorname{Ind}_{H_n(0)\otimes H_m(0)}^{H_{n+m}(0)}(N \otimes M)$, in exact analogy with the S_n representation ring above.

By Norton's theorem, G is spanned by $\{L_\alpha : \alpha \text{ is a composition}\}$.

403 Theorem (Duchamp-Krob-Leclere-Thibon, 1996)
$\mathcal{F} \colon G \to \mathrm{QSYM}$ *given by* $[L_\alpha] \mapsto F_\alpha$ *is a graded Hopf algebra isomorphism.*

404 Proposition
$H_n(0)$ *acts on the standard immaculate tableaux via*

$$\pi_i(T) = \begin{cases} 0 & i, i+1 \text{ in first column} \\ T & \text{if } i \text{ is in a row weakly below } i+1 \\ s_i(T) & \text{otherwise} \end{cases}$$

405 Example
Let $T = 129/3567/48$; $\pi_1(T) = T$; $\pi_2(T) = 139/2567/48$; $\pi_3(T) = 0$, etc.

406 Notation
Let $\boxed{\mathrm{SIT}(\alpha)}$ denote the set of standard immaculate tableaux of shape α.

407 Proposition
There is a natural partial order on $\mathrm{SIT}(\alpha)$, *where* $S < T$ *if there exists* $\sigma \in S_n$ *such that* $\pi_\sigma(T) = S$. *Take a linear extension and list* $\mathrm{SIT}(\alpha)$ *as* $\{T_1, T_2, \ldots, T_p\}$.

408 Example
$T = 1234/56/789$ is maximal for $\alpha = (4, 2, 3)$.

Let M_j be the span of T_i for $i \le j$. We have a filtration

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_p := V^\alpha,$$

where $\dim(M_j/M_{j-1}) = 1$ *(spanned by T_j). It follows that*

$$\pi_i(T_j + M_{j-1}) = \begin{cases} 0 & i \in D(T_j) \\ T_j & \textit{otherwise} \end{cases}$$

So, $[M_j/M_{j-1}] = [L_{comp(D(T_j))}]$.

409 Remark
Different linear orders may give rise to different successive quotients, but the collection of them is unique up to isomorphism.

410 Homework
Count $\#\,\mathrm{SIT}(\alpha)$. (Hopefully reasonable.)

411 Theorem (BBSSZ)
In the above notation, $\mathcal{F}[V^\alpha] = \sum_{T \in \mathrm{SIT}(\alpha)} F_{D(T)}$.

412 Corollary
V^α is a cyclic module.

413 Theorem
V^α is indecomposable as an $H_n(0)$-module.

Definition 414. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_m)$, a $\boxed{\text{Y-word}}$ Y-word is a word in $[m]$ with j appearing α_j times such that the first instance of j appears before the first instance of $j+1$ for every j.

To each Y-word w, there is a unique corresponding standard immaculate tableau whose shape is given by the word's content and which has a j in row w_j.

415 Example
The SIT $T = 129/3567/48$ corresponds to the Y-word 11232223 (where we read off the row numbers of $1, 2, \ldots$ successively).

416 Remark
They define a space Words_n^α given by the span of words $w_1 \cdots w_n$ for $w_i \in [m]$ of content $\alpha = (\alpha_1, \ldots, \alpha_m)$. There is an $H_n(0)$-action on this space

$$\pi(w_1 \cdots w_n) = \begin{cases} w & w_{i+1} \ge w_i \\ ws_i & w_{i+1} < w_i \end{cases}$$

where ws_i is w with the i and $i+1$st letters interchanged. What is $\mathcal{F}[\mathrm{Words}_n^\alpha]$? What if we know

$$[\mathrm{Words}_n^\alpha] = [\mathrm{Ind}_{H_{\alpha_1} \otimes \cdots \otimes H_{\alpha_n}(0)} L_{(\alpha_1)} \otimes \cdots \otimes L_{(\alpha_m)}]?$$

(Exercise: prove this.) Then $\mathcal{F}[\mathrm{Words}_n^\alpha] = F_{(\alpha_1)} \cdots F_{(\alpha_m)} = h_\alpha$.

June 2nd, 2014—Plethysms; P vs. NP vs. $\#P$; Standardization and Matrices

Summary Sara is lecturing again today. Unfortunately Ed Witten had to cancel Wednesday's lecture. Sara will be presenting some lectures for Bogata on quasisymmetric functions instead.

417 Remark

There are numerous analogues between SYM and QSYM: the representations of S_n and GL_n vs. the 0-Hecke algebra; the monomials m_λ and the fundamentals F_α both expand nicely in terms of monomial quasisymmetric functions; the Schurs decompose very nicely using the fundamental basis.

Definition 418. Given two symmetric functions $f(x_1, x_2, \ldots)$ and $g(x_1, x_2, \ldots) = x^a + x^b + x^c + \cdots$, define the $\boxed{\text{plethysm}}$ of f and g to be the function

$$\boxed{f[g]} := f(x^a, x^b, x^c, \ldots).$$

Then $f[g]$ is again a symmetric function.

419 Example

Expand $h_2[e_2(X)]$ on $X = \{x_1, x_2, x_3\}$. Then

$$\begin{aligned} h_2[e_2(x_1, x_2, x_3)] &= h_2(x_1x_2 + x_1x_3 + x_2x_3) \\ &= h_2(x_1x_2, x_1x_3, x_2x_3) \\ &= (x_1x_2)^2 + (x_1x_3)^2 + (x_2x_3)^2 + x_1^2x_2x_3 + x_1x_2^2 + x_3 + x_1x_2x_3^2 \\ &= s_{(2,2)}(x_1, x_2, x_3). \end{aligned}$$

These computations get nasty by hand quickly. Using the infinite Schur functions (for instance), Sage tells us there is an $s_{(1,1,1,1)}$ term, but we didn't have enough variables for this to show up. Doing some more examples, we see these plethysms seem to be Schur-positive, not multiplicity free, and the number of terms can get horrendous quickly. Schur-positivity follows from showing $s_\mu[s_\nu]$ is the character of a GL_N representation: see Fulton-Harris, 1991.

420 Open Problem

Find a combinatorial or "optimal" formula for the coefficients $\boxed{d^\lambda_{\mu,\nu}}$ in the exapansion

$$s_\mu[s_\nu] = \sum_\lambda d^\lambda_{\mu,\nu} s_\lambda.$$

421 Theorem (Thrall, 1942)

$h_2[h_n(X)]$ *has a very nice explicit formula.*

Definition 422. P is the set of all questions which can be decided in polynomial time depending on the input size. NP is the set of all questions for which one can test a proposed solution in polynomial time depending on the input size. $\#P$ is the set of all questions of the form, "How many solutions does X have?" where X is in NP.

423 Example

1. Does a permutation $w \in S_n$ contain another $v \in S_k$? NP-complete; Bose-Buss-Lubiw, 1998. In NP.
2. How many instances of the permutation $v \in S_k$ does $w \in S_n$ have? Bose-Buss-Lubiw, 1998. This is in $\#P$.
3. Does a permutation $w \in S_n$ contain a fixed $v \in S_k$? Guillemot-Marx, 2013. In P. (Found a linear time algorithm!)
4. Does a graph G have a planar embedding? Kuratowski 1930, in P. Hopcroft-Tarjan 1974.
5. Does G have a 3-coloring? NP-complete, Gary-Johnson-Stockmeyer, 1976.

6. How many k-colorings does G have for $k = 1, 2, 3, \ldots$? $\#P$.

7. What are the coefficients of the chromatic polynomial? Jaeger-Vertifan-Welsh, 1990. Harder than any $\#P - complete$ problem.

