(Compact) Lie Groups and Representation Theory Lecture Notes

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Abstract

The following notes were taking during a course on (Compact) Lie Groups and Representation Theory at the University of Washington in Fall 2014. Please send any corrections to `dsmatth@uw.edu`. Thanks!

Contents

September 24th, 2014: Intro to Lie Algebras (Knapp Ch.1 §1)

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1 Remark

Textbooks: the official text is "Representations of Compact Lie Groups" by Bröcker and tom Dieck; another is "Compact Lie Groups" by Sepanski—available online through UW, can buy it cheap through SpringerLink; we will begin with chapter one of "Lie Groups: Beyond an Introduction" by Knapp; also "Intro to Lie Algebras" by Humphreys. These are on reserve in the math research library.

Sara Billy teaches a related course in the Spring.

We'll have a set-aside Q&A session for the first few minutes of Friday lectures, so please come with questions.

2 Notation

Throughout, our base field will be $\boxed{\mathbb{F}} := \mathbb{R}$ or $\mathbb{C}$. More general statements are frequently possible.

Definition 3. A $\boxed{\text{Lie algebra}}$ is an $\mathbb{F}$-vector space $\mathfrak{g}$ with a map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the $\boxed{\text{Lie bracket}}$ (or sometimes $\boxed{\text{commutator}}$) and notated $\boxed{[-,-]}$ which is $\mathbb{F}$-bilinear, skew symmetric (so $[X, Y] = -[Y, X]$), and it satisfies the $\boxed{\text{Jacobi identity}}$,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

A $\boxed{\text{Lie algebra homomorphism}}$ $\mathfrak{g} \to \mathfrak{h}$ is an $\mathbb{F}$-linear transformation which preserves brackets,

$$\phi([X, Y]_{\mathfrak{g}}) = [\phi(X), \phi(Y)]_{\mathfrak{h}}.$$

A $\boxed{\text{Lie algebra isomorphism}}$ is additionally an isomorphism of vector spaces. (It follows that its inverse also preserves brackets.)

A $\boxed{\text{Lie subalgebra}}$ $\mathfrak{h} \subset \mathfrak{g}$ is a subspace closed under brackets.

4 Example

1. Let M be a smooth manifold, $\mathfrak{X}(M)$ the set of smooth vector fields on M, and $[-,-]$ as usual; $\mathfrak{X}(M)$ is a Lie algebra.

2. Let a be any associative $\mathbb{F}$-algebra. Define $\boxed{\mathrm{Lie}(a)}$ as the associated Lie algebra: as a set, it is just a; use commutator as bracket, $[\alpha, \beta] := \alpha\beta - \beta\alpha$. This satisfies the suggested axioms.

   An important special case is the following. Let V be a vector space and set $a = \mathrm{End}(V)$ with algebra operation given by composition. Denote $\boxed{\mathfrak{gl}(V)} := \mathrm{Lie}(\mathrm{End}(V))$.

   For instance, letting $V = \mathbb{R}^n$, $\mathfrak{gl}(n, \mathbb{R})$ is just $M_n(\mathbb{R})$ with commutator of matrices as the Lie bracket. Likewise with $\mathfrak{gl}(n, \mathbb{C})$.

3. Let $\mathfrak{g}$ be a $\mathbb{C}$-Lie algebra. We can view this as an $\mathbb{R}$-vector space; denote $\boxed{\mathfrak{g}_\mathbb{R}}$ as the induced $\mathbb{R}$-Lie algebra. The commutator is unaffected. Note that

$$\dim_\mathbb{R} \mathfrak{g}_\mathbb{R} = 2 \dim_\mathbb{C} \mathfrak{g}.$$

4. Let G be a Lie group. Let $\mathfrak{g}$ be the Lie algebra of G. The associated Lie algebra can be viewed in two ways. We can let $\mathfrak{g}$ be the set of left-invariant vector fields on G under Lie bracket. This is a subalgebra of $\mathfrak{X}(G)$ from above. As a vector space, $\mathfrak{g}$ is T_eG, the tangent space at the identity (so for instance $\dim \mathfrak{g} = \dim G$).

   Since Lie groups are real manifolds, $\mathfrak{g}$ is a real Lie algebra. There is a theory of complex Lie groups we mostly will not develop. They're roughly complex manifolds with a smooth (i.e. analytic) group structure.

5. Let V be a finite dimensional $\mathbb{F}$-vector space. Then $\mathrm{Aut}(V) = \mathrm{GL}(V)$ is the vector space of invertible linear transformations $V \to V$, and indeed we can consider $\mathrm{GL}(V)$ as a Lie group. Its Lie algebra is $\mathfrak{gl}(V)$.

   Note from a previous example that the Lie agebra of $\mathrm{End}(V)$ is $\mathfrak{gl}(V)$, but we have just seen that the Lie algebra of $\mathrm{Aut}(V)$ is also $\mathfrak{gl}(V)$. This makes sense since $\mathrm{End}(V)$ is an algebra (with addition and multiplication) so the bracket operation does not generate elements outside of $\mathrm{End}(V)$. However, $\mathrm{Aut}(V)$ is a group (under matrix multiplication) so roughly the Lie bracket operation generates elements outside of $\mathrm{Aut}(V)$ and indeed generates all matrices in $M_n(\mathbb{F})$, hence we have the same Lie algebra.

6. A $\boxed{\text{linear Lie algebra}}$ is a (Lie) subalgebra of $\mathfrak{gl}(V)$ where V is a finite-dimensional $\mathbb{F}$-vector space. (More concretely though equivalently, we could say they are Lie subalgebras of $\mathfrak{gl}(n, \mathbb{F})$ if a we choose a basis.)

   Some special cases:

   (a) $\boxed{\mathfrak{sl}(n, \mathbb{F})}$ is the zero trace matrices in $\mathfrak{gl}(n, \mathbb{F})$. (Note that $\operatorname{Tr} XY = \operatorname{Tr} YX$, so they are closed under brackets.) This is the Lie algebra of $\mathrm{SL}(n, \mathbb{F})$, i.e. the matrices of determinant 1.

   (b) $\boxed{\mathfrak{so}(n, \mathbb{F})}$ is the skew symmetric matrices in $\mathfrak{gl}(n, \mathbb{F})$, namely we require $X + X^T = 0$. This is closed under brackets since

$$\begin{aligned}[X, Y]^T &= (XY - YX)^T = Y^T X^T - X^T Y^T \\ &= YX - XY = -[X, Y].\end{aligned}$$

   This is the Lie algebra of the Lie group $\mathrm{O}(n, \mathbb{F})$ (orthogonal matrices, i.e. $X^T X = I$).

(c) $\mathfrak{u}(n)$ as an $\mathbb{R}$-Lie subalgebra of $\mathfrak{gl}(n,\mathbb{C})_{\mathbb{R}}$, namely we require $X + \overline{X}^T = 0$ (skew-Hermitian matrices). It's basically the same computation. This is the Lie algebra of the Lie group $U(n)$ (unitary matrices, i.e. $\overline{X}^T X = I$). This is not a $\mathbb{C}$-algebra since it isn't closed under multiplication by i, for instance (diagonal elements are pure imaginary).

(d) $\mathfrak{su}(n) := \mathfrak{u}(n) \cap \mathfrak{sl}(n,\mathbb{C})$. This is the Lie algebra of the Lie group $\mathrm{SU}(n)$ (complex matrices of determinant 1).

7. $(\mathbb{R}^3, \times)$: here $\times$ is the vector cross product. Claim: this is a $\mathbb{R}$-Lie algebra. You can hammer through the Jacobi identity: remember the identity

$$u \times (v \times w) = (w \cdot u)v - (u \cdot v)w.$$

The Jacobi identity then cancels in pairs. The commutator relations on the standard basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are

$$[\boldsymbol{i}, \boldsymbol{j}] = \boldsymbol{k}, \qquad [\boldsymbol{k}, \boldsymbol{i}] = \boldsymbol{j}, \qquad [\boldsymbol{j}, \boldsymbol{k}] = \boldsymbol{i}.$$

Preview of next time: $(\mathbb{R}^3, \times) \cong \mathfrak{so}(3,\mathbb{R}) \cong \mathfrak{su}(2)$.

September 26th, 2014: Draft

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5 Aside

Is there some nice relationship between Lie algebras (and whatever else we'll discuss) and the representation theory of finite groups, or finite groups in general? Yes; for instance, solvability, nilpotence, etc. exist in both contexts, and the representation theory of compact Lie groups is semisimple in analogy with semisimplicity for finite groups (Maschke's theorem).

What's the connection between representation theory and harmonic analysis? Well, one can consider Fourier series on a circle as giving a decomposition of $L^1(S^1)$ as a direct sum of irreducible one-dimensional representations.

6 Example

Recall our example from last time, $(\mathbb{R}^3, \times)$, R^3 viewed as a Lie algebra with bracket given by the cross product. Recall the relations

$$[\boldsymbol{i}, \boldsymbol{j}] = \boldsymbol{k}, \qquad [\boldsymbol{k}, \boldsymbol{i}] = \boldsymbol{j}, \qquad [\boldsymbol{j}, \boldsymbol{k}] = \boldsymbol{i}.$$

7 Proposition

$(\mathbb{R}^3, \times) \cong \mathfrak{so}(3,\mathbb{R}) \cong \mathfrak{su}(2)$ *as Lie algebras.*

Proof $\left[(\mathbb{R}^3, \times) \cong \mathfrak{so}(3,\mathbb{R})\right]$ We really just need to find basis vectors for $\mathfrak{so}(3,\mathbb{R})$ and $\mathfrak{su}(2)$ satisfying the above "cyclic" relations. For $v \in \mathbb{R}^3$, set

$$A_v := \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \in \mathfrak{so}(3,\mathbb{R}).$$

One can check by hand that $A_{u \times v} = [A_u, A_v] = A_u A_v - A_v A_u$. Hence $v \mapsto A_v$ is a Lie algebra homomorphism and the dimensions work out, so it's an isomorphism. This is the same as:

$$\boldsymbol{i} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{j} \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{k} \mapsto \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\left[(\mathbb{R}^3, \times) \cong \mathfrak{su}(2)\right]$ Recall $\mathfrak{so}(2)$ consists of trace zero skew-Hermitian matrices. To construct the isomorphism, set

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These are the $\boxed{\text{Pauli spin matrices}}$. They are a basis for the trace zero Hermitian matrices. We want skew-Hermitian, so just multiply each by i. Fact:

$$[\sigma_1, \sigma_2] = 2i\sigma_3, \qquad [\sigma_2, \sigma_3] = 2i\sigma_1, \qquad [\sigma_3, \sigma_1] = 2i\sigma_2.$$

It follows very quickly that the desired isomorphism of Lie algebras $(\mathbb{R}^3, \times) \to 2)$ is given by

$$\boldsymbol{i} \mapsto \frac{\sigma_1}{2i}, \qquad \boldsymbol{j} \mapsto \frac{\sigma_2}{2i}, \qquad \boldsymbol{k} \mapsto \frac{\sigma_3}{2i}$$

8 Remark

Our next goal is to understand different types of Lie algebras, based on "how degenerate" the bracket is:

$$\text{abelian} \subset \text{nilpotent} \subset \text{solvable}.$$

We will also consider "how non-degenerate" the bracket is:

$$\text{simple} \subset \text{semisimple} \subset \text{reductive}.$$

Definition 9. An $\boxed{\text{abelian Lie algebra}}$ $\mathfrak{g}$ is one for which $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. These are essentially just vector spaces.

A $\boxed{\text{nilpotent Lie algebra}}$ $\mathfrak{g}$ is one for which there is some $k \in \mathbb{N}$ where if we iterate bracketing k times, we always get zero. More formally,

$$[X_1, [X_2, [\ldots [X_k, X_{k+1}] \ldots]]] = 0$$

for all $X_1, \ldots, X_{k+1} \in \mathfrak{g}$. If k is minimal with this property, we call $\mathfrak{g}$ $\boxed{k\text{-step nilpotent}}$. For instance, $\mathfrak{g}$ is abelian if and only if it is 1-step nilpotent.

10 Example

Let $\mathfrak{g} = \operatorname{Span}_{\mathbb{F}}(X, Y, Z)$. Define $[X, Z] = 0, [Y, Z] = 0, [X, Y] = Z$. Extend this by skew-symmetry and $\mathbb{F}$-linearity. Check the Jacobi identity holds. (The "abstract approach" in terms of "structure constants" generalizes this procedure; see the first homework.)