8. What is the determinant of an $n \times n$ matrix? (Williams 2012, Cohn-Umans 2013.) In P.

9. What is the permanent of an $n \times n$ matrix? (Valiant 1979) $\#P - complete$. (Compute permanent by using cofactor expansion without negative signs.)

424 Open Problem
Does $P = NP$? Does $NP = \#P$? Does $P = \#P$? All unknown. Millenium prize: \$1,000,000 in US dollars for P vs NP problem.

425 Remark
Mulmuley-Sohoni (2001-2002) have an approach to $P \neq NP$:

1. Homogeneous degree n polynomials form a vector space with a GL_N action. Here N is the number of monomials of degree n.

2. The determinant of an $n \times n$ matrix is a homogeneous polynomial of degree n^2 which is computable in $O(n^3)$ time, perhaps $O(n^{2+\epsilon})$. (Cohn-Kleinberg-Szegedy-Umans 2005.)

3. The permanent of an $n \times n$ matrix is a homogeneous polynomial of degree n^2. Its computation is a $\#P$-complete problem. (Valiant, 1979a.)

4. Every formula f of size u can be written as a determinant of some $k \times k$ matrix M_f with entries depending linearly on the original inputs where $k \leq 2u$. (Valiant, 1979b.)

5. Use GL_N representation theory to study the orbit of the permanent vs. determinant.

426 Theorem (Loehr-Warrington, 2012)
For any two partitions μ, ν,

$$s_\mu[s_\nu(X)] = \sum_{A \in S_{(a,b)}(\mu,\nu)} F_{\mathrm{Des}(\mathrm{rw}(A)^{-1})}.$$

(See their paper for definitions.)

427 Remark
The plethysm $s_\mu[s_\nu]$ is the generating function for semistandard tableaux with entries which are semistandard tableaux. The weight of such a tableau is the product of the weights of each entry. If ν is a fixed partition shape, we can identify $T \in \mathrm{SSYT}(\nu)$ with its reading word, eg. $T = 6/23/113$ goes to 623113. For a tableau $\mu = (2, 2)$ with entries which are tableaux of shape $\nu = (3, 2, 1)$, we might have $V' = (633222)(735244)/(423112)(423112)$. Let $\mathrm{SSYT}_{W(\nu)}(\mu)$ be the set of all tableaux of shape μ with entries which are words associated to semistandard Young tableaux of shape ν. One an obtain a matrix from such an element by sticking the words into rows of a matrix. Let $M(\mu, \nu)$ be the matrices obtained in this way. (We order the inner tableaux using lexicographic order.)

428 Remark
Let Words_n be the words of length n in $\mathbb{P}$. Define weight in the obvious way. Clearly

$$f(X) := \sum_{w \in \mathrm{Words}_n} x^w$$

is a symmetric function. How do we collect terms into a finite sum? (Question: what's the Schur expansion?)

Goal: partition the set Words_n by $\boxed{\text{standardizing}}$ them as above:

$$\mathrm{std}\colon \mathrm{Words}_n \to S_n.$$

429 Example

$w = 331241123$ maps to $[671492358]$. In steps,

$$\begin{aligned}331241123 &\mapsto ..1..11.. \mapsto ..1..23..\\ &\mapsto \cdots\end{aligned}$$

What are the fibers of this map? We can "lift" every letter by some amount so long as the relative ordering is the same, as in the quasisymmetric function shifting definition.

430 Lemma

We have

$$f(X) := \sum_{w \in \mathrm{Words}_n} x^w = \sum_{\pi \in S_n} F_{\mathrm{Des}(\pi^{-1})}(X).$$

The left-hand side is the generating function for Words_n*. The right-hand side is the generating function for pairs* (π, u) *where* $\pi \in S_n$ *and* $u = u_1 \ldots u_n$ *is a weakly increasing word of length* n *such that* $u_i < u_{i+1}$ *whenever* $i \in \mathrm{Des}(\pi^{-1})$*. Let* I_π *be the set of such pairs* (π, u)*. Construct a weight preserving bijective function*

$$\begin{aligned} g\colon\ \mathrm{Words}_n &\to \cup_{\pi \in S_n} I_\pi \\ w = w_1 \ldots w_n \in \mathrm{Words}_n &\mapsto (\pi, u) = (\mathrm{std}(w), \mathrm{sort}(w)).\end{aligned}$$

431 Example

$g(331241123) = ([671492358], 111223334).$

Question: does g^{-1} *exist? A moment's thought says yes.* $\mathrm{sort}(w)$ *is really just telling us how many of each letter there were.*

432 Proposition

We have

$$f(X) := \sum_{w \in \mathrm{Words}_n} x^w = \sum_{\pi \in S_n} F_{\mathrm{Des}(\pi^{-1})}(X) = \sum_{\lambda \vdash n} f^\lambda s_\lambda$$

where f^λ *is the number of standard tableaux of shape* λ*.*

Proof Bijective proof relies on two important facts:

1. $s_\lambda = \sum_{T \in \mathrm{SYT}(\lambda)} F_{\mathrm{Des}(T)}$
2. RSK is a bijection roughly preserving descents.

433 Remark

We would like to standardize elements of $M(\mu, \nu)$ by standardizing the reading word. The most obvious way does not land in $M(\mu, \nu)$. With more thought, there is a standardization algorithm which does work. An example computation:

434 Example

Let

$$A = \begin{pmatrix} 8 & 7 & 2 & 10 \\ 6 & 1 & 9 & 4 \\ 12 & 5 & 11 & 3 \end{pmatrix} \in S_{3,4}.$$

Then $\mathrm{rw}(A) = 6, 12, 8; 7, 1, 5; 11, 9, 2; 10, 4, 3.$

Similarly we can produce a standardization of a matrix, for instance via

435 Example

$$M = \begin{pmatrix} 1 & 1 & 3 & 3 & 5 \\ 1 & 2 & 2 & 2 & 4 \\ 2 & 2 & 3 & 3 & 3 \end{pmatrix} \quad \Rightarrow \quad \mathrm{std}(M) = \begin{pmatrix} 1 & 3 & 10 & 12 & 15 \\ 2 & 5 & 7 & 8 & 14 \\ 4 & 6 & 9 & 11 & 13 \end{pmatrix}$$

See Loehr-Warrington for more details.

Proof (of Loehr-Warrington's theorem.) Bijective proof. Two things to check

1. The map S taking $M \in M_{a,b}$ to $(\mathrm{std}(M), \mathrm{sort}(M))$ is invertible.
2. The map std on $M_{a,b}$ maps $M(\mu, \nu)$ into $S_{a,b}(\mu, \nu)$. ($S_{a,b}(\mu, \nu)$ refers to the standard matrices.)

From here neither step is particularly involved!

436 Open Problem
Expand $s_\mu[s_\nu(X)]$ in Schur basis and relate back to P vs NP.

437 Remark
Recently established methods:

1. Use Dual Equivalence Graphs. (Assaf 2008-2013, Roberts 2013-2014.)
2. Flip F_α to $s_\alpha \in \{1, 0, -1\} s_{\mathrm{sort}(alpha)}$. (Egge-Loehr-Warrington, 2010.)
3. Find a quasi-Schur expansion: $s_\lambda = \sum_{\alpha:\mathrm{sort}(\alpha)=\lambda} S_\alpha$. (Haglund-Luoto-Mason-van Willigenburg 2011, book 2013.)

438 Open Problem
Other similar open problems:

1. Find the Schur expansion of Macdonald polynomials

$$H_\mu(X; q, t) := \sum_{\pi \in S_n} q^{\mu^{-1}(\pi)} t^{\mathrm{maj}_\mu(\pi)} F_{\mathrm{Des}(\pi^{-1})}$$

(See Macdonald 1988, Haiman-Haglund-Loehr 2005.)

2. See also LLT polynomial expansions in same paper.
3. Find the Schur expansion of k-Schur functions

$$S_\lambda^{(k)}(X; q) = \sum_{S^* \in \mathrm{SST}(\mu, k)} q^{\mathrm{spin}(S^*)} F_{D(S^*)}.$$

(Lam-Lapointe-Morse-Shimozono + Lascoux, 2003-2010, Assaf-Billey 2012.) (Remark: there are currently five different definitions of k-Schurs that are conjecturally the same.)

439 Open Problem
Easier open problem: for a partition λ, find the Schur expansion of

$$\sum_{\pi \in S_n : \mathrm{cycle}(\pi) = \lambda} F_{\mathrm{Des}(\pi)}.$$

June 4th, 2014—Counting Reduced Words; Stanley Symmetric Functions; Little Bumps; k-Vexillary Permutations

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Summary Sara is giving another Colombia lecture.

Definition 440. Every permutation $w \in S_\infty := \cup_{n>0} S_n$ can be written as a finite product of adjacent transpositions $s_i = (i, i+1)$, sometimes in many ways. Her favorite permutation is

$$\nu = [2,1,5,4,3] = s_1 s_3 s_4 s_3 = s_1 s_4 s_3 s_4 = s_4 s_1 s_3 s_4 = s_4^3 s_1 s_3 s_4.$$

There are evidently some relations involved: $s_i^2 = 1$ ("involution"), $s_i s_j = s_j s_i$ when $|i-j| > 1$ ("commutation"), and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ ("braid"). Indeed, these are the only relations in the sense that they give a presentation of S_∞.

A minimal length expression for w is said to be $\boxed{\text{reduced}}$. The $\boxed{\text{length}}$ $\ell(w)$ is the number of inversions. If $w = s_{a_1} \cdots s_{a_p}$ is reduced, we say the sequence $(a_1, \ldots, a_p)$ is a $\boxed{\text{reduced word}}$ for w. Let $\boxed{R(w)}$ denote the set of reduced words for w.

441 Remark

Today's questions: how many reduced words are there? What structure do they have?

442 Theorem (Tit's Theorem)

The graph with vertices indexed by reduced words in $R(w)$ (for fixed w) and edges connecting two words if they differ by a commutation move or a braid move is connected. If $w = [2,1,5,4,3]$, then $R(w)$ has 8 elements arranged in a cycle,

$$1343 - 1434 - 4134 - 4314 - 4341 - 3431 - 3413 - 3143 - 1343.$$

443 Example

$[7,1,2,3,4,6]$ has reduced word $(6,5,4,3,2,1)$, and we clearly can't apply braid relations or commutations, so there is precisely one such word. While we can count the number of reduced words this way, we have to generate all of them, which is inefficient.

444 Remark

There is one reduced word for the identity. For any other permutation,

$$\#R(w) = \sum_{i \in \mathrm{Des}(w)} \#R(ws_i).$$

This is not quite as "efficient as possible", since we might compute $\#R(w')$ multiple times while we recurse.

445 Theorem

Stanley's first observation: $R([n, n-1, \ldots, 3, 2, 1])$ is the same as the number of standard tableaux of the staircase shape $(n-1, n-2, \ldots, 2, 1)$, which is

$$\frac{\binom{n}{2}!}{1^{n-1} 3^{n-2} \cdots (2n-3)!}.$$

Definition 446. (Stanley 1984.) For any $w \in S_n$, define

$$G_w(X) := \sum_{a \in R(w)} F_{\mathrm{Des}(a)}(X).$$

447 Example

For $w = [7,1,2,3,4,5,6]$, $G_w(X) = F_{(1,2,3,4,5)}(X) = F_{(1^6)}(X) = s_{1^6}(X)$.

448 Theorem

Stanley's second observation: G_w is a symmetric function with positive Schur expansion. They are now called $\boxed{\textit{Stanley symmetric functions}}$.

449 Theorem (Edelman-Greene 1987)
G_w has positive Schur expansion,

$$G_w = \sum_\lambda a_{\lambda,w} s_\lambda, \qquad a_{\lambda,w} \in \mathbb{N}.$$

Corollary: $|R(w)| = \sum_\lambda a_{\lambda,w} f^\lambda$ where f^λ is the number of standard tableaux of shape λ.

450 Example
If $w = w_0$, then $G_w = s_\delta$ where δ is the staircase shape with $n-1$ rows.

Proof They construct an injective map from $R(w)$ to pairs of tableaux (P, Q) of the same shape where P is row- and column-strict and Q is standard. For each P every single standard tableaux of the same shape occurs as Q in the image.

Algorithm: Edelman-Greene insertion is a variation on RSK. The only difference is when inserting i into a row with i and $i+1$ already, skip that row and insert $i+1$ into the next row.

451 Example
For 4314, we get 4, 4/3, 4/3/1, 4/3/14 = P with $Q = 3/2/14$.

If a maps to (P, Q) under Edelman-Greene, then $\mathrm{Des}(w) = \mathrm{Des}(Q)$.

Definition 452. The $\boxed{\text{Coxeter-Knuth graph}}$ for w has vertices $R(w)$. If two reduced words differ by a braid relation, they have an edge between them. We only allow edges for "witnessed commutations", where roughly two reduced words are connected by an edge labeled i if they agree in all positions except for a single Coxeter-Knuth relation starting in position i.

453 Theorem (Edelman-Greene 1987)
Two words $a, b \in R(w)$ have the same P-tableau if and only if they are in the same component of the Coxeter-Knuth graph for w.

454 Theorem
The Coxeter-Knuth graphs in type A are dual equivalence graphs and the isomorphism is given by the Q tableaux in Edelman-Greene insertion. Furthermore, descent sets are preserved.

In type A this is a nice corollary of (Roberts, 2014) and (Hamaker-Young, 2014).

455 Notation
Let $1 \times w = [1, w_1 + 1, \dots, w_n + 1]$. There is a bijection from $R(w)$ to $R(1 \times w)$ that preserves descent sets, so $G_w = G_{1\times w}$.

456 Theorem (Lascoux-Schuztenberger)
If w is vexillary, then $G_w = s_{\lambda(w)}$. Otherwise,

$$G_w = \sum G_{w'}$$

where the sum is over all w' such that $\ell(w) = \ell(w')$ and $w' = t_{ir} t_{rs} w$ with $0 < i < r$. Call this set $T(w)$. In the case $T(w)$ is empty, replace w by $1 \times w$.

457 Remark
We can apply this formula recursively, forming a tree where the leaves correspond to vexiillary permutations.

Some nice corollaries: if w doesn't contain any 2143 pattern, then $T(w)$ has one element w' which also avoids 2143. If w is vexillary, $\#R(w) = f^{\lambda(w)}$ as before. The proof Sara's outlined isn't bijective.

Is there a bijection from $R(w)$ to $\cup_{w' \in T(w)} R(w')$ which preserves the descent set, the Coxeter-Knuth classes, and the Q-tableau? Yes! It's called Little's bijection, named for David Little (Little, 2003) + (Hamaker-Young, 2013).

Definition 458 (Little Bump Algorithm). Given a reduced word, there is an associated reduced wiring diagram. Very roughly, "push" crossings, checking if the resulting word is reduced and stopping when it is. The Little bijection initiates a Little bump at the crossing (r, s) of the lexicographically largest inversion.

459 Example

Little's bijection could be used to prove the transition equation. There is an identity (Macdonald 1991, Fomin-Stanley 1994, Young 2014). For w_0 the longest word,

$$\sum_{a_1 \ldots a_p \in R(w_0)} a_1 \cdots a_p = \binom{n}{2}!$$

(Young, 2014) There exists an algorithm based on the Little bijection to choose a reduced word $a = a_1 \cdots a_p \in R(w_0)$ with probability distribution $a_1 \cdots a_p / \binom{n}{2}!$.