Note that any commutator gives a scalar multiple of Z. Applying any second commutator then gives 0 since Z commutes with everything. Thus this example, the $\boxed{\text{Heisenberg algebra}}$ (over $\mathbb{R}$ or $\mathbb{C}$) is 2-step nilpotent.

Definition 11. Let $\mathfrak{g}$ be a Lie algebra. If $X \in \mathfrak{g}$, define

$$\boxed{\operatorname{ad} X} \colon \mathfrak{g} \to \mathfrak{g}$$

by

$$(\operatorname{ad} X)(Y) := [X, Y].$$

Notice that $\operatorname{ad}\colon \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$ given by $X \mapsto \operatorname{ad} X$ is a linear transformation. Moreover, recall $\operatorname{End}(\mathfrak{g})$ is a Lie algebra under the commutator. In fact,

12 Proposition

ad *is a homomorphism of Lie algebras.*

Proof Expanding the definitions, this is equivalent to the Jacobi identity:

$$\begin{aligned} \operatorname{ad}([X,Y])(Z) &= [[X,Y],Z] \\ [\operatorname{ad} X, \operatorname{ad} Y](Z) &= (\operatorname{ad} X \operatorname{ad} Y - \operatorname{ad} Y \operatorname{ad} X)(Z) = [X,[Y,Z]] - [Y,[X,Z]] \\ &= -[[Y,Z],X] - [[Z,X],Y]. \end{aligned}$$

Definition 13. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The $\boxed{\text{Killing form}}$ of $\mathfrak{g}$ is defined as the map

$$B \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$$

given by

$$B(X,Y) := \operatorname{Tr}(\operatorname{ad} X \circ \operatorname{ad} Y).$$

B is clearly bilinear. It is symmetric since trace commutes with matrix multiplication. (Computationally, this is messy to do. See the homework.)

Definition 14. Suppose $\mathfrak{g}$ is a Lie algebra. A subspace $\mathfrak{h} \subset \mathfrak{g}$ is an $\boxed{\text{ideal}}$ if it is strongly closed under commutators, namely

$$[X,Y] \in \mathfrak{h}$$

for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$.

15 Remark

Note that an ideal is in particular a subalgebra, but the reverse does not hold. Ideals are also called "invariant subalgebras", since $\operatorname{ad} Y$ for all $Y \in \mathfrak{h}$ maps into $\mathfrak{h}$.

Ideals are conceptually equivalent to normal subgroups in group theory. For instance, the kernel of any Lie algebra homomorphism is an ideal. Likewise, quotients exist: if $\mathfrak{a}$ is an ideal, then $\mathfrak{g}/\mathfrak{a}$ has a natural Lie algebra structure. (Subalgebra, like non-normal subgroup, is not enough.) In particular,

$$[X+\mathfrak{a}, Y+\mathfrak{a}] := [X,Y] + \mathfrak{a}$$

is well-defined. The projection map

$$\mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$$

is surjective with kernel $\mathfrak{a}$, so ideals are precisely kernels of some Lie algebra homomorphism.

Definition 16. If $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ are sets in a Lie algebra $\mathfrak{g}$, define

$$[\mathfrak{a},\mathfrak{b}] := \operatorname{Span}\{[X,Y] : X \in \mathfrak{a}, Y \in \mathfrak{b}\} \subset \mathfrak{g}.$$

This is a subspace of $\mathfrak{g}$, though not necessarily more.

17 Proposition

If $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ are ideals, so are $\mathfrak{a}+\mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b}$, and $[\mathfrak{a},\mathfrak{b}]$.

Proof They're all subspaces. $\mathfrak{a}+\mathfrak{b}$ is closed under commutators by linearity; $\mathfrak{a} \cap \mathfrak{b}$ trivially. For $[\mathfrak{a},\mathfrak{b}]$, we'll use the Jacobi identity: informally,

$$\begin{aligned} [\mathfrak{g},[\mathfrak{a},\mathfrak{b}]] &\subset [\mathfrak{a},[\mathfrak{b},\mathfrak{g}]] + [\mathfrak{b},[\mathfrak{g},\mathfrak{a}]] \\ &= [\mathfrak{a},\mathfrak{b}] + [\mathfrak{b},\mathfrak{a}] \subset [\mathfrak{a},\mathfrak{b}]. \end{aligned}$$

18 Example

Let $\mathfrak{g}$ be a Lie algebra. Some examples of ideals:

1. $\{0\}$.
2. $\mathfrak{g}$.

3. $\mathfrak{z}$ (or $\boxed{\mathfrak{z}_{\mathfrak{g}}}$), the $\boxed{\text{center}}$ of $\mathfrak{g}$, namely $\{X \in \mathfrak{g} : [X, Y] = 0, \forall Y \in \mathfrak{g}\} = \ker \mathrm{ad}$.

4. $[\mathfrak{g}, \mathfrak{g}]$, sometimes called the $\boxed{\text{commutator ideal}}$ or $\boxed{\text{derived algebra}}$.

19 Remark

We can now reformulate what it means for $\mathfrak{g}$ to be nilpotent. Define a sequence

$$\begin{aligned} \mathfrak{g}_0 &:= \mathfrak{g} \\ \mathfrak{g}_1 &:= [\mathfrak{g}, \mathfrak{g}] \\ &\cdots \\ \mathfrak{g}_{k+1} &:= [\mathfrak{g}, \mathfrak{g}_k]. \end{aligned}$$

Since the bracket of two ideals is again an ideal, we have that each $\mathfrak{g}_k$ is an ideal inductively. Further,

$$\mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots.$$

This is called the $\boxed{\text{lower descending series}}$ of $\mathfrak{g}$. Hence $\mathfrak{g}$ is nilpotent if the lower decending series for $\mathfrak{g}$ terminates at 0 at some finite point.

Definition 20. We likewise define the $\boxed{\text{derived series}}$ of $\mathfrak{g}$ as

$$\begin{aligned} \mathfrak{g}^0 &:= \mathfrak{g} \\ \mathfrak{g}^1 &:= [\mathfrak{g}, \mathfrak{g}] \\ &\vdots \\ \mathfrak{g}^{k+1} &:= [\mathfrak{g}^k, \mathfrak{g}^k]. \end{aligned}$$

These again are ideals, and again

$$\mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \cdots.$$

We say that $\mathfrak{g}$ is $\boxed{\text{solvable}}$ if $\mathfrak{g}^k = 0$ for some $k \in \mathbb{P}$. It is clear that $\mathfrak{g}^k \subset \mathfrak{g}_k$, so $\mathfrak{g}$ nilpotent implies $\mathfrak{g}$ solvable.

21 Example

Let $\mathfrak{n} \subset \mathfrak{gl}(n, \mathbb{F})$ consist of all strictly upper triangular matrices. The product of two upper strictly triangular matrices is strictly upper triangular, so $\mathfrak{n}$ is at least a Lie subalgebra. Indeed, we add an extra diagonal of zeros above the main diagonal. Iterating such products, we continue adding diagonals of zeros. Hence the derived series terminates in 0, so $\mathfrak{n}$ is solvable, while we can check it's not nilpotent.

September 29th, 2014: Draft

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22 Proposition

Here are some general linear algebra considerations.

1. *If $\phi \colon \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras, then $\ker \phi \subset \mathfrak{g}$ is an ideal and $\mathrm{im}\, \phi \subset \mathfrak{h}$ is a subalgebra. If $\mathfrak{a} \subset \mathfrak{g}$ is an ideal, we saw that $\mathfrak{g}/\mathfrak{a}$ inherits a natural Lie algebra structure. If $\mathfrak{a} \subset \ker \phi$, there is an induced map (on the level of linear algebra) $\overline{\phi} \colon \mathfrak{g}/\mathfrak{a} \to \mathfrak{h}$, which in fact is a homomorphism of Lie algebras.*

2. *If $\mathfrak{a} = \ker \phi$, then $\overline{\phi}$ is injective.*

3. *If in addition ϕ is surjective, then $\overline{\phi}$ is an isomorphism (of either vector spaces or Lie algebras).*

4. *If $\mathfrak{a} \subset \mathfrak{g}$ is an ideal, then there is a one-to-one correspondence between ideals in $\mathfrak{g}$ containing $\mathfrak{a}$ and ideals in $\mathfrak{g}/\mathfrak{a}$. More precisely, $\mathfrak{a} \subset \mathfrak{b} \subset \mathfrak{g}$ corresponds to $\mathfrak{b}/\mathfrak{a} \subset \mathfrak{g}/\mathfrak{a}$.*

5. *If $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ are ideals, then*
$$\frac{\mathfrak{a}+\mathfrak{b}}{\mathfrak{a}} \cong \frac{\mathfrak{b}}{\mathfrak{a}\cap\mathfrak{b}}$$
as Lie algebras. Explicitly, the map is $x + y + \mathfrak{a} \mapsto y + (\mathfrak{a} \cap \mathfrak{b})$ for $x \in \mathfrak{a}, y \in \mathfrak{b}$.

6. *If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Lie algebras, their (vector space) $\boxed{\text{direct sum}}$ $\boxed{\mathfrak{g}_1 \oplus \mathfrak{g}_2}$ has a natural Lie algebra structure,*
$$[(X_1, Y_1), (X_2, Y_2)] := ([X_1, X_2], [Y_1, Y_2]).$$
Notice that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ideals in $\mathfrak{g}_1 \oplus \mathfrak{g}_2$.

7. *If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ are ideals such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ using the internal direct sum of vector spaces, then $\mathfrak{g} \cong \mathfrak{a} \oplus \mathfrak{b}$ using the external direct sum of Lie algebras.*

   Warning: $\mathfrak{a}$ and $\mathfrak{b}$ must both be ideals; they cannot simply be subspaces. For instance, if $\mathfrak{a}$ is an ideal and $\mathfrak{b}$ is just a subalgebra so that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ as vector spaces, then $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ as Lie algebras if and only if $\mathfrak{b}$ is also an ideal.

Summary Last time we were talking about nilpotent and solvable ideals. We constructed two decreasing series of ideals by taking successive commutators in different ways. Recall $\mathfrak{g}^0 = \mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}^1 = \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$. The lower central series is defined inductively by $\mathfrak{g}_{k+1} = [\mathfrak{g}, \mathfrak{g}_k]$. The derived series is defined inductively by $\mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k]$. In general $\mathfrak{g}^k \subset \mathfrak{g}_k$.

Recall $\mathfrak{g}$ nilpotent means $\mathfrak{g}_k = 0$ for some k, while $\mathfrak{g}$ solvable means $\mathfrak{g}^k = 0$ for some k.

Definition 23. If $\mathfrak{a} \subset \mathfrak{g}$ is an ideal, we say that $\mathfrak{a}$ is a $\boxed{\text{solvable ideal}}$ if it is a solvable Lie algebra.

Our goal today is to show that if $\mathfrak{g}$ is a finite-dimensional Lie algebra, the $\mathfrak{g}$ contains a unique solvable ideal that contains all solvable ideals. This ideal is called the $\boxed{\text{radical}}$ of $\mathfrak{g}$, denoted $\boxed{\operatorname{rad}\mathfrak{g}}$.

24 Proposition

A subalgebra of a solvable Lie algebra is solvable. A homomorphic image of a solvable Lie algebra is solvable.

Proof If $\mathfrak{h} \subset \mathfrak{g}$, then $\mathfrak{h}^k \subset \mathfrak{g}^k$, which gives the first statement. For the second, if $\phi \colon \mathfrak{g} \to \mathfrak{h}$ is a surjective Lie algebra homomorphism, then $\phi(\mathfrak{g}^k) = \mathfrak{h}^k$. (In general, we only have $\phi(\mathfrak{g}^k) \subset \mathfrak{h}^k$, whereas we get the other inclusion since ϕ is surjective.)

25 Proposition

Let $\mathfrak{g}$ a Lie algebra, $\mathfrak{a} \subset \mathfrak{g}$ a solvable ideal. $\mathfrak{g}/\mathfrak{a}$ is solvable if and only if $\mathfrak{g}$ is solvable.

Proof Consider the projection map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$. If $\mathfrak{g}$ is solvable, then so is $\mathfrak{g}/\mathfrak{a}$. On the other hand, if $\mathfrak{g}/\mathfrak{a}$ is solvable, we see that $\pi(\mathfrak{g}^k) = (\mathfrak{g}/\mathfrak{a})^k = 0$ for k large enough. Hence $\mathfrak{g}^k \subset \mathfrak{a}$. Now choose l lage enough that $\mathfrak{a}^l = 0$. It follows that $\mathfrak{g}^{k+l} = (\mathfrak{g}^k)^l = 0$.

26 Proposition

If $\mathfrak{a}$ and $\mathfrak{b}$ are solvable ideals in a Lie algebra, then so is $\mathfrak{a} + \mathfrak{b}$.

Proof $\mathfrak{a}$ is an ideal in $\mathfrak{a} + \mathfrak{b}$ and is solvable. The quotient is $\frac{\mathfrak{a}+\mathfrak{b}}{\mathfrak{a}} \cong \frac{\mathfrak{b}}{\mathfrak{a}\cap\mathfrak{b}}$. $\mathfrak{b}$ is solvable, so the quotient on the right is; apply the preceding proposition.

27 Theorem

If $\mathfrak{g}$ is a finite dimensional Lie algebra (over any field), then $\mathfrak{g}$ contains a unique solvable ideal that contains all solvable ideals.

Proof Uniqueness is clear by maximality. For existence, let $\mathfrak{a}$ be a solvable ideal. If $\mathfrak{b}$ is a solvable ideal not contained in $\mathfrak{a}$, then $\mathfrak{a}+\mathfrak{b}$ is a larger solvable ideal containing both, and the dimension as vector spaces strictly increases. Induct.

Alternatively, let $\mathfrak{a}$ be a solvable ideal of maximal dimension (possibly zero). Then if $\mathfrak{b}$ is any other solvable ideal, $\mathfrak{a}+\mathfrak{b}$ is a solvable ideal of larger dimension unless $\mathfrak{b} \subset \mathfrak{a}$.

28 Remark

We'll next attack the inclusions

$$\text{simple} \subset \text{semisimple} \subset \text{reductive}.$$

Until we say otherwise, *from now on* $\mathfrak{g}$ *will be finite dimensional.*

Definition 29. $\mathfrak{g}$ is $\boxed{\text{simple}}$ if $\mathfrak{g}$ is *not abelian* and there are no nontrivial proper ideals. Equivalently, $\dim \mathfrak{g} > 1$ and $\mathfrak{g}$ has no nontrivial proper ideals.

For instance, $\mathfrak{g} = 0$ *is not* simple.

Definition 30. $\mathfrak{g}$ is $\boxed{\text{semisimple}}$ if $\operatorname{rad} \mathfrak{g} = 0$, or equivalently if there are no nontrivial solvable ideals, or equivalently if there are no nonzero abelian ideals. (Take $\mathfrak{g}^{k+1} = 0$ for k minimal; then $\mathfrak{g}^k$ is an abelian ideal since $\mathfrak{g}^{k+1} := [\mathfrak{g}^k, \mathfrak{g}^k]$.)

For instance, $\mathfrak{g} = 0$ *is* semisimple but not simple.

31 Proposition

Let $\mathfrak{g}$ be a simple Lie algebra.

1. $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. *(This motivates the "non-abelian" requirement in the definition of "simple".)*
2. $\mathfrak{g}$ *is semisimple. (*$\operatorname{rad} \mathfrak{g} = 0$ *or* $\mathfrak{g}$*; if* $\operatorname{rad} \mathfrak{g} = \mathfrak{g}$*, then* $\mathfrak{g}$ *is solvable, but then* $\mathfrak{g} \neq [\mathfrak{g}, \mathfrak{g}]$*, contradiction.)*

Let $\mathfrak{g}$ be a semisimple Lie algebra.

1. $\mathfrak{z}_{\mathfrak{g}} = 0$. *($\mathfrak{z}_{\mathfrak{g}}$ is abelian, so it must be 0.)*

October 1st, 2014: Draft

Summary Last time, we defined the radical of a finite dimensional Lie algebra. Namely, it is the largest solvable ideal and contains all other solvable ideals. We defined simple and semisimple Lie algebras. Recall that $\mathfrak{g}$ is simple if $\mathfrak{g}$ is not abelian and it has no nonzero proper ideals. Likewise, $\mathfrak{g}$ is semisimple if $\operatorname{rad} \mathfrak{g} = 0$, or equivalently if $\mathfrak{g}$ has no nontrivial solvable or abelian ideals. We ended with the proposition above, which says that for a simple Lie algebra $\mathfrak{g}$, the derived ideal is $\mathfrak{g}$ itself; that $\mathfrak{g}$ is semisimple; and that the center $\mathfrak{z}_{\mathfrak{g}}$ of $\mathfrak{g}$ is 0.

32 Proposition

For any $\mathfrak{g}$, $\mathfrak{g}/\operatorname{rad} \mathfrak{g}$ is semisimple.

Proof If $\mathfrak{a} \subset \mathfrak{g}/\operatorname{rad} \mathfrak{g}$ is solvable, we'll show $\mathfrak{a} = 0$. Let $\pi \colon \mathfrak{g} \to \mathfrak{g}/\operatorname{rad} \mathfrak{g}$. Note that $\pi^{-1}(\mathfrak{a})$ is an ideal in $\mathfrak{g}$ and we must show $\pi^{-1}(\mathfrak{a})$ is solvable (since then $\pi^{-1}(\mathfrak{a}) \subset \operatorname{rad} \mathfrak{g}$, so $\mathfrak{a} = 0$). The restriction

$$\pi|_{\pi^{-1}(\mathfrak{a})} \colon \pi^{-1}(\mathfrak{a}) \to \mathfrak{a}$$

is a surjective homomorphism of Lie algebras. Hence $\mathfrak{a} \cong \pi^{-1}(\mathfrak{a})/\ker$. Now $\ker = \pi^{-1}(\mathfrak{a}) \cap \operatorname{rad} \mathfrak{g}$ is solvable, so since $\mathfrak{a}$ is solvable, $\pi^{-1}(\mathfrak{a})$ is solvable.

33 Remark

Hence, to any (finite dimensional) $\mathfrak{g}$, we associate the pair $(\operatorname{rad}\mathfrak{g}, \mathfrak{g}/\operatorname{rad}\mathfrak{g})$ with $\operatorname{rad}\mathfrak{g}$ solvable and $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$ semisimple.

34 Fact

Roughly, there exists a subalgebra $\mathfrak{g}_{SS} \subset \mathfrak{g}$ so that

$$\mathfrak{g} = \operatorname{rad}(\mathfrak{g}) \oplus \mathfrak{g}_{SS}$$

as vector spaces. (A homework problem shows that every Lie algebra is a semidirect product of $\operatorname{rad}\mathfrak{g}$ and $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$.) This is called a $\boxed{\text{Levi decomposition}}$. Abstractly, $\mathfrak{g}_{SS}$ is always isomorphic to $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$, though it is not strictly speaking unique.

Proof Appendix of Fulton and Harris; also in Varadarajan. Be careful of possible extra assumptions required in general.

35 Example

Low dimensional Lie algebras up to isomorphism:

1. $\dim\mathfrak{g} = 1$: abelian; 1 of them.
2. $\dim\mathfrak{g} = 2$: exactly two; one solvable, one avelian.
3. $\dim\mathfrak{g} = 3$: claim: $\mathfrak{g}$ is either solvable or simple. If $\mathfrak{g}$ is not simple, it has a proper non-zero ideal $\mathfrak{a}$ with $\dim\mathfrak{a} = 1, 2$. By the previous points, $\mathfrak{a}$ is solvable, and $\mathfrak{g}/\mathfrak{a}$ has dimension $2, 1$, so the quotient is also solvable. Hence $\mathfrak{g}$ is solvable. Further, one can check that $\mathfrak{g}$ is simple if and only if $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, and $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$. Hence we have:

   36 Proposition

   $(\mathbb{R}^3, \times) \cong 3, \mathbb{R}) \cong \mathfrak{su}(2)$ *is simple. Further* $3, \mathcal{C})$ *is simple and* $\mathfrak{sl}(2, \mathbb{F})$ *is simple.*

   Proof The proof above of these isomorphisms actually computes the commutator ideal as the whole Lie algebra. The same reasoning generalizes from $3, \mathbb{R})$ to $3, \mathbb{C})$.

   The "standard basis" for $\mathfrak{sl}(2, \mathbb{F})$ is given by

   $$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

   with commutation relations

   $$[x, y] = h \qquad [h, x] = 2x \qquad [h, y] = -2y.$$

37 Theorem

Up to isomorphism,

a) . . . there are exactly 2 *simple* $\mathbb{R}$*-Lie algebras of dimension* 3,

$$3, \mathbb{R}) \cong \mathfrak{sl}(2, \mathbb{R}).$$

b) . . . there is exactly 1 *simple* $\mathbb{C}$*-Lie algebra of dimension* 3,

$$\mathfrak{sl}(2, \mathbb{C}) \cong 3, \mathbb{C}).$$

c) . . . There are uncountably many distinct 3*-dimensional solvable Lie algebras.*

38 Remark

We'll next describe an alternate characterization of semisimple Lie algebras which avoids direct discussion of solvable Lie algebras. It will say $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}$ is a (finite) direct sum of simple Lie algebras. (Zero is the special case with no summands, say.)

We'll begin with a characterization of solvable and semisimple Lie algebras in terms of the Killing form. Recall that if $\mathfrak{g}$ is a finite dimensional Lie algebra, the Killing form is a symmetric bilinear form $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ given by

$$B(X, Y) := \operatorname{Tr}(\operatorname{ad} X \circ \operatorname{ad} Y).$$

Recall the following basic linear algebra. Let V be a finite dimensional vector space, $B\colon V \times V \to V$ a symmetric bilinear form. Then $\ker B := \{v \in V : B(v, w) = 0 \text{ for all } w \in V\}$. (This is sometimes written $B^\perp$.) B is called nondegenerate if and only if $\ker B = 0$. If $v_1, \ldots, v_n$ is a basis for V, then the matrix

$$(B(v_i, v_j))_{i,j}$$

is the $\dim V \times \dim V$ matrix with entries in $\mathbb{F}$ of B. Over $\mathbb{C}$, there exists a basis for V so that the matrix is just some number $\leq n$ of 1's along the main diagonal with zeros elsewhere, where nondegeneracy says the whole main diagonal consists of 1's. Over $\mathbb{R}$, there exists a basis consisting of some number of 1's followed by some number of -1's followed by some number of 0's along the main diagonal. If there are p one's and q negative one's, then one says B has "signature" (p, q).

39 Theorem (**Cartan's Criterion**)

Let $\mathfrak{g}$ be a (finite dimensional) Lie algebra with Killing form B (over $\mathbb{R}$ or $\mathbb{C}$ only? check Humphreys). Then

(1) $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}] \subset \ker B$.

(2) $\mathfrak{g}$ is semisimple if and only if B is nondegenerate.

Proof (1) is deferred to Sara's course in the Spring. It's also in the references mentioned above. We'll derive (2) from (1). We'll also use some facts:

40 Fact

If $\mathfrak{g}$ is nilpotent, then $B \equiv 0$. If $B \equiv 0$, then $\mathfrak{g}$ is solvable (by Cartan (1)). There exists $\mathfrak{g}$ solvable but not nilpotent with $B \equiv 0$. Hence $B \equiv 0$ does not characterize nilpotence.

We'll first argue that $\mathfrak{g}$ semisimple implies B nondegenerate. The main step is showing that for any Lie algebra $\mathfrak{g}$, $\ker B \subset \operatorname{rad} \mathfrak{g}$. Assuming this, if $\mathfrak{g}$ is semisimple, then $\operatorname{rad} \mathfrak{g} = 0$, so $\ker B \subset 0$, so B is nondegenerate.

41 Lemma

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{a} \subset \mathfrak{g}$ an ideal. Let $B_\mathfrak{a}$ denote the Killing form of $\mathfrak{a}$ and $B_\mathfrak{g}$ the Killing form of $\mathfrak{g}$. Then

$$B_\mathfrak{a} = B_\mathfrak{g}|_{\mathfrak{a} \times \mathfrak{a}}.$$

More explicitly, if $X, Y \in \mathfrak{a}$, then

$$\operatorname{Tr}(\operatorname{ad}_\mathfrak{a}(X) \circ \operatorname{ad}_\mathfrak{a}(Y)) = \operatorname{Tr}(\operatorname{ad}_\mathfrak{g}(X) \circ \operatorname{ad}_\mathfrak{g}(Y)).$$

Proof Next time.

October 3rd, 2014: Draft

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42 Remark

Friday question session discussion:

- We can go from Lie groups to Lie algebras. Can we always go the other way? That is, can we go from a Lie algebra over $\mathbb{R}$ to a Lie group over $\mathbb{R}$? Jack Lee's Smooth Manifolds book gives a one-to-one correspondence between connected simply-connected Lie groups and Lie algebras over $\mathbb{R}$. (Pretty sure it also works with $\mathbb{C}$-manifolds and $\mathbb{C}$-Lie algebras.)