(Macdonald 1991, Fomin-Stanley 1994.) For any permutation w of length p,

$$\sum_{a_1 \cdots a_p \in R(w)} a_1 \cdots a_p = p! \mathfrak{S}_w(1, 1, \ldots, 1)$$

where $\mathfrak{S}_w(x_1, \ldots, x_n)$ is the Schubert polynomial for w. Open problem: find a bijective proof.

460 Remark

The Schubert polynomial

$$\mathfrak{S}_w = \sum_{a \in R(w)} \sum_{i_1 \leq \cdots \leq i_p} x_1 \cdots x_p = \sum_{T \in RC(w)} x^T$$

where the sum is over $i_j < i_{j+1}$ if $a_j < a_{j+1}$, $i_j \leq a_j$. Lascoux and Schutzenberger originally defined these, and Billey-Jokusch-Stanley proved the above equivalent formula. The second equality uses rc-graphs; see Billey-Bergeron's rc-graphs paper for more on that.

Definition 461. A permutation is $\boxed{\text{k-vexillary}}$ k-vexillary if $F_w = \sum a_{\lambda,w} s_\lambda$ and $a_{\lambda,w} \leq k$.

462 Theorem (Billey-Pawlowski)

A permutation w is k-vexillary iff w avoids a particular finite set of patterns V_k.

463 Example

2-vexillary permutations have expansion $F_w = s_{\lambda(w)} + s_{\lambda(w^{-1})'}$. 3-vexillary permutations are multiplicity free.

PROOF One step: James-Peel use generalized Specht modules. For $D(w)$ the diagram of w, it happens that

$$S^{D(w)} = \oplus (S^\lambda)^{a_{\lambda,w}}.$$

June 7th, 2014—QSYM*, the Concatenation Hopf Algebra, and Solomon Descent Algebras

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Summary Josh is presenting on Malvenuto and Reutenauer's "Duality between Quasi-Symmetric Functions and the Solomon Descent Algebra".

There is a bialgebra structure $(\mathrm{QSYM}_{\mathbb{Q}}, m, u, \gamma, \epsilon)$ which dualizes to give a bialgebra

$$(\mathrm{QSYM}^*_{\mathbb{Q}}, \gamma^*, \epsilon^*, m^*, u^*).$$

This is isomorphic as a bialgebra to the bialgebra $\mathbb{Q}\langle t_1, t_2, \ldots\rangle$ where $\deg t_i = i$ and each t_i is primitive. They are both Hopf algebras; the latter is called the concatenation Hopf algebra. Note that the main difference between

$$\mathrm{NSYM}_{\mathbb{Q}} := \mathbb{Q}\langle H_1, H_2, \ldots\rangle$$

and the concatenation Hopf algebra is that the coproducts differ (rock-breaking vs. primitive).

There is a second coproduct γ' on QSYM with counit ϵ'. Dualizing gives a ring structure $(\mathrm{QSYM}^*, (\gamma')^*, (\epsilon')^*)$. Gessel showed the Solomon Descent Algebras Σ_n have a ring structure isomorphic to $(\mathrm{QSYM}_n^*, (\gamma')^*, (\epsilon')^*)$. (Indeed, this second coproduct and counit give a second bialgebra structure $(\mathrm{QSYM}, m, u, \gamma', \epsilon')$, which gives a second bialgebra structure $(\mathrm{QSYM}^*, (\gamma')^*, (\epsilon')^*, m^*, u^*)$.)

There is a different multiplication $*$ and a coproduct Δ on Σ (with unit and counit) such that $(\Sigma, *, \Delta)$ forms a Hopf algebra isomorphic to $(\mathrm{QSYM}^*, \gamma^*, \epsilon^*, m^*, u^*)$.

464 Remark

Outline:

1. Define QSYM^* with bialgebra structure.
2. Show (1) is naturally isomorphic to the concatenation Hopf algebra $\mathbb{Q}\langle T\rangle$.
3. Corollaries: $\mathrm{QSYM}_{\mathbb{Q}}$ is a free algebra and a free $\mathrm{SYM}_{\mathbb{Q}}$-module; antipode formula for QSYM.
4. Define QSYM^* with second algebra structure.
5. Define Solomon Descent Algebra Σ, which is naturally isomorphic to (4).
6. Define a Hopf algebra structure on Σ agreeing with (1).

(Note: only 1, 2, and part of 3 were presented in class.)

Definition 465. Let T be a countable totally ordered set. $\boxed{\mathrm{QSYM}(T)}$ is the subring of formal power series $F(T)$ over $\mathbb{Z}$ in commuting variables T which are of bounded degree and which have the property that, if $t_1^{c_1} \cdots t_k^{c_k}$ is a monomial in $F(T)$ (here $c_i \geq 1$), then $u_1^{c_1} \cdots u_k^{c_k}$ is a monomial in $F(T)$ with the same coefficient, for any $u_1 < \cdots < u_k$ in T. An element of QSYM is a $\boxed{\text{quasisymmetric function}}$.

To each (strong) composition $C = (c_1, \ldots, c_k)$, we associate a quasisymmetric function $\boxed{M_C^T}$ in $\mathrm{QSYM}(T)$ given by $\sum_{t_1 < \cdots < t_k} t_1^{c_1} \cdots t_k^{c_k}$. These are the monomial quasisymmetric functions, and they form a $\mathbb{Z}$-basis for $\mathrm{QSYM}(T)$.

Note that $\mathrm{QSYM}(X)$ and $\mathrm{QSYM}(Y)$ are canonically isomorphic, for any totally ordered sets X and Y (even if there is no order-preserving bijection between them). In particular, send M_C^X to M_C^Y.

466 Remark

Consider the tensor product $\mathrm{QSYM}(X) \otimes \mathrm{QSYM}(Y)$ (tensored over $\mathbb{Z}$). An element $x_1^{a_1} \cdots \otimes y_1^{b_1} \cdots$ with $x_1 < x_2 < \cdots$, $y_1 < y_2 < \cdots$ can naturally be identified with the element $x_1^{a_1} \cdots y_1^{b_1} \cdots$ of $\mathrm{QSYM}(X \cup Y)$ where $X \cup Y$ is totally ordered by declaring $x_i < y_j$ for all i, j. This gives a ring isomorphism $\mathrm{QSYM}(X) \otimes \mathrm{QSYM}(Y) \cong \mathrm{QSYM}(X \cup Y)$. It operates via

$$M_C^{X\cup Y} \mapsto \sum_{C=AB} M_A^X \otimes M_B^Y$$

where AB denotes the concatenation of compositions A and B.

Definition 467. Define $\boxed{\gamma}$: $\mathrm{QSYM}(T) \to \mathrm{QSYM}(X) \otimes \mathrm{QSYM}(Y)$ to be the composite

$$\mathrm{QSYM}(T) \xrightarrow{\sim} \mathrm{QSYM}(X \cup Y) \to \mathrm{QSYM}(X) \otimes \mathrm{QSYM}(Y) \xrightarrow{\sim} \mathrm{QSYM}(T) \otimes \mathrm{QSYM}(T).$$

We call γ the $\boxed{\text{outer coproduct}}$. One can check

$$\gamma(M_C^T) = \sum_{C=AB} M_A^T \otimes M_B^T.$$

From now on, we drop the alphabet from the notation when disambiguation is not needed.

468 Remark

The counit ϵ of γ is evaluation at 0, i.e. it gives the constant coefficient. Indeed, $(\mathrm{QSYM}, m, u, \gamma, \epsilon)$ is a bialgebra, where m denotes the usual multiplication and u the natural inclusion $\mathbb{Z} \to \mathrm{QSYM}$.