- Is compactness of the underlying Lie group reflected in the Lie algebra? We'll get to an intrinsic criterion for determining if the Lie algebra comes from a compact Lie group. Recall over $\mathbb{R}$ that we have $\mathfrak{sl}(2, \mathbb{R})$ and 2), which by the homework are non-isomorphic. One comes from a compact Lie group while the other does not.

- Let $\mathfrak{g} = \mathbb{R}^n$ (an abelian Lie algebra). This is the Lie algebra of the Lie group $(S^1)^n$, the n-torus, which is compact but not simply-connected. We can also use $\mathbb{R}^n$ as Lie group which is connected and simply-connected but not compact. We'll show that if you have a semisimple real Lie algebra, then the corresponding connected simply-connected Lie group is compact.

Summary Recall the main results from last time: if $\mathfrak{g}$ is a (finite dimensional) Lie algebra, $B(X, Y) := \operatorname{Tr}(\operatorname{ad} X \circ \operatorname{ad} Y)$ is the Killing form, then Cartan's criterion says (1) $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}] \subset \ker B$ and (2) $\mathfrak{g}$ is semisimple if and only if B is nondegenerate. A corollary to Cartan (1) is that $B \equiv 0$ implies $\mathfrak{g}$ is solvable.

43 Proposition

For all X, Y, Z in a Lie algebra $\mathfrak{g}$,

$$B([X, Y], Z) = B(X, [Y, Z]).$$

(This is sometimes called "associativity.)

Proof

$$\begin{aligned} B([X, Y], Z) &= \operatorname{ad}[X, Y] \circ \operatorname{ad} Z \\ &= \operatorname{ad} X \operatorname{ad} Y \operatorname{ad} Z - \operatorname{ad} Y \operatorname{ad} X \operatorname{ad} Z \end{aligned}$$

and

$$\begin{aligned} B(X, [Y, Z]) &= \operatorname{ad} X \operatorname{ad}[Y, Z] \\ &= \operatorname{ad} X \operatorname{ad} Y \operatorname{ad} Z - \operatorname{ad} X \operatorname{ad} Z \operatorname{ad} Y. \end{aligned}$$

Letting $A = \operatorname{ad} X, B = \operatorname{ad} Y, C = \operatorname{ad} Z$, the result follows if $\operatorname{Tr}(BAC) = \operatorname{Tr}(ACB)$. Indeed it's a standard result (which we've already used) that the order of a product does not affect the trace.

44 Remark

We next prove the lemma from the very end of last time, roughly that the Killing form of an ideal in a Lie algebra is the restriction of the Killing form of the Lie algebra.

Proof We'll use the following linear algebra fact:

45 Fact

Suppose $L\colon V \to V$ is a linear transformation, V is a vector space. Let $W \subset V$ be a subspace and $L(V) \subset L(W)$. Then

$$\operatorname{Tr}_W(L|_W) = \operatorname{Tr}_V(L).$$

Proof Decompose $V = W \oplus U$ for some subspace U. Pick a basis for W and a basis for U and imagine writing L as a matrix in this basis. The result is of the form

$$\begin{pmatrix} L|_W & * \\ 0 & 0 \end{pmatrix},$$

so the trace is as described.

In our situation, if $X, Y \in \mathfrak{a}$, we let $L = \operatorname{ad} X \circ \operatorname{ad} Y$ in the above lemma. The conclusion is then $\operatorname{Tr}_{\mathfrak{g}} L = \operatorname{Tr}_{\mathfrak{a}} L|_{\mathfrak{a} \times \mathfrak{a}}$, which by definition says $B_{\mathfrak{a}}(X, Y) = B_{\mathfrak{g}}(X, Y)$.

46 Proposition

$\ker B$ *is a solvable ideal, i.e.* $\ker B \subset \operatorname{rad} \mathfrak{g}$.

Proof Suppose $X \in \ker B$, $Y \in \mathfrak{g}$. One can check $[X, Y] \subset \ker B$ using associativity of the Killing form. Hence $\ker B$ is indeed an idea. But from the lemma at the end of last time,

$$B_{\ker B} = B_{\mathfrak{g}}|_{\ker B \times \ker B} \equiv 0,$$

so by Cartan's criterion, $\ker B$ is solvable.

47 Remark

We next continue the proof of Cartan's criterion started last lecture. We were proving that $\mathfrak{g}$ is semisimple if and only if B is nondegenerate. We have $\mathfrak{g}$ semisimple if and only if $\operatorname{rad} \mathfrak{g} = 0$ which implies by the preceding proposition that $\ker B = 0$ which occurs if and only if B is nondegenerate; this is the $\Rightarrow$ direction.

For $\Leftarrow$, suppose $\mathfrak{g}$ is not semisimple. There is then a nonzero abelian ideal, say $\mathfrak{a}$. We'll show that any abelian ideal is contained in $\ker B$, so $\ker B \neq 0$, so B is degenerate.

Let $x \in \mathfrak{a}$, $y \in \mathfrak{g}$. We just need to show $B(X, Y) = 0$, i.e. $\operatorname{Tr}(\operatorname{ad} X \circ \operatorname{ad} Y) = 0$. Let U be a vector space complement to $\mathfrak{a}$ in $\mathfrak{g}$, so $\mathfrak{g} = \mathfrak{a} \oplus U$. Using this decomposition, the matrix of $\operatorname{ad} X$ is of the form

$$\operatorname{ad} X = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}.$$

Likewise $\operatorname{ad} Y$ is of the form

$$\operatorname{ad} Y = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

We then compute

$$\begin{aligned} B(X, Y) &= \operatorname{Tr}\left(\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right) \\ &= \operatorname{Tr}\left(\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \right) \\ &= 0. \end{aligned}$$

48 Theorem

$\mathfrak{g}$ *is semisimple if and only if there are ideals* $\mathfrak{g}_1, \ldots, \mathfrak{g}_m$ *which are simple Lie algebras so that*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m.$$

In this case, the only ideals of $\mathfrak{g}$ *are indexed by subsets* $I \subset [m]$*, namely they are* $\mathfrak{a}_I := \oplus_{i \in I} \mathfrak{g}_i$.

49 Remark

The $\mathfrak{g}_i$ must be simple for the ideal parameterization to work out. Since the $\mathfrak{g}_i$ are ideals, there is no actual ambiguity in whether $\oplus$ refers to a Lie algebra direct sum or to a vector space direct sum: the one implies the other.

The statement at the end is horribly false for abelian Lie algebras: for instance, if $\mathfrak{g} = \mathbb{R}^2$, then $\mathfrak{g} = \mathbb{R} \oplus \mathbb{R}$, but each line through the origin is an ideal in $\mathfrak{g}$ which is not one of the summands.

Proof We'll first show the statement at the end. It is clear that $\mathfrak{a}_I$ is an ideal for each I. Let $\pi_i \colon \mathfrak{g} \to \mathfrak{g}_i$ be the projection map; it is a Lie algebra homomorphism (either directly, or because it is isomorphic to the natural projection map associated to quotienting $\mathfrak{g}$ by $\oplus_{j \neq i} \mathfrak{g}_j$). π_i is surjective. If $\mathfrak{a}$ is an ideal in $\mathfrak{g}$, then $\pi_i \mathfrak{a} \subset \mathfrak{g}_i$ is an ideal, so either $\pi_i \mathfrak{a} = 0$ or $\pi_i \mathfrak{a} = \mathfrak{g}_i$.

We claim that $\mathfrak{g}_i \subset \mathfrak{a}$. (This fails horribly in the abelian example above.) Since $\mathfrak{g}_i$ is simple, $\mathfrak{g}_i = [\mathfrak{g}_i, \mathfrak{g}_i]$, which is $[\mathfrak{g}_i, \pi_i \mathfrak{a}]$. This is $[\mathfrak{g}_i, \mathfrak{a}]$ since $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ for $j \neq i$. Since $\mathfrak{a}$ is an ideal, this is a subset of $\mathfrak{a}$.

Hence we conclude that for each i, either $\pi_i\mathfrak{a} = 0$ or $\mathfrak{g}_i \subset \mathfrak{a}$; let I be the set of indexes of the latter type. Thus $\mathfrak{a} \subset \oplus_{i\in I}\mathfrak{g}_i = \mathfrak{a}_I \subset \mathfrak{a}$. This proves the statement at the end. In this situation, suppose $\mathfrak{a}$ is an ideal. Then $\mathfrak{a} = \mathfrak{a}_I$ for some $I \subset [m]$. In particular, $[\mathfrak{a}, \mathfrak{a}] = \mathfrak{a}$ (since this is true for each $\mathfrak{g}_i$), so $\mathfrak{a}$ cannot be solvable (unless it's 0). This finishes the $\Leftarrow$ direction.

More next time.

October 6th, 2014: Draft

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50 Remark

Today we continue the proof of the theorem from the end of last lecture.

PROOF We had shown $\Leftarrow$, that a decomposition into a sum of simple ideals gives a classification of all ideals of $\mathfrak{g}$ and shows that $\mathfrak{g}$ is semisimple.

Some remarks before $\Rightarrow$: the only simple ideals of $\mathfrak{g}$ are the $\mathfrak{g}_i$'s, so the decomposition is unique up to reordering; it can happen that $\mathfrak{g}_i \cong \mathfrak{g}_j$ for $i \neq j$. For instance, consider $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1$. Then $\Delta = \{(x, x) : x \in \mathfrak{g}_1\}$ is a subalgebra of $\mathfrak{g}$, but it is not an ideal.

For $\Rightarrow$, suppose $\mathfrak{g}$ is semisimple. By Cartan's criterion, we must show the Killing form $B_\mathfrak{g}$ is nondegenerate. Recall the following definition from linear algebra:

Definition 51. Let V be a finite dimensional vector space, B a non-degenerate symmetric bilinear form on V. If $W \subset V$ is a subspace, we define

$$\boxed{W^\perp} := \{v \in V : B(v, w) = 0 \text{ for all } w \in W\}.$$

One may check that

$$\dim W + \dim W^\perp = \dim V,$$

though it is entirely possible that $W \cap W^\perp = 0$ (this happens frequently over finite fields, for instance).

52 Proposition

In the notation of the previous definition, the following are equivalent:

(1) $W \cap W^\perp = 0$

(2) $V = W \oplus W^\perp$

(3) $\ker B|_{W\times W} = 0$

(4) $B|_{W\times W}$ *is nondegenerate.*

We now induct on $\dim \mathfrak{g}$, so we suppose that any semisimple Lie algebra of lower dimension than $\mathfrak{g}$ is a direct sum of simple ideals. Choose a minimal nonzero ideal $\mathfrak{a}$ in $\mathfrak{g}$ (which exists since $\mathfrak{g}$ is finite dimensional). Note that $\mathfrak{a}$ is simple. If $\mathfrak{a} = \mathfrak{g}$, we're done. If $\mathfrak{a} \neq \mathfrak{g}$, consider $\mathfrak{a}^\perp$ relative to the Killing form. We claim that $\mathfrak{a}^\perp$ is an ideal of $\mathfrak{g}$: if $X \in \mathfrak{a}^\perp$ and $Y \in \mathfrak{g}$, then $[X, Y] \in \mathfrak{a}^\perp$ since for all $Z \in \mathfrak{a}$,

$$B([X, Y], Z) = B(X, [Y, Z]) = 0.$$

Consider $\mathfrak{a} \cap \mathfrak{a}^\perp \subset \mathfrak{a}$. By minimality, this is either 0 or $\mathfrak{a}$. If it is $\mathfrak{a}$, then everything in $\mathfrak{a}$ is orthogonal to everything in $\mathfrak{a}$, i.e. $B|_{\mathfrak{a}\times\mathfrak{a}} = 0$, so by our lemma from last time, $B_\mathfrak{a} = 0$. That is, $\mathfrak{a}$ is solvable, contradicting the fact that $\mathfrak{a}$ was assumed semisimple. Hence $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$. We may decompose $\mathfrak{a}^\perp$ inductively, though there are two minor verifications left: first, $\mathfrak{a}^\perp$ is semisimple; second, any simple ideal $\mathfrak{b} \subset \mathfrak{a}$ is a simple ideal of $\mathfrak{g}$. The second is straightforward—ideals $0 \subset \mathfrak{b} \subset \mathfrak{a}$ correspond precisely to ideals between 0 and $\mathfrak{a}$ in $\mathfrak{g}$. The second is also easy: a solvable ideal in $\mathfrak{b}$ is solvable in $\mathfrak{g}$, hence is 0.