469 Remark

QSYM is a graded $\mathbb{Z}$-algebra, with homogeneous $\mathbb{Z}$-basis consisting of M_C of degree $|C| := \sum c_i$. (We allow $M_\varnothing = 1$.) Moreover each homogeneous component QSYM_n of QSYM is finite dimensional.

Definition 470. The $\boxed{\text{graded dual}}$ of QSYM is

$$\boxed{\mathrm{QSYM}^*} := \bigoplus_{n=0}^{\infty} \mathrm{QSYM}_n^*$$

as a $\mathbb{Z}$-module, where $\mathrm{QSYM}_n^* := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{QSYM}_n, \mathbb{Z})$.

471 Remark

The maps m, u, γ, ϵ above respect the grading, so we may dualize them as well. (Note that $(\mathrm{QSYM} \otimes \mathrm{QSYM})_n := \oplus_{p+q=n} \mathrm{QSYM}_p \otimes \mathrm{QSYM}_q$.) Since each component has finite rank, the natural map $\mathrm{QSYM}_p^* \otimes \mathrm{QSYM}_q^* \to (\mathrm{QSYM}_p \otimes \mathrm{QSYM}_q)^*$ is an isomorphism, allowing us to view m^* as a coproduct with counit u^*. Similarly $(\mathrm{QSYM}^*, \gamma^*, \epsilon^*, m^*, u^*)$ is a bialgebra.

472 Remark

Since QSYM_n is finite dimensional, QSYM_n^* is isomorphic as a $\mathbb{Z}$-module to QSYM_n. Similarly, QSYM^{**} is canonically isomorphic to QSYM as a bialgebra. Graded duals V^* of other graded $\mathbb{Z}$-modules, algebras, or coalgebras, with finite rank in each (free) component, are defined in the same way. Note that $V^* \otimes V^*$ is canonically isomorphic to $(V \otimes V)^*$ for such an object.

473 Remark

We have a dual $\mathbb{Z}$-basis $\{M_C^*\}$ of QSYM^*, which is the $\mathbb{Z}$-linear function $\mathrm{QSYM} \to \mathbb{Z}$ which is 1 on M_C and 0 on M_D for $D \neq C$. Throughout, we use the isomorphism (of $\mathbb{Z}$-modules) $\mathrm{QSYM} \xrightarrow{\sim} \mathrm{QSYM}^*$ given by $M_C \mapsto M_C^*$.

As usual, the dual is non-canonically isomorphic (as a $\mathbb{Z}$-module) to the original object: we seem to like the monomial basis, so we choose it to give a "pseduo-canonical" isomorphism. However, we could theoretically use the fundamental basis, giving a different isomorphism, and neither choice is clearly "correct".

Definition 474. If V is a $\mathbb{Z}$-module, define the $\boxed{\text{pairing}}$ (i.e. $\mathbb{Z}$-bilinear map)

$$\begin{aligned} \langle -, - \rangle \colon V^* \times V &\to \mathbb{Z} \\ \langle \phi, v \rangle &\mapsto \phi(v). \end{aligned}$$

If the graded dual of V exists in the above sense, this naturally induces a pairing

$$\begin{aligned} \langle - \otimes -, - \otimes - \rangle \colon (V^* \otimes V^*) \times (V \otimes V) &\to \mathbb{Z} \\ \langle \phi \otimes \psi, v \otimes w \rangle &\mapsto \langle \phi, v \rangle \langle \psi, w \rangle \\ &= \phi(v)\psi(w). \end{aligned}$$

475 Remark

Suppose V, W have graded duals V^*, W^*. If $\Gamma\colon V \to W$, the dual $\Gamma^*\colon W^* \to V^*$ is defined by

$$(\phi\colon W \to \mathbb{Z}) \mapsto (\phi \circ \Gamma\colon V \to W \to \mathbb{Z}).$$

In particular, $\Gamma^*(\phi)(v) = \phi(\Gamma(v))$. In terms of the pairing,

$$\langle \Gamma^*(\phi), v\rangle = \langle \phi(\Gamma), v\rangle,$$

and this condition characterizes Γ^*.

476 Remark

Denote the product $\gamma^*\colon$ QSYM$^* \otimes$ QSYM$^* \to$ QSYM* by juxtaposition. It is characterized by

$$\langle \phi\psi, F\rangle = \langle \phi \otimes \psi, \gamma(F)\rangle$$

for all F. Likewise, the coproduct m^* can be defined by requiring

$$\langle \boxed{m^*(\phi)}, F \otimes G\rangle = \langle \phi, FG\rangle$$

for all $F \otimes G$. The unit remains obvious. The counit is given by $u^*(\phi) = \phi(1)$.

Definition 477. Let T be a set of *non*commuting variables and consider the free associative $\mathbb{Z}$-algebra $\mathbb{Z}\langle T\rangle$. Define coproduct $\delta(t_i) = t_i \otimes 1 + 1 \otimes t_i$, counit $t_i \mapsto 0$, $1 \mapsto 1$, and antipode $S(t_1 \cdots t_n) = (-1)^n t_n \cdots t_1$, which gives the $\boxed{\text{concatenation Hopf algebra}}$ over $\mathbb{Z}$. (This is similar to NSYM, except we use a primitive rather than rock-breaking coproduct.)

478 Remark

$\mathbb{Z}$ in the above may be replaced by $\mathbb{Q}$ (or any field) with no change whatsoever. We use SYM$_{\mathbb{Q}}$, QSYM$_{\mathbb{Q}}$, QSYM$^*_{\mathbb{Q}}$ and the like to denote these versions.

479 Theorem (2.1)

QSYM$^*_{\mathbb{Q}}$ *is canonically isomorphic as a bialgebra to the concatenation Hopf algebra* $\mathbb{Q}\langle T\rangle$, *where* $T = \{t_n : n \geq 1\}$ *and* $\deg t_n = n$.

Explicitly, the isomorphism is given by

$$t_n \mapsto \sum_{|C|=n} \frac{(-1)^{\ell(C)-1}}{\ell(C)} M^*_C =: \boxed{P_{(n)}},$$

where $\ell(C) = k$ *for* $C = (c_1, \ldots, c_k)$.

480 Remark

Note that $\mathbb{Q}\langle T\rangle$ is finitely generated in each component, and indeed $\dim(\mathbb{Q}\langle T\rangle)_n$ is the number of (strong) compositions of size n.

Proof There are three main steps:

(1) $M^*_A M^*_B = M^*_{AB}$.

(2) $\Delta(M^*_{(n)}) = \sum_{p+q=n} M^*_{(p)} \otimes M^*_{(q)}$ for $n \geq 1$.

(3) The elements $P^*_{(n)} \in$ QSYM$^*_{\mathbb{Q}}$, $n \geq 1$ from the theorem statement are primitive.

Note that from (1) we may define an isomorphism of algebras NSYM$_{\mathbb{Q}} \to$ QSYM$^*_{\mathbb{Q}}$ given by $t_n \mapsto M^*_{(n)}$. Jose asserted this is an isomorphism of Hopf algebras. (2) is a weak form of the coalgebra isomorphism. Roughly, (3) goes from the rock-breaking coproduct on NSYM to the primitive coproduct on $\mathbb{Q}\langle T\rangle$.