53 Corollary

The preceding characterization of semisimplicity and in particular the ideal structure of semisimple (finite dimensional) Lie algebras $\mathfrak{g}$ implies the following. Let $\mathfrak{a} \subset \mathfrak{g}$ be an ideal.

1. $[\mathfrak{a}, \mathfrak{a}] = \mathfrak{a}$. *In particular,* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.
2. $\mathfrak{a}^\perp$ *is an ideal and* $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$.

Proof (1): this is true on the simple ideals $\mathfrak{g}_i$, so it is true of their direct sums. (2): this was actually proven in the theorem, namely when we showed $\mathfrak{a} \cap \mathfrak{a}^\perp = 0$.

Definition 54. A Lie algebra $\mathfrak{g}$ is $\boxed{\text{reductive}}$ if for any ideal $\mathfrak{a} \subset \mathfrak{g}$, there is an ideal $\mathfrak{g} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$.

It is clear from the above corollary that $\mathfrak{g}$ semisimple implies $\mathfrak{g}$ reductive. Note that $\mathfrak{g}$ abelian also implies $\mathfrak{g}$ reductive (using the linear algebraic complement). Note that $\mathfrak{b}$ is unique for $\mathfrak{g}$ semisimple though not for $\mathfrak{g}$ abelian.

55 Theorem

A Lie algebra $\mathfrak{g}$ is reductive if and only if there is a semisimple ideal $\mathfrak{g}_{ss}$ and an abelian ideal $\mathfrak{g}_{ab}$ such that

$$\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{g}_{ab}.$$

Proof We begin with $\Leftarrow$. Suppose $\mathfrak{a} \subset \mathfrak{g}$ is an ideal. We claim that $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ where $\mathfrak{a}_1$ is an ideal in $\mathfrak{g}_{ss}$ and $\mathfrak{a}_2$ is an ideal in $\mathfrak{g}_{ab}$. Let $\mathfrak{a}_1$ be the projection of $\mathfrak{a}$ onto $\mathfrak{g}_{ss}$ and let $\mathfrak{a}_2$ be the projection of $\mathfrak{a}$ onto $\mathfrak{g}_{ab}$. $\mathfrak{a}_1$ is an ideal in $\mathfrak{a}$, and since $\mathfrak{a}$ is semisimple, $\mathfrak{a}_1 = [\mathfrak{a}_1, \mathfrak{a}_1]$. However, $[\mathfrak{a}_1, \mathfrak{a}_1] = [\mathfrak{a}, \mathfrak{a}]$ since for all $X_1, X_2 \in \mathfrak{g}_{ss}$ and $Y_1, Y_2 \in \mathfrak{g}_{ab}$,

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], 0).$$

Hence $\mathfrak{a}_1 \subset \mathfrak{a}$. Since for all $X \in \mathfrak{a}$, X is the sum of its projections onto $\mathfrak{a}_1$ and $\mathfrak{a}_2$, it follows that $\mathfrak{a}_2 \subset \mathfrak{a} + \mathfrak{a}_1 \subset \mathfrak{a}$. Since $\mathfrak{g}_{ss}$ and $\mathfrak{g}_{ab}$ are reductive, we can write $\mathfrak{g}_{ss} = \mathfrak{a}_1 \oplus \mathfrak{b}_1$ and $\mathfrak{g}_{ab} = \mathfrak{a}_2 \oplus \mathfrak{b}_2$. Hence $\mathfrak{g} = (\mathfrak{a}_1 \oplus \mathfrak{a}_2) \oplus (\mathfrak{b}_1 \oplus \mathfrak{b}_2)$ gives the required decomposition, so $\mathfrak{g}$ is reductive.

For $\Rightarrow$, suppose $\mathfrak{g} \neq 0$ is reductive. Let $\mathcal{I}_1$ be the set of all minimal nonzero ideals in $\mathfrak{g}$; it is non-empty since $\mathfrak{g}$ is finite dimensional. Let $\mathcal{I}_2$ be the set of all finite sums of ideals in $\mathcal{I}_1$. $\mathcal{I}_2$ is a subset of the set of ideals in $\mathfrak{g}$. ($\mathcal{I}_2$ includes 0 as the empty sum.) Choose an ideal $\mathfrak{a} \in \mathcal{I}_2$ of maximal dimension. We claim $\mathfrak{a} = \mathfrak{g}$. If not, then since $\mathfrak{g}$ is reductive, we can write $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ for a non-zero ideal $\mathfrak{b}$. $\mathfrak{b}$ contains a minimal non-zero $\mathfrak{b}' \in \mathcal{I}_1$, so $\mathfrak{a} \oplus \mathfrak{b}'$ is a larger ideal in $\mathcal{I}_2$, a contradiction. Hence write $\mathfrak{g} = \mathfrak{a}_1 \oplus \cdots \mathfrak{a}_n$ for minimal non-zero ideals $\mathfrak{a}_i$. Since we've written $\mathfrak{g}$ as a sum of $\mathfrak{a}_i$, each $\mathfrak{a}_i$ is either simple or of dimension 1 (that is, each $\mathfrak{a}_i$ has no non-zero proper ideals, either in $\mathfrak{a}_i$ or in $\mathfrak{g}$). Collect the simples together to get a semisimple ideal; collect the dimension 1's together to get an abelian ideal; done.

October 8th, 2014: Draft

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56 Remark

In the homework due today, the last problem defined the externial semidirect product slightly incorrectly: it should use $[X, Y] = -\operatorname{ad}(Y)X$.

Also, notes are available online (but you already knew that).

Summary Last time, we showed that a Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form $B_{\mathfrak{g}}$ is nondegenerate, if and only if $\mathfrak{g}$ is a direct sum of simple ideals (with all ideals being sums of some subcollection of the simple ideals). We defined $\mathfrak{g}$ to be reductive if each ideal $\mathfrak{a} \subset \mathfrak{g}$ has a "complement" ideal $\mathfrak{b}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$; semisimple Lie algebras have this property. Finally we showed that $\mathfrak{g}$ is reductive if and only if there is a semisimple ideal $\mathfrak{g}_{ss}$ and an abelian ideal $\mathfrak{g}_{ab}$ such that $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{g}_{ab}$.

Indeed, the proof showed more:

57 Proposition

If $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{g}_{ab}$ is reductive and $\mathfrak{a} \subset \mathfrak{g}$ is any ideal, then $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ where $\mathfrak{a}_1 \subset \mathfrak{g}_{ss}$ and $\mathfrak{a}_2 \subset \mathfrak{g}_{ab}$ are the ideals obtained by projecting $\mathfrak{a}$.

Note that if $\mathfrak{g}$ is reductive, from the above decomposition,

$$\mathfrak{g}_{ss} = [\mathfrak{g}, \mathfrak{g}] \qquad \mathfrak{g}_{ab} = \mathfrak{z}_{\mathfrak{g}}.$$

In particular the decomposition is unique.

58 Remark

Recall that if $\mathfrak{g}$ is a $\mathbb{C}$-Lie algebra, we defined $\mathfrak{g}_{\mathbb{R}}$ as the Lie algebra given by restricting scalars.

Definition 59. Suppose V is a $\mathbb{C}$-vector space. An $\mathbb{R}$-subspace $V_0 \subset V$ is $\boxed{\text{totally real}}$ if $V_0 \cap iV_0 = 0$. This is equivalent to saying V_0 contains no non-zero $\mathbb{C}$-subspaces.

60 Example

$\mathbb{R}^n \subset \mathbb{C}^n$ is a totally real subspace.

Note that $\dim_{\mathbb{R}} V_0 \leq \dim_{\mathbb{C}} V$ since $V_0 \cap iV_0 = 0$.

We say that V_0 is a $\boxed{\text{real form}}$ of V if $V = V_0 \oplus iV_0$. This is equivalent to saying V_0 is totally real and $\dim_{\mathbb{R}} V_0 = \dim_{\mathbb{C}} V$. Roughly, a real form is a maximal totally real subspace.

61 Proposition

Suppose V_0 is a real form of V.Define a map $\sigma\colon V \to V$ as follows. Given $z \in V$, we have unique $x, y \in V_0$ such that $z = x + iy$. Define $\sigma(z) := x - iy$. Then σ is conjugate linear and an involution, i.e. $\sigma^2 = \mathrm{id}$. Further, $V_0 = \{\sigma(z) = z\}$.

Proof Since V_0 is a real subspace, for $a \in \mathbb{R}$, $\sigma(az) = a\sigma(z)$ and if $z = x + y$ then $iz = -y + ix$ so that $\sigma(iz) = -y - ix = -i\sigma(z) = -i(x - iy)$, giving conjugate linearity. That $\sigma^2 = \mathrm{id}$ and $V_0 = \{\sigma(z) = z\}$ are immediate.

62 Exercise

If V is a $\mathbb{C}$-vector space and $\sigma\colon V \to V$ is a conjugate linear involution, then $\{z \in V : \sigma(z) = z\}$ is a real form.

Definition 63. Let $\mathfrak{g}$ be a complex Lie algebra. A real subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ is a $\boxed{\text{real form}}$ if $\mathfrak{g}$ is a vector space real form. That is, $\mathfrak{g}_0$ is a real subalgebra of the complex Lie algebra $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ as vector spaces.

(Almost always $i\mathfrak{g}_0$ is not even a subalgebra.)

64 Example

We have

1. $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{g}_0 = \mathfrak{gl}(n, \mathbb{R})$ is a real form immediately.

2. $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{g}_0 = \mathfrak{u}(n)$. Recall that $\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) : X = -X^*\}$ where $X^* := \overline{X}^T$. Then $i\,\mathfrak{u}(n) = \{X : X = X^*\}$. Any $X \in \mathfrak{gl}(n, \mathbb{C})$ can be written as

$$X = \frac{X - X^*}{2} + i\left(\frac{X + X^*}{2i}\right).$$

3. $\mathfrak{g} = (n, \mathbb{C})$, $\mathfrak{g}_0 = (n, \mathbb{R})$ or $\mathfrak{g}_0 = \mathfrak{su}(n, \mathbb{R})$.

4. $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$, $\mathfrak{g} = \mathfrak{so}(n, \mathbb{R})$.

65 Remark

1. It's not true that every $\mathbb{C}$-Lie algebra has a real form; see homework.

2. If $\mathfrak{g}$ is a complex semisimple Lie algebra, then $\mathfrak{g}$ has at least two nonisomorphic real forms. We'll get to "compact" and "simple" real forms. For instance, $(2, \mathbb{C})$, there are two real forms $(2, \mathbb{R})$ (split) and $\mathfrak{su}(2)$ (compact).

66 Proposition

Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be a real form. Then $\mathfrak{g}_0$ is semisimple if and only if $\mathfrak{g}$ is semisimple.

Proof Choose a(n $\mathbb{R}$-)basis $X_1, \ldots, X_n$ for $\mathfrak{g}_0$. The matrix for $\operatorname{ad} X_j$ relative to this basis has real entries. Hence $B_{\mathfrak{g}_0}(X, Y) = \operatorname{Tr}(\operatorname{ad} X \circ \operatorname{ad} Y)$ can be calculated over the reals. Since $\mathfrak{g}_0$ is a real form, $X_1, \ldots, X_n$ is also a ($\mathbb{C}$-)basis for $\mathfrak{g}$. We conclude that the matrix of $B_{\mathfrak{g}}$ is the same as the matrix of $B_{\mathfrak{g}_0}$, i.e.

$$B_{\mathfrak{g}_0} = B_{\mathfrak{g}}|_{\mathfrak{g}_0 \times \mathfrak{g}_0}.$$

Hence $B_{\mathfrak{g}_0}$ is nondegenerate if and only if $B_{\mathfrak{g}}$ is.

67 Remark

We'll see shortly that if $\mathfrak{g}_0 \subset \mathfrak{g}$ is a real form, then $\mathfrak{g}$ simple implies $\mathfrak{g}_0$ simple, while $\mathfrak{g}_0$ simple does not imply that $\mathfrak{g}$ is simple.

68 Proposition

If $\mathfrak{g}_0 \subset \mathfrak{g}$ is a real form, let $\sigma \colon \mathfrak{g} \to \mathfrak{g}$ be the associated conjugate-linear involution. Then σ is a conjugate-linear isomorphism of ($\mathbb{C}$-)Lie algebras.

Definition 69. (A conjugate-linear homomorphism of Lie algebras is just a conjugate-linear map which preserves brackets.)

Proof We must show that σ preserves brackets. Suppose $Z = X + iY$, $W = U + iV$. Then

$$\begin{aligned} [Z, W] &= [X + iY, U + iV] \\ &= ([X, U] - [Y, V]) + i([X, V] + [Y, U]). \end{aligned}$$

If we computed $[\sigma(Z), \sigma(W)]$ instead, the real parts are the same but the imaginary parts pick up a negative sign, which is the desired conclusion.

70 Homework

If σ is a conjugate linear involutive isomorphism of $\mathfrak{g}$, then $\{z \in \mathfrak{g} : \sigma(z) = z\}$ is a real form of $\mathfrak{g}$. Together with the preceding proposition, this produces an alternate characterization of real forms.

71 Example

1. $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{g}_0 = \mathfrak{gl}(n, \mathbb{R})$; then $\sigma(Z)$ is given by conjugating each entry of the matrix Z, i.e. $\sigma(Z) = \overline{Z}$.

2. $\mathfrak{g}_0 = \mathfrak{u}(n)$ implies $\sigma(Z) = -Z^*$. Note that $\sigma(Z) = Z^*$ does not preserve brackets:

$$[Z^*, W^*] = Z^* W^* - W^* Z^* = ZW - WZ \neq [Z, W]^*.$$

72 Remark

Given a $\mathbb{R}$-Lie algebra $\mathfrak{g}_0$, we will construct a $\mathbb{C}$-Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}_0$ is a real form in $\mathfrak{g}$. This is called "complexification"; it's convenient to describe in terms of tensor products.

Definition 73. If V, W are $\mathbb{R}$-vector spaces, we can form the $\boxed{\text{tensor product}}$ of V and W, $\boxed{V \otimes_{\mathbb{R}} W}$, as the vector space with basis $v_i \otimes w_j$ where $\{v_i\}$ and $\{w_j\}$ are bases of V and W.

If V is a real vector space, take $W = \mathbb{C}$ as a real vector space and defined

$$V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}.$$

Now $V^{\mathbb{C}}$ becomes a $\mathbb{C}$-vector space using

$$\alpha(v \otimes z) := v \otimes (\alpha z).$$

If $\{v_i\}$ is a basis over $\mathbb{R}$ for V, then $\{v_j \otimes 1\}$ is a basis over $\mathbb{C}$ for $V^{\mathbb{C}}$. Similarly $\{v_j \otimes 1\} \cup \{v_j \otimes i\}$ is a basis over $\mathbb{R}$ for $V^{\mathbb{C}}$. V is a real form for $V^{\mathbb{C}}$, where we identify V with its image under $V \hookrightarrow V \otimes_{\mathbb{R}} \mathbb{C}$ given by $v \mapsto v \otimes 1$.

If $\mathfrak{g}_0$ is a real Lie algebra, then $\mathfrak{g}_0^{\mathbb{C}} := \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ has a natural Lie algebra structure,

$$[X \otimes z, Y \otimes w] := [X, Y] \otimes (zw).$$

Indeed, $\mathfrak{g}_0 \hookrightarrow g = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ shows that $\mathfrak{g}_0$ is a real form for $\mathfrak{g}$.

October 10th, 2014: Draft

Summary Last time, we defined real forms and discussed their basic properties. If $\mathfrak{g}$ is a $\mathbb{C}$-Lie algebra, a real subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ is a real form of $\mathfrak{g}$ if $\mathfrak{g}_0 \oplus i\mathfrak{g}_0 = \mathfrak{g}$. We also defined the "complexification" of a real Lie algebra $\mathfrak{g}_0$, namely $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$. The corresponding decomposition is $(\mathfrak{g}_0 \otimes 1) \oplus (\mathfrak{g}_0 \otimes i)$, so $\mathfrak{g}_0 = \mathfrak{g}_0 \otimes 1$ is a real form of $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$.

74 Proposition

Suppose $\mathfrak{g}$ is a $\mathbb{C}$-Lie algebra with a real subalgebra $\mathfrak{g}_0$. There is a natural map $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} \to \mathfrak{g}$ given by $x \otimes z \mapsto zx$. $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$ if and only if $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} \to \mathfrak{g}$ is an isomorphism.

Proof $\mathfrak{g}_0$ maps into $\mathfrak{g}_0$ and $\mathfrak{g}_0 \otimes i$ maps into $i\mathfrak{g}_0$. Hence the nautral map, which is injective, maps onto $\mathfrak{g}_0 \oplus i\mathfrak{g}_0$. Hence this map is an isomorphism if and only if $\mathfrak{g}_0 \oplus i\mathfrak{g}_0 = \mathfrak{g}$.

Definition 75. (Alternate definition of "real form".) Suppose $\mathfrak{g}$ is a $\mathbb{C}$-Lie algebra and $\mathfrak{g}_0$ is a $\mathbb{R}$-Lie algebra (not necessarily a subalgebra of $\mathfrak{g}$). We say $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$ if

$$\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}$$

as $\mathbb{C}$-Lie algebras.

76 Remark

Using the preceding proposition, one can show our previous definition is equivalent to this one "up to isomorphism". The only real difference between the two is that in this one we allow real subalgebras isomorphic to subalgebras of $\mathfrak{g}$ while with the previous one we did not.

77 Example

Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ for $\mathbb{C}$-Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$. Take $\mathfrak{g}_0 = (\mathfrak{g}_1)_{\mathbb{R}}$ map $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$; ; $X + Y \otimes i$, $X, Y \in (\mathfrak{g}_1)_{\mathbb{R}}$.

Another question: let $\mathfrak{g}$ be a $\mathbb{C}$-Lie algebra. What is $\mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$?

Definition 78. Let V be a $\mathbb{C}$ vector space. We define a "conjugate" vector space $\overline{V}$ as follows. The abelian group structure is the same, but scalar multiplication is "twisted" by complex conjugation: αv in $\overline{V}$ is defined to be $\overline{\alpha} v$ in V.

Likewise given a $\mathbb{C}$-Lie algebra $\mathfrak{g}$, we can construct a new $\mathbb{C}$-Lie algebra $\boxed{\overline{\mathfrak{g}}}$ using the twisted $\mathbb{C}$-vector space structure and the same bracket as before. If $\mathfrak{g}$ has structure constants c_{ij}^k with respect to some basis, then the structure constants of $\overline{\mathfrak{g}}$ in the same basis are $\overline{c_{ij}^k}$.

79 Remark

If $\mathfrak{g}$ has a real form, then $\mathfrak{g} \cong \overline{\mathfrak{g}}$. We can see this in several ways; say σ is the conjuecation associated to the real form $\mathfrak{g}_0$; then $\sigma\colon \mathfrak{g} \to \mathfrak{g}$ is a conjugate linear isomorphism, whence $\sigma\colon \mathfrak{g} \to \overline{\mathfrak{g}}$ is a linear isomorphism of Lie algebras.

It can happen that $\mathfrak{g} \ncong \overline{\mathfrak{g}}$.

80 Proposition

Suppose $\mathfrak{g}$ is a $\mathbb{C}$-Lie algebra. Then

$$\mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g} \oplus \overline{\mathfrak{g}}.$$

(Note that this had to be symmetric with respect to $\overline{g}$, since $\mathfrak{g}_{\mathbb{R}} = \overline{\mathfrak{g}}_{\mathbb{R}}$ trivially.) In particular, if $\mathfrak{g}$ has a real form, then this is isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$.

Proof Multiplication by i is a $\mathbb{C}$-linear transformation $\mathfrak{g} \to \mathfrak{g}$. Similarly let $J\colon \mathfrak{g}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}$ where $J(x) = iX \in \mathfrak{g} \cong \mathfrak{g}_{\mathbb{R}}$. THe isomorphism is the following: $X + Y \otimes i$ for $X, Y \in \mathfrak{g}_R$ is mapped to $(X + JY, X - JY)$. One may check the remaining properties: this is an isomorphism of $\mathbb{R}$-vector spaces; indeed, an isomorphism of $\mathbb{C}$-vector spaces; preserves brackets.

81 Remark

Last time, we showed that if $\mathfrak{g}_0$ is a real form on a $\mathbb{C}$-Lie algebra $\mathfrak{g}$, then $\mathfrak{g}_0$ is semisimple if and only if $\mathfrak{g}$ is semisimple. Equivalently, complexification preserves semisimplicity. The next proposition gives a converse.

82 Proposition

If $\mathfrak{g}$ is a $\mathbb{C}$-Lie algebra and $\mathfrak{g}$ is semisimple, then $\mathfrak{g}_{\mathbb{R}}$ is semisimple (as a $\mathbb{R}$-Lie algebra).

Proof We will again use Cartan's criterion. We will show that $B_{\mathfrak{g}}$ is nondegenerate if and only of $B_{\mathfrak{g}_{\mathbb{R}}}$ is nondegenerate. This is a quick consequence of the next proposition:

83 Proposition

If $\mathfrak{g}$ is a $\mathbb{C}$-Lie algebra and $X, Y \in \mathfrak{g}_{\mathbb{R}}$, then

$$B_{\mathfrak{g}_{\mathbb{R}}}(X, Y) = 2\Re B_{\mathfrak{g}}(X, Y).$$

Proof For $X, Y \in \mathfrak{g}$, $B_{\mathfrak{g}}(X, Y) = \operatorname{Tr}(\operatorname{ad} X \circ \operatorname{ad} Y)$. Choose a basis for $\mathfrak{g}$ over $\mathbb{C}$, $Z_1, \ldots, Z_n$. Hence $Z_1, \ldots, Z_n, iZ_1, \ldots, iZ_n$ form a $\mathbb{R}$-basis for $\mathfrak{g}_{\mathbb{R}}$. Define an $\mathbb{R}$-linear transformation $\mathbb{R}^{2n} \to \mathbb{C}^n$ given by sending $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$ to $(x_1 + iy_1, \ldots, x_n + iy_n)$. More generally, let $\Phi\colon\ \mathfrak{gl}(n, \mathbb{C}) \to \mathfrak{gl}(2n, \mathbb{R})$ be given by

$$\Phi(M) = \begin{pmatrix} \Re M & -\Im M \\ \Im M & \Re M \end{pmatrix}.$$

One may check the following properties: (1) $\Phi(M, N) = \Phi(M)\Phi(N)$; (2) $\Phi(M^*) = M^T$; (3) $\operatorname{Tr} \Phi(M) = 2\Re \operatorname{Tr}(M)$; (4) $\det \Phi(M) = |\det M|^2$. The last is the least obvious; it suffices to check the result on elementary matrices (those representing elementary row operations) and apply (1). In any case, from (3), we have

$$\begin{aligned} B_{\mathfrak{g}_{\mathbb{R}}}(\operatorname{ad}_{\mathbb{R}} X \circ \operatorname{ad}_{\mathbb{R}} Y) &= \operatorname{Tr}(\Phi(\operatorname{ad}_{\mathbb{C}} X) \circ \Phi(\operatorname{ad}_{\mathbb{C}} Y)) \\ &= 2\Re \operatorname{Tr}(\operatorname{ad}_{\mathbb{C}} X \circ \operatorname{ad}_{\mathbb{C}} Y) \\ &= 2\Re B_{\mathfrak{g}}(X, Y). \end{aligned}$$

Suppose that $B_{\mathfrak{g}_{\mathbb{R}}}$ is nondegenerate. If $X \in \mathfrak{g}$ and $B_{\mathfrak{g}}(X, Y) = 0$ for all $Y \in \mathfrak{g}$, then $B_{\mathfrak{g}_{\mathbb{R}}}(X, Y) = 0$ for all $Y \in \mathfrak{g}_{\mathbb{R}}$, so $X = 0$. On the other hand, if $B_{\mathfrak{g}}$ is nondegenerate, $X \in \mathfrak{g}_b R$, and $B_{\mathfrak{g}_{\mathbb{R}}}(X, Y) = 0$ for all $Y \in \mathfrak{g}_{\mathbb{R}}$, then $\Re B_{\mathfrak{g}}(X, Y) = 0$ for all $Y \in \mathfrak{g}$. But then $\Re B_{\mathfrak{g}}(X, iY) = 0$ which implies $\Im B_{\mathfrak{g}}(X, Y) = 0$, so $X = 0$ for all $Y \in \mathfrak{g}$.