(1) Follows since

$$\begin{aligned}
\langle M_A^* M_B^*, M_C\rangle &= \langle M_A^* \otimes M_B^*, \gamma(M_C)\rangle \\
&= \sum_{C=A'B'} \langle M_A^* \otimes M_B^*, M_{A'} \otimes M_{B'}\rangle \\
&= \sum_{C=A'B'} \delta_{AA'}\delta_{BB'} = \delta_{AB,C} \\
&= \langle M_{AB}^*, M_C\rangle.
\end{aligned}$$

Thus $\mathrm{QSYM}_{\mathbb{Q}}^*$ is freely generated as a $\mathbb{Z}$-algebra by $\{M_{(i)}^* : i \geq 1\}$.

(2) is similar to (1) and not worth the time to discuss.

(3) begins by defining the $P_{(n)}^*$ through a generating function,

$$\sum_{n\geq 1} P_{(n)}^* t^n = \log(1 + M_{(1)}^* t + M_{(2)}^* t^2 + \cdots),$$

where these expressions live in $\mathrm{QSYM}^*[[t]]$. By comparing coefficients and using (1), the formula from the theorem statement follows, so in particular P_n is homogeneous of degree n. These elements are primitive:

$$\begin{aligned}
\sum_{n\geq 1} \Delta(P_{(n)}^*)t^n &= \Delta(\log \sum_{i\geq 0} M_{(i)}^* t^i) = \log\left(\sum_{i\geq 0} \Delta(M_{(i)}^*)t^i\right) \\
&= \log\left(\sum_{p,q\geq 0} M_{(p)}^* t^p \otimes M_{(q)}^* t^q\right) \\
&= \log\left((\sum_{p\geq 0}(M_{(p)}^* t^p \otimes 1)(1 \otimes \sum_{q\geq 0} M_{(q)}^* t^q)\right) \\
&= \log\left(\sum_{p\geq 0} M_{(p)}^* t^p \otimes 1\right) + \log\left(1 \otimes \sum_{q\geq 0} M_{(q)}^* t^q\right) \\
&= \log\left(\sum_{p\geq 0} M_{(p)}^* t^p\right) \otimes 1 + 1 \otimes \log\left(\sum_{q\geq 0} M_{(q)}^* t^q\right) \\
&= \sum_{n\geq 1} P_{(n)}^* t^n \otimes 1 + 1 \otimes \sum_{n\geq 1} P_{(n)}^* t^n.
\end{aligned}$$

(The fifth equality uses the fact that $\log(ab) = \log(a) + \log(b)$ when a, b commute.)

Applying exp to the generating function defining P_n, we find

$$M_{(n)}^* = \sum_{|C|=n} \frac{1}{\ell(C)!} P_C^*.$$

Hence each $M_{(i)}^*$ is a polynomial combination of P_n's, so we've found a primitive generating set. Apparently it's free; they say this follows from the formula in the theorem statement, but I don't see it immediately.

481 Corollary
$\mathrm{QSYM}_{\mathbb{Q}}$ *has a free generating set containing a free generating set of* $\mathrm{SYM}_{\mathbb{Q}}$. *In particular,* $\mathrm{QSYM}_{\mathbb{Q}}$ *is a free commutative algebra and a free* $\mathrm{SYM}_{\mathbb{Q}}$*-module.*

Proof Identify $\mathrm{QSYM}^*_{\mathbb{Q}}$ and $\mathbb{Q}\langle T\rangle$ as in the theorem. Define $t_C := t_{c_1}\cdots t_{c_k}$ for $C = (c_1,\ldots,c_k)$. Note $\{t_C\}$ is a $\mathbb{Q}$-basis, essentially by definition. Let $\{P_C\}$ denote its dual basis in $\mathrm{QSYM}_{\mathbb{Q}}$. Take L to be the set of $\boxed{\text{Lyndon compositions}}$, which is to say, the set of compositions which are $\boxed{\text{Lyndon words}}$ (with respect to the natural ordering on $\mathbb{P}$), which is to say the set of compositions which are lexicographically smaller than all of their rotations. Then $\{P_\ell : \ell \in L\}$ is a free generating set for QSYM as an algebra: see Reutenauer, "Free Lie Algbras", Theorem 6.1(i).

We claim $P_{(n)} = M_{(n)}$. Since this is just the usual power symmetric functions of degree n, these are free generators of $\mathrm{SYM}_{\mathbb{Q}}$, giving the first part of the corollary. To prove the claim, roughly, use the formula for $M^*_{(n)}$ in terms of P^*_C to write M^*_D as a certain sum of P^*_C, where the sum is over $C \le D$ (ordered by refinement). Hence the transition matrix from M^*_D to P^*_C is upper triangular. Taking duals just transposes the matrix, whence P_C is a sum over $C \le D$ of M_D. Since $C = (n)$ is as coarse as possible for n-compositions, the sum has one term, and in fact the coefficient is 1.

The second part of the corollary is just saying

$$\mathrm{QSYM}_{\mathbb{Q}} = \mathbb{Q}[\{P_\ell\}] = \mathbb{Q}[\{P_{(n)}\}][\{P_\ell\} - \{P_{(n)}\}] = \mathrm{SYM}_{\mathbb{Q}}[\{P_\ell\} - \{P_{(n)}\}],$$

so $\mathrm{QSYM}_{\mathbb{Q}}$ is a free $\mathrm{SYM}_{\mathbb{Q}}$-algebra, and in particular a free $\mathrm{SYM}_{\mathbb{Q}}$-module.

Definition 482. Let F_C denote the fundamental quasisymmetric function indexed by C, let I be the usual $\boxed{\text{stars and bars bijection}}$ between (strong) compositions of n and subsets of $[n-1]$. Courting ambiguity, we denote both I and its inverse by the same letter I, relying on context to disambiguate. Let $\overline{C}$ denote the reverse of the composition C. Define $\boxed{\omega}$ to be the involution on n-compositions given by applying I, taking the complement in $[n-1]$, applying I, and reversing the resulting composition.

483 Example

$\omega((2,1,3,2,1)) = (2,2,1,3,1)$:

$$\begin{aligned} 2+1+3+2+1 &\mapsto **\,|\,*\,|\,***\,|\,**|* \\ &\mapsto *|\,***\,|\,*\,|\,**|\,** \\ &\mapsto 1+3+1+2+2 \\ &\mapsto 2+2+1+3+1. \end{aligned}$$

484 Corollary

$\mathrm{QSYM}_{\mathbb{Q}}$ *is a Hopf algebra with antipode S equivalently defined by either*

$$S(M_C) := \sum_{C\le D} (-1)^{\ell(C)} M_{\overline{D}} \qquad \textit{or} \qquad S(F_C) := (-1)^{|C|} F_{\omega(C)}.$$

Proof The concatenation Hopf algebra $\mathbb{Q}\langle T\rangle$ from the theorem has antipode S^* given by $S^*(t_1\cdots t_k) = (-1)^k t_k\cdots t_1$, which is the unique anti-automorphism of $\mathbb{Q}\langle T\rangle$ such that $S^*(t_n) = -t_n$. Hence $\mathrm{QSYM}^*_{\mathbb{Q}}$ is a Hopf algebra with antipode S^* determined by $S^*(P^*_{(n)}) = -P^*_{(n)}$. Since $\mathrm{QSYM}^{**}_{\mathbb{Q}}$ is canonically isomorphic to $\mathrm{QSYM}_{\mathbb{Q}}$ as a bialgebra, $\mathrm{QSYM}_{\mathbb{Q}}$ is a Hopf algebra with antipode $S^{**} = S$, which we now compute.

Using generating functions, one may show

$$S^*(M^*_{(n)}) = \sum_{|C|=n} (-1)^{\ell(C)} M^*_C.$$

It follows that

$$S^*(M^*_D) = \sum_{C\le D} (-1)^{\ell(C)} M^*_{\overline{C}};$$

the $\overline{C}$ comes from the fact that S^* is an *anti*automorphism; eg. try the $D = (n, m)$ case. Applying duality (and reversing all compositions in sight) gives the first formula. The second formula follows with a little more work; see the paper for details.