84 Example

Begin with $\mathbb{R}$-Lie algebras of matrices $\mathfrak{gl}(n,\mathbb{R}), \mathfrak{gl}(n,\mathbb{C}), \mathfrak{gl}(n,\mathbb{H})$ where $\boxed{\mathbb{H}}$ are the $\boxed{\text{quaternions}}$. (Recall that $\mathbb{H}$ is the set of elements $a+bi+cj+dk$ for $a,b,c,d \in \mathbb{R}$ subject to the multiplication relations $ij=k, jk=i, ki=j$, $ji=-k, kj=-i, ik=-j$, and $i^2=j^2=k^2=-1$.)

We will not use Lie algebras over $\mathbb{H}$, though we will use "vector spaces" over $\mathbb{H}$. ($\mathbb{H}$ is a division ring, not a field; that's alright.) Here $\boxed{\mathfrak{gl}(n,\mathbb{H})}$ is defined as a set to consist of $n \times n$ matrices with entries in $\mathbb{H}$. We can multiply quaternions and define matrix multiplication in $\mathfrak{gl}(n,\mathbb{H})$. We can also form the commutator of such matrices, and the resulting bracket is bilinear *only* over $\mathbb{R}$; the non-commutativity of quaternion multiplication prevents us from using $\mathbb{C}$ or $\mathbb{H}$. More precisely, $\mathfrak{gl}_n(\mathbb{H})$ is a $\mathbb{R}$-algebra but not a $\mathbb{C}$- or $\mathbb{H}$-algebra.

October 13th, 2014: Draft

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85 Remark

Today we'll look at subalgebras of $\mathfrak{gl}(n,\mathbb{R})$, $\mathfrak{gl}(n,\mathbb{C})$, and $\mathfrak{gl}(n,\mathbb{H})$. We first briefly review standard properties of $\mathbb{H}$, the ring of quaternions. The multiplication structure in terms of the usual "i,j,k" basis were defined last time. For $z=a+bi+cj+dk$, we let $\Re z := a$ and $\Im z := bi+cj+dk$. Likewise, $\overline{z} = a-bi-cj-dk$ and $|z|^2 = z\overline{z} = \overline{z}z = a^2+b^2+c^2+d^2$. If $z \neq 0$, then $z^{-1} = \overline{z}/|z|^2$. Hence $\mathbb{H}$ is formally a "division algebra", which is a field except multiplication need not be commutative (and here isn't). One may verify $\overline{zw} = \overline{w}\overline{z}$. We consider $\mathbb{C} \subset \mathbb{H}$ as the elements of the form $a+bi$. We identify $\mathbb{R} \subset \mathbb{H}$ as the elements of the form a; indeed, $\mathbb{R}$ is the center of $\mathbb{H}$, namely the set of elements of $\mathbb{H}$ with commute all other elements.

$\mathbb{H}$ is an $\mathbb{R}$-associative algebra, but it is *not* a $\mathbb{C}$-associative algebra: bilinearity requires commuting scalars, but $\mathbb{C}$ is not central in $\mathbb{H}$. That is, left and right multiplying a quaternion by a complex number in general gives different results. Hence $\mathbb{H}$ is a $\mathbb{C}$-vector space in at least two ways: through left or right multiplication, and neither way gives $\mathbb{C}$-bilinear multiplication.

As before, $\mathfrak{gl}(n,\mathbb{H})$ is the set of all $n \times n$ matrices with entries in $\mathbb{H}$. It is a $\mathbb{R}$-associative algebra and hence an $\mathbb{R}$-Lie algebra through the usual commutator bracket. (It is not a $\mathbb{C}$-Lie algebra; the $n=1$ case just gives $\mathbb{H}$.)

For $X \in \mathfrak{gl}(n,\mathbb{H})$, we define $X^* := \overline{X}^T$, so $X^{**} = X$. One must check that $(XY)^* = Y^*X^*$:

$$(XY)^*_{ik} = \overline{\sum_j X_{kj}Y_{ji}} = \sum_j \overline{Y_{ji}} \cdot \overline{X_{kj}} = (Y^*X^*)_{ik}.$$

It is *not* true that $\operatorname{Tr} XY = \operatorname{Tr} YX$ for $X,Y \in \mathfrak{gl}(n,\mathbb{H})$. It is true that

$$\Re \operatorname{Tr}(XY) = \Re \operatorname{Tr}(YX),$$

roughly since $\Re z_1z_2 = \Re z_2z_1$ for quaternions z_1, z_2.

86 Theorem

Let $\mathfrak{g}$ be a $\mathbb{R}$-Lie subalgebra of $\mathfrak{gl}(n,\mathbb{R})$, $\mathfrak{gl}(n,\mathbb{C})$, or $\mathfrak{gl}(n,\mathbb{H})$. If $\mathfrak{g}$ is closed under $X \mapsto X^*$, then $\mathfrak{g}$ is reductive.

Proof Roughly, we will build a replacement for the Killing form argument above which produced a complement to a given ideal. Define

$$\langle X,Y \rangle := \Re \operatorname{Tr}(XY^*).$$

This is a $\mathbb{R}$-bilinear form. (It is called the Hilbert-Schmidt inner product in the complex case.) We claim it is a real inner product.

$$(XY^*)_{ik} = \sum_j x_{ij}\overline{y}_{kj},$$

so

$$\operatorname{Tr}(XY^*) = \sum_{ij} x_{ij}\overline{y}_{ij}.$$

It follows that $\Re\operatorname{Tr}(XY^*)$ is the Euclidean inner product on $\mathfrak{gl}(n, \mathbb{H})$ viewed as $\mathbb{R}^{4n^2}$. Hence it is a positive-definite real quadratic form.

Let $\mathfrak{a} \subset \mathfrak{g}$ be an ideal. Let $\mathfrak{a}^\perp$ be the orthogonal complement with respect to this inner product. Then $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ as vector spaces from standard Euclidean geometry. It suffices to show $\mathfrak{a}^\perp$ is an ideal.

Let $X \in \mathfrak{a}^\perp, Y \in \mathfrak{g}$. We must show $[X, Y] \in \mathfrak{a}^\perp$. Take some $Z \in \mathfrak{a}$; we must show $\langle [X, Y], Z\rangle = 0$. This is

$$\begin{aligned}
\langle [X, Y], Z\rangle &= \Re\operatorname{Tr}[X, Y]Z^* \\
&= \Re\operatorname{Tr}(XYZ^* - YXZ^*) \\
&= \Re\operatorname{Tr}(XYZ^* - XZ^*Y) \\
&= \Re\operatorname{Tr} X(ZY^* - Y^*Z)^* \\
&= \Re\operatorname{Tr} X[Z, Y^*]^* \\
&= \langle X, [Z, Y^*]\rangle.
\end{aligned}$$

$Y^* \in \mathfrak{g}$ by the main assumption above and $Z \in \mathfrak{a}$, so $[Z, Y^*] \in \mathfrak{a}$, and since $X \in \mathfrak{a}^\perp$, we get zero, as required.

87 Example

We will apply the preceding theorem to some of our previous examples. Recall that if $\mathfrak{g}$ is reductive, then $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{g}_{ab}$ where $\mathfrak{g}_{ss}$ is semisimple and $\mathfrak{g}_{ab} = \mathfrak{z}_{\mathfrak{g}}$ is the center of $\mathfrak{g}$. If $\mathfrak{g}$ is as in the theorem and $\mathfrak{z}_{\mathfrak{g}} = 0$, then $\mathfrak{g}$ is semisimple.

Note: there are many examples of matrices which are not closed under conjugation. For instance, upper triangular or strictly upper triangular matrices are examples. They are not reductive; they are instead solvable or nilpotent, respectively.

1. $\mathfrak{gl}(n, \mathbb{R})$, $\mathfrak{gl}(n, \mathbb{C})$, and $\mathfrak{gl}(n, \mathbb{H})$ are reductive $\mathbb{R}$-Lie algebras. Recall the center of $\mathfrak{gl}(n, \mathbb{R}) = \mathbb{R}I_n$, $\mathfrak{gl}(n, \mathbb{C}) = \mathbb{C}I_n$, and $\mathfrak{gl}(n, \mathbb{R}) = \mathbb{R}I_n$. In particular, these examples are not semisimple.
2. $\mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{sl}(n, \mathbb{C})$ have trivial centers and are semisimple.
3. What is $\mathfrak{sl}(n, \mathbb{H})$? It is
$$\boxed{\mathfrak{sl}(n, \mathbb{H})} := \{X \in \mathfrak{gl}(n, \mathbb{H}) : \Re\operatorname{Tr} X = 0\}.$$
This is a $\mathbb{R}$-Lie subalgebra of $\mathfrak{gl}(n, \mathbb{H})$.

88 Remark

We showed last week that if $\mathfrak{g}$ is a $\mathbb{C}$-Lie algebra, then $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}_{\mathbb{R}}$ is semisimple. Likewise we showed that if $\mathfrak{g}$ is a $\mathbb{R}$-Lie algebra, then $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is semisimple if and only if $\mathfrak{g}$ is semisimple.

For instance, this allows us to use the preceding theorem to deduce that $\mathfrak{sl}(n, \mathbb{C})$, the complexification of $\mathfrak{sl}(n, \mathbb{R})$, is semisimple as a $\mathbb{C}$-Lie algebra.

Definition 89. A $\mathbb{R}$-Lie algebra $\mathfrak{g}$ is $\boxed{\text{compact}}$ (or of "compact type") if it is the Lie algebra of a compact Lie group.

We will discuss Lie groups more systematically starting Friday.

90 Example

Here are some examples of compact Lie algebras. Notation note: if we don't write the field, it is assumed to be $\mathbb{R}$.

1. $\mathfrak{so}(n) = \{X \in \mathfrak{gl}(n, \mathbb{R}) : X + X^* = 0\}$.
2. $\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) : X + X^* = 0\}$.
3. $\boxed{\mathfrak{sp}(n)} := \{X \in \mathfrak{gl}(n, \mathbb{H}) : X + X^* = 0\}$.

The theorem above shows these are reductive $\mathbb{R}$-algebras. A homework problem is to show the center of $\mathfrak{so}(n)$ is 0 for $n \geq 3$ large. What is the center of $\mathfrak{u}(n)$? It certainly contains $i\mathbb{R}I_n$. Hence $\mathfrak{su}(n) = \{X \in \mathfrak{su}(n) : \operatorname{Tr} X = 0\}$; one may check this is center-less, and $\mathfrak{u}(n) = i\mathbb{R}I_n \oplus \mathfrak{su}(n)$. Hence $\mathfrak{u}(n)$ is reduced but not semisimple.

$\mathfrak{so}(1) = 0$; $\mathfrak{so}(2)$ has dimension 1, so is abelian. $\mathfrak{so}(3)$ is simple, so it certainly has trivial center. $\mathfrak{so}(4)$ is semisimple, but is not simple: from homework, it is $3) \oplus 3)$. Fact: $n)$ for $n \geq 5$ is simple.

$\mathfrak{su}(1) = 0$, $\mathfrak{su}(2)$ is simple, and indeed $\mathfrak{su}(n)$ for $n \geq 2$ is simple.

For $\mathfrak{sp}(n)$, the defining condition forces $\operatorname{Tr} X + \overline{\operatorname{Tr} X} = 0$, so $\Re \operatorname{Tr} X = 0$. Hence $\mathfrak{sp}(n) \subset (n, \mathbb{H})$. $\dim_{\mathbb{R}}^{(} 1) = 3$; it happens that for all $n \geq 1$, $\mathfrak{sp}(n)$ is simple. On the homework, $\mathfrak{sp}(1) \cong \mathfrak{su}(2)$.

October 15th, 2014: Draft

Summary Today, we will discuss groups and Lie algebras preserving a bilinear form. We will briefly review elementary Lie group theory starting Monday.

We first discuss vector spaces over $\mathbb{F} = \mathbb{R}, \mathbb{C}$. We will discuss quaternionic analogues at the end.