Definition 485. Define $\boxed{\omega}$: $\mathrm{QSYM}_{\mathbb{Q}} \to \mathrm{QSYM}_{\mathbb{Q}}$ to be the linear map given by $\omega(F_C) = F_{\omega(C)}$.

486 Corollary

ω is an antiautomorphism of $\mathrm{QSYM}_{\mathbb{Q}}$ which extends the usual conjugation automorphism of $\mathrm{SYM}_{\mathbb{Q}}$.

Proof Since $\omega(F_C) = (-1)^{|C|}S(F_C)$, ω is an antiautomorphism. Since $\omega(n) = (1^n)$, we have $\omega(F_{(n)}) = F_{1^n}$, which is $\omega(h_n) = e_n$ using the complete homogeneous and elementary symmetric polynomials. This property characterizes the usual conjugation automorphism.

487 Remark

The theorem was useful. Our next goal is to define the Solomon Descent Algebras, recall previous results, and endow it with a new Hopf algebra structure making it isomorphic to $\mathrm{QSYM}_{\mathbb{Q}}^*$ from above.

Definition 488. Let X, Y be countable totally ordered sets. Define XY as $X \times Y$ with lecicographic order. Recall the canonical map $\mathbb{Z}[XY] \to \mathbb{Z}[X \cup Y]$ given by $M_C^{XY} \mapsto M_C^{X \cup Y}$. Suppressing other similar canonical maps, define a second coproduct γ' on QSYM to be the composite

$$\begin{aligned} \boxed{\gamma'}\colon \mathrm{QSYM}(T) &\to \mathrm{QSYM}(T) \otimes \mathrm{QSYM}(T) \\ \mathrm{QSYM}(XY) &\to \mathrm{QSYM}(X \cup Y) \to \mathrm{QSYM}(X) \otimes \mathrm{QSYM}(Y) \\ &\to \mathrm{QSYM}(Y) \otimes \mathrm{QSYM}(X). \end{aligned}$$

Let $\boxed{\epsilon'}$ be the counit determined by $\epsilon'(F_{(n)}) = 1$ and $\epsilon'(F_C) = 0$ for $\ell(C) \geq 2$.

489 Remark

More concretely, Gessel showed (and Jair said)

$$\gamma'(F_{D(\pi)}) = \sum_{\sigma\tau=\pi} F_{D(\sigma)} \otimes F_{D(\tau)}$$

where $D(\pi) := I(\mathrm{Des}(\pi))$.

These operations give QSYM^* a second bialgebra structure $(\mathrm{QSYM}, m, u, \gamma', \epsilon')$.

Definition 490. Let $n \geq 0$ and $I \subset [n-1]$. Define

$$\boxed{D_I} := \sum_{\substack{\sigma \in S_n \\ \mathrm{Des}(\sigma)=I}} \sigma \in \mathbb{Z}S_n.$$

Say $\deg D_i := n$. Let $\boxed{\Sigma_n} := \mathrm{Span}_{\mathbb{Z}}\{D_I\}$ be a $\boxed{\text{Solomon Descent Algebra}}$. (Note: it is not obvious that this is closed under multiplication.)

491 Theorem (Solomon, 1976)

Σ_n is a subalgebra of $\mathbb{Z}S_n$.

492 Remark

See for instance Schocker 2004, "The Descent Algebra of the Symmetric Group", for a survey of relatively recent work, a formula for the expansion coefficients, and much more. Note he uses $(\mathbb{Z}S_n)^{\mathrm{op}}$. This algebra is also implemented in Sage; see "Descent Algebras". Solomon in fact defined similar algebras for all Coxeter groups.

493 Remark

Malvenuto and Reutenauer define $\Sigma := \oplus_{n\geq 0}\Sigma_n$ with a (non-unital) ring structure given by $\sigma\tau = 0$ if σ, τ do not belong to the same S_n. They claim in Theorem 3.2 that Σ is isomorphic to $\mathrm{QSYM}^*((\gamma')^*)$ as a not-necessarily-unital ring, but since γ' had a counit ϵ', the latter is a unital ring, forcing the former to be as well, a contradiction. More concretely, if you compute the product of two elements from different homogeneous components of QSYM^* using their bijection, you always get 0 on the Σ side, which is nonsense.

They almost surely meant the slight variation below, given by (Gessel, 1984). They do not use this theorem for anything more than motivation.

494 Theorem (Gessel, 1984)

$(\mathrm{QSYM}^*_n, (\gamma')^*)$ is isomorphic as a ring to Σ_n, with $F^*_C \leftrightarrow D_C$.

495 Remark

Letting juxtaposition denote $(\gamma')^*$, as before this product on QSYM^*_n is characterized by

$$\langle \phi\psi, F\rangle = \langle \phi \otimes \psi, \gamma'(F)\rangle.$$

Using $\phi = F^*_A, \psi = F^*_B, F = F_C$, and applying the theorem, the right-hand side is the coefficient of $F_A \otimes F_B$ in $\gamma'(F_C)$ and the left-hand side is the coefficient of D_C in the product $D_A D_B$. Compactly,

$$(\gamma'(F_C))_{A\otimes B} = (D_A D_B)_C.$$

496 Theorem

Let $\mathbb{Z}S := \oplus_{n\geq 0}\mathbb{Z}S_n$ as a $\mathbb{Z}$-module. There is a product $*$ and coproduct Δ on $\mathbb{Z}S$ (with unit and counit) which make $\mathbb{Z}S$ into a Hopf algebra. Indeed, $\Sigma := \oplus_{n\geq 0}\Sigma_n \subset \mathbb{Z}S$ is a Hopf subalgebra, and $(\mathrm{QSYM}^*, \gamma^*, \epsilon^*, m^*, u^*)$ is isomorphic to Σ as a Hopf algebra via $F^*_C \leftrightarrow D_C$.

497 Remark

Note that Σ no longer has a product induced by the group algebra in any sense.

Proof Lengthy; main tool is the "shuffle Hopf algebra"; they connect $*$ to the convolution in $\mathrm{End}(\mathbb{Z}\langle T\rangle)$ and consider a second (ultimately dual) bialgebra structure on $\mathbb{Z}S$; see their paper for details. We merely define the operations involved.

Definition 498. Let $\boxed{\mathrm{str}}$ denote the $\boxed{\text{straightening}}$ in S_n of a word (of length n) on a totally ordered alphabet. Call $\mathrm{str}(w)$ the $\boxed{\text{standard permutation}}$ of w.

499 Example

$\mathrm{PIAZZA} \mapsto - - 1 - -2 \mapsto -31 - -2 \mapsto 431 - -2 \mapsto 431562 = \mathrm{str}(\mathrm{PIAZZA})$.

Definition 500. For $\sigma \in S_n$ and $I \subset [n]$, let $\boxed{\sigma|I}$ denote the word obtained from σ (viewed as a word on $[n]$ in one-line notation) where only letters in I are kept.

Definition 501. Define a coproduct Δ on $\mathbb{Z}S$ by

$$\boxed{\Delta}(\sigma) := \sum_{i=0}^{n} \sigma|[1,i] \otimes \mathrm{str}(\sigma|[i+1,n]).$$

502 Example

$$\begin{aligned}\Delta(3124) &= \lambda \otimes 3124 + 1 \otimes \mathrm{str}(324) + 12 \otimes \mathrm{str}(34) + 312 \otimes \mathrm{str}(4) + 3124 \otimes \lambda \\ &= \lambda \otimes 3124 + 1 \otimes 213 + 12 \otimes 12 + 312 \otimes 1 + 3124 \otimes \lambda.\end{aligned}$$

Here $\lambda \in S_0$ is the empty word. The associated counit is given by $\lambda \mapsto 1$ and $\sigma \mapsto 0$ for $\sigma \in S_n$ with $n \geq 1$.

Definition 503. Define a product $\boxed{*}$ on $\mathbb{Z}S$ as follows. For $\sigma \in S_n, \tau \in S_m$, let

$$\sigma * \tau = \sum uv$$

where the sum is over words u, v in $[n+q]$ such that u, v together are a disjoint union of $[n+q]$, $\mathrm{str}(u) = \sigma$, and $\mathrm{str}(v) = \tau$.

504 Example

$12 * 12 = 1234 + 1324 + 1423 + 2314 + 2413 + 3412.$

Evidently λ is the identity.

List of Symbols

S Antipode, page 36

$S(\phi, \psi)$ Equalizer Subcoalgebra, page 66

$S_+(H, \zeta)$ Even Subalgebra, page 66

$S_-(H, \zeta)$ Odd Subalgebra, page 66

$U \otimes V$ Tensor Product of Vector Spaces, page 15

$V(x_1, \ldots, x_n)$ Vandermonde Determinant, page 47

X_G Chromatic Polynomial, page 54

$Y\uparrow_H^G : G \to \mathrm{GL}(V)$ Induced Representation, page 6

Δ Coproduct on Combined Solomon Descent Algebra, page 94

$\Delta_2(c)$ Iterated Comultiplication, page 29

$\Delta_{C\otimes C}$ Tensor Product Coalgebra Comultiplication, page 32

Δ_{n-1} Iterated Comultiplication, page 29

$\mathrm{Des}(T)$ Descent Set of Standard Young Tableau T, page 49

$\mathrm{Ext}^1_H(M, N)$ An Ext Group, page 45

$\mathrm{GKdim}\, H$ Gelfand-Kirillov or GK dimension, page 41

$\mathrm{Hom}(C, A)$ Algebra of Morphisms from a Coalgebra to an Algebra, page 37

NSYM Non-Commutative Symmetric Functions, page 64

Perms Permutation Bialgebra, page 35

QSYM Quasisymmetric Functions, page 49

QSYM(T) Quasisymmetric Functions, page 88

QSYM* Dual of QSYM, page 89

SIT(α) Standard Immaculate Tableaux of Shape α, page 79

SSYT(λ) Semistandard Young Tableaux of Shape λ, page 17

SSYT(λ/μ) Semistandard Skew Young Tableaux, page 27

Σ_n Solomon Descent Algebras, page 93

$\alpha(S)$ Count of Maximal Chains in Rank-Selected Poset, page 62

$\mathbb{C}A_k(n)$ Partition Algebra, page 67

$\mathbb{Z}DS$ (Span of) Special Double Posets, page 71

$\Bbbk[G]$ Global Sections, page 73

$\mathcal{C}(P)$ Incidence Coalgebra of P, page 57

$\mathcal{F}([V])$ Frobenius Characteristic Map, page 23

$\mathcal{G}$ Finite Simple Graph Hopf Algebra, page 54

$\mathcal{H}$ Alternate Notation for $\mathcal{I}_{\mathrm{ranked}}$, page 60

$\mathcal{H}$ Non-Commutative Symmetric Functions, page 64

$\mathcal{I}$ (Unranked) Reduced Indcidence Bialgebra, page 34

$\mathcal{I}_{\text{ranked}}$ Reduced Incidence Hopf Algebra, page 34

$\mathrm{ch}(\chi)$ Frobenius Characteristic Map, page 23

χ Even Induced Character, page 64

$\chi(H)$ Set of Characters of H, page 63

$\chi\downarrow_H^G : H \to \mathrm{GL}(V)$ Restricted Representation, page 6

$\chi_+(H)$ Even Characters in $\chi(H)$, page 64

$\chi_-(H)$ Odd Characters in $\chi(H)$, page 64

$\chi_G(n)$ Chromatic Number, page 54

ϵ' Alternate Counit on QSYM, page 93

$\epsilon(w)$ Sign of w, page 75

$\epsilon : C \to \Bbbk$ Counit of Coalgebra C, page 26

$\epsilon_{C\otimes C}$ Tensor Product Coalgebra Counit, page 32

$\mathfrak{S}_\alpha$ Immaculate Basis Element for NSYM, page 78

$\mathfrak{S}_a^*$ Dual Immaculate Basis Element, page 78

fl Flattening Operator, page 35

γ Outer Coproduct in QSYM, page 88

γ' Alternate Coproduct on QSYM, page 93

$\operatorname{gldim} H$ Global Dimension of Ring H, page 41

$\hat{e}_\lambda$ Scaled Basis for SYM_n, page 51

$\operatorname{injdim} M$ Injective Dimension of Module M, page 41

$\int^\ell$ Left Integrals of Hopf Algebra H, page 44

$\int_H^\ell$ Homological Integral, page 45

λ/μ Skew Tableaux, page 27

$\mu(P)$ Möbius function on a Poset, page 60

ν Odd Induced Character, page 64

ω Involution on $\mathrm{QSYM}_{\mathbb{Q}}$, page 93

ω Involution on Compositions, page 92

$\overline{(\cdot)}$ Natural Involution for a Graded Hopf Algebra, page 64

$\overline{\Delta}$ (, page 51

$\operatorname{projdim} M$ Projective Dimension of Module M, page 40

ψ^2 Hopf Squared Map, page 50

ρ (Normalized) Rank Function for a Poset, page 61

$\rho(P)$ Rank of a Poset, page 61

str Straightening Operator, page 94

$\widetilde{E}$ Opposite Double Poset, page 72

$\widetilde{M_a}$ Sign Toggled Fundamental Quasisymmetric Function, page 61

ξ_μ^λ Generating Function for Characters of Specht Modules, page 8

$\{T\}$ Tabloid of Tableau T, page 8

$\{f_\lambda\}$ Forgotten Basis for SYM, page 22

a_T Sum of Row Stabilizers of Tableau T, page 9

a_λ Count of Stable Partitions, page 54

b_T Sum of Column Stabilizers of Tableau T, page 9

$c = \sum_{(c)} c_{(1)} \otimes c_{(2)}$ Sweedler Notation, page 29

$c_{\mu\nu}^\lambda$ Littlewood-Richardson Coefficients, page 23

$d_{\mu,\nu}^\lambda$ Schur Plethysm Expansion Coefficients, page 81

e_T Specht Module Basis Element, page 9

$f[g]$ Plethysm of f and g, page 81

f^λ $\#\,\mathrm{SYT}(\lambda)$, page 11

$g_{\lambda,\mu,\nu}$ Kronecker Coefficients, page 24

h_k Complete Homogeneous Symmetric Functions, page 4

$m^*(\phi)$ Coproduct in QSYM*, page 90

m_λ Monomial Symmetric Function, page 8

$m_{A\otimes A}$ Tensor Product Algebra Multiplication, page 32

p_μ Power Symmetric Function, page 8

p_i Power Sum, page 8

q_w Weight of w, page 75

s_λ Schur Function, page 17

$s_{\lambda/\mu}$ Skew Schur Function, page 27

$u\colon \Bbbk \to A$ Unit of the $\Bbbk$-algebra A, page 24

$u \sqcup\!\sqcup v$ Shuffle Sum, page 35

$u_{A\otimes A}$ Tensor Product Algebra Unit, page 32

$x <_L w$ Left Antisymmetric Kazhdan-Lusztig Order, page 76

$x \leq_L w$ Left Kazhdan-Lusztig Order, page 76

$x \leq_R w$ Right Kazhdan-Lusztig Order, page 76

$x \leq_{LR} w$ Two-Sided Kazhdan-Lusztig Order, page 76

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