Definition 91. Let V be a finite dimensional vector space over $\mathbb{F}$. Take $\operatorname{Aut}(V) := \operatorname{GL}(V)$ to be the group of invertible linear transformations $V \to V$. Let $q \colon V \times V \to \mathbb{F}$ be a bilinear, non-degenerate, symmetric or skew-symmetric form. Consider the subset of $\operatorname{GL}(V)$ which preserve the form q,

$$\begin{aligned}&\{L \in \operatorname{GL}(V) : q(Lv, Lw) = q(v, w), v, w \in V\}\\ &= \begin{cases} O(V, q) & \text{orthogonal group of } \mathfrak{q} \text{ if } \mathfrak{q} \text{ is symmetric} \\ \operatorname{Sp}(V, q) & \text{symplectic group of } \mathfrak{q} \text{ if } \mathfrak{q} \text{ is skew-symmetric} \end{cases}\end{aligned}$$

Let L_t be a one-parameter family in $O(V, q)$ with $L_0 = \mathrm{id}$. Since $q(L_t v, L_t w) = q(v, w)$, apply the Leibniz rule to see that $X := \frac{d}{dt} L_t\big|_{t=0}$ satisfies

$$\begin{aligned}&\{X \in \mathfrak{gl}(V) : q(Xv, w) = q(v, Xw) = 0\}\\ &:= \begin{cases} \mathfrak{o}(V, q) & \text{orthogonal algebra of } q \text{ if } q \text{ is symmetric} \\ \mathfrak{sp}(V, q) & \text{symplectic algebra of } q \text{ if } q \text{ is skew-symmetric} \end{cases}\end{aligned}$$

One may check that this is a Lie subalgebra of $\mathfrak{gl}(V)$.

Given a basis $v_1, \ldots, v_n$ for V, set $Q_{ij} : +q(v_i, q_j)$ and $Q := (Q_{ij}) \in \mathbb{F}^{n \times n}$. The non-degeneracy of q says that Q is nonsignular. The symmetry (anti-symmetry) of $\mathfrak{q}$ says $Q = Q^T$ ($Q = -Q^T$). We identify $V \cong \mathbb{F}^n$ by sending $v \in V$ to the column vector of its coefficients $x_1, \ldots, x_n$ in the basis $v_1, \ldots, v_n$. In this way,

$$q(v, w) = \sum_{i,j} x_i Q_{ij} y_j = x^T Q y.$$

If $L \in \mathfrak{gl}(V)$, then L has a corresponding matrix $M \in \mathbb{F}^{n\times n}$ where if $v \mapsto x$ as above, Lv maps to Mx. Hence $q(Lv, Lw) = x^T M^T Q M y$, so $L \in \mathrm{O}(v, q)$ or $\mathrm{Sp}(v, q)$ if and only if $M^T Q M = Q$. By differentiating, the corresponding condition for the corresponding Lie algebra is $X^T Q + QX = 0$.

92 Proposition

In the notation of the preceding definition, we can choose a basis on V so that Q has a particular form:

- $\mathbb{F} = \mathbb{F}$, *Q symmetric:*

$$Q = \begin{pmatrix} I_i & 0 \\ 0 & -I_j \end{pmatrix}$$

where $i + j = n := \dim_{\mathbb{R}} V$. In this case (p, q) is called the $\boxed{\text{signature}}$ *of the form q. We write* $\mathrm{O}(i, j)$ *for the orthogonal group corresponding to a bilinear form of signature (i, j); concretely, we may use the above Q matrix. Likewise $\overset{p}{\underset{,}{(q)}}$ is the Lie algebra associated to $O(p, q)$.*

- $\mathbb{F} = \mathbb{C}$, *Q symmetric:*

$$Q = (I_n)$$

where $n = \dim_{\mathbb{C}} V$. This is called $O(n, \mathbb{C})$, with corresponding Lie algebra $\overset{n}{\underset{,}{(\mathbb{C})}}$.

- $\mathbb{F} = \mathbb{R}, \mathbb{C}$, *$\dim_{\mathbb{F}} V = 2m$ even:*

$$Q = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$$

(or a sequence of 2×2 blocks down the main diagonal of the form $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$)). The notation is non-standard here: the corresponding group is either $\mathrm{Sp}(m, \mathbb{F})$ *or* $\mathrm{Sp}(n, \mathbb{F})$. *The algebra is* $\mathfrak{sp}(n, \mathbb{F})$.

Warning: there are multiple normal forms in common use, so be sure you know which one is used in any given context. Similarly $\mathrm{Sp}(m, \mathbb{R}), \mathrm{Sp}(m, \mathbb{C}), \mathrm{Sp}(n)$ *are all different.*

93 Example

Let $X \in \mathfrak{gl}(p + q, \mathbb{R})$ be partitioned into blocks according to the signature (p, q), namely

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is $p \times p$, D is $q \times q$. Then $X \in \mathfrak{o}(p, q)$ if and only if $A^T = -A, D^T = -D, B^T = C$.

If $X \in \mathfrak{gl}(m + m, \mathbb{R})$ is partitioned in the same way, then $X \in \mathfrak{sp}(m, \mathbb{F})$ if and only if $A^T = -D$, $B^T = B$, $C^T = C$.

94 Remark

Claim: if $L \in \mathrm{O}(V, q)$, then $\det L = \pm 1$.

Proof Let M be the matrix of L, Q the matrix of q. Then $M^T Q M = Q$ implies upon taking det

$$(\det M)^2 \det Q = \det Q.$$

We set $\mathrm{SO}(V, q) := \mathrm{O}(V, q) \cap \{\det 1\}$. Claim: $X \in \mathfrak{o}(V, q)$ implies $\operatorname{Tr} X = 0$:

Proof $X^T Q + QX = 0$ implies $X = -Q^{-1} X^T Q$, whence $\operatorname{Tr} X = -\operatorname{Tr} X$.

Hence $\mathfrak{so}(V,q) = \underset{,}{\overset{V}{(}}q)$. Homework: $L \in \mathrm{Sp}(V,q)$ implies $\det L = 1$. Use the "Pfaffian".

All of the algebras defined today have trivial center and are semisimple.

Definition 95. Let V be a finite-dimensional $\mathbb{C}$-vector space. A function $q \colon V \times V \to \mathbb{C}$ is $\boxed{\text{sesquilinear}}$ if it is $\mathbb{R}$-linear and

$$q(\lambda v, \mu w) = \overline{\lambda}\mu q(v,w).$$

(Note: mathematicians typically put the conjugate on the second argument, though physicists tpyically put it on the first argument; we'll use the physicist version.)

q is $\boxed{\text{Hermitian-symmetric}}$ (or just "Hermitian") if additionally

$$\overline{q(v,w)} = q(w,v)$$

and is $\boxed{\text{skew-Hermitian}}$ if

$$\overline{q(v,w)} = -q(w,v).$$

Since q is Hermitian if and only if iq is skew-Hermitian, we focus on the Hermitian case.

Definition 96. Suppose q is a Hermitian form on V. We set

$$\mathrm{U}(V,q) := \{L \in \mathrm{GL}(V) : q(Lv, Lw) = q(v,w)\},$$

$$\mathfrak{u}(V,q) := \{X \in \mathfrak{gl}(V) : q(Xv,w) + q(v,Xw) = 0\}.$$

If we choose a basis $v_1, \ldots, v_n$ for V, set $Q_{ij} := q(v_i, v_j)$, so $\overline{Q_{ij}} = Q_{ji}$. Fixing an isomorphism $V \cong \mathbb{C}^n$ as before, sending $v \mapsto x$ where x is the column vector of coefficients of v in the $v_1 \ldots, v_n$ basis,

$$q(v,w) = \overline{x}^T Q y = x^* Q y.$$

97 Proposition

Under the notation of the previous definition, we again have normal forms: if $p + q = \dim_{\mathbb{C}} V$, then

$$Q = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

We define $\boxed{\mathrm{U}(p,q)}$ as the unitary group associated to Q of signature (p,q). The corresponding Lie algebra is $\mathfrak{u}(p,q)$.

98 Remark

If $M \in \mathrm{U}(p,q)$, then $|\det M|^2 = 1$. $M \in \mathfrak{u}(p,q)$ implies $\operatorname{Tr} M \in i\mathbb{R}$. $\mathfrak{su}(p,q)$ has the additional condition that the trace is zero. $\mathfrak{u}(p,q)$ is reductive while $\mathfrak{su}(p,q)$ is semisimple.

October 17th, 2014: Draft

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99 Remark

Today we'll discuss quaternionic matrix groups and algebras. We will turn towards Lie groups in subsequent lectures. This is essentially the analogue of the previous lecture for the quaternions.

Definition 100. What is a quaternionic vector space? Quaternions are not abelian, so it requires a little finesse.

We can use scalar multiplication on the left or right. If V is a set and $\lambda, \mu \in \mathbb{H}$, then we can use either

$$\mu(\lambda v) = (\mu\lambda)v$$

(this is a "left quaternionic vector space") or

$$(\lambda\mu)v = \mu(\lambda v)$$

which can also be written as

$$(v\mu)\lambda = v(\mu\lambda)$$

(this is a "right quaternionic vector space). By conjugating scalars, we can translate between left and right quaternionic vector spaces.

We will use right quaternionic vector spaces. If $M \in \mathbb{H}^{n\times n}$, then the map $\mathbb{H}^n \to \mathbb{H}^n$ given by $X \mapsto MX$ where X is a column vector and MX is matrix multiplication as usual is linear with respect to right quaternion multiplication. (Conversely, every such linear map comes from some unique matrix M.)

101 Remark

If $x, y \in \mathbb{H}^n$, is the map $q\colon \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}$ given by $q(x,y) = \sum_i x_i y_i$ "bilinear", where we use either left multiplication or right multiplication? Nope; we would need to commute arbitrary quaternions in either case.

Definition 102. Let V be a (finite dimensional) right quaternionic vector space. A $\boxed{\text{sesquilinear form}}$ on V is a map $q\colon V \times V \to \mathbb{H}$ such that

$$q(v\lambda, w\mu) = \overline{\lambda} q(v,w)\mu$$

(and it distributes over addition as usual).

We'll say a sesquilinear form q is $\boxed{\text{Hermitian}}$ if $\overline{q(v,w)} = q(w,v)$. We'll say it is $\boxed{\text{skew-Hermitian}}$ if $\overline{q(v,w)} = -q(w,v)$.

For instance, $q(x,y) = \sum_\ell \overline{x_\ell} i y_\ell$ is skew-Hermitian.

Definition 103. Let (V,q) be a quaternionic vector space with q either Hermitian or skew-Hermitian, and also non-degenerate. We set

$$\boxed{\mathrm{GL}(V)} := \{\text{quaternionic linear transformations } V \to V\}.$$

We set

$$\begin{aligned}\{L \in \mathrm{GL}(V) : q(Lv, Lw) = q(v,w), v, w \in V\} \\ = \begin{cases} U_{\mathbb{H}}(V,q) & \text{if } \mathfrak{q} \text{ is Hermitian} \\ \mathrm{Sk}(V,q) & \text{if } \mathfrak{q} \text{ is skew-Hermitian}\end{cases}\end{aligned}$$

The corresponding algebra is

$$\mathfrak{u}_{\mathbb{H}}(V,q) := \{X \in \mathfrak{gl}(V) : q(Xv,w) + q(v,Xw) = 0\}.$$

Choose a basis $v_1, \ldots, v_n$ for V. Set $Q_{ij} := q(v_i, v_j)$, $Q := (Q_i j) \in \mathbb{H}^{n\times n}$. One sees quickly that q is Hermitian if and only if $Q = Q^*$, and q is skew-Hermitian if and only if $Q = -Q^*$. If $v = \sum_i v_i x_i$ and $w = \sum_j v_j y_j$, we find

$$q(v,w) = q(\sum v_i x_i, \sum v_j y_j) = \sum_{i,j} \overline{x_i} q(v_i, v_j) y_j,$$

so that

$$q(x,y) = x^* Q y.$$

One checks that a matrix M preserves q if and only if $M^* Q M = Q$, $X^* Q + QX = 0$.

104 Proposition

If (V, q) is a quaternionic vector space with q either Hermitian or skew-Hermitian and non-degenerate, there is a basis for V such that the matrix Q of q is:

- *Hermitian case: $Q = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$; the corresponding group is $U_{\mathbb{H}}(p,q)$, and (p,q) is the analogue of the signature.*
- *Skew-Hermitian case: $Q = iI$; this is written* $\mathrm{Sk}(n, \mathbb{H})$.

105 Remark

The notation for these objects is unfortunately not terribly standard, there are a lot of variants, and some of them conflict abstractly.

106 Remark

There are embeddings

$$\mathfrak{gl}(n, \mathbb{H}) \hookrightarrow \mathfrak{gl}(2n, \mathbb{C}) \hookrightarrow \mathfrak{gl}(4n, \mathbb{R}).$$

As setup, we first define a complex-linear map $\mathbb{H} \hookrightarrow \mathbb{C}^2$ given by sending $z = a + bi + cj + dk$ to $(\alpha, \beta) := (a + ib, c - id)$. Here $z = \alpha + j\beta$. This induces a complex-linear map $\mathbb{H}^n \to \mathbb{C}^{2n}$. Note that if $\beta \in \mathbb{C}$, then $j\beta = \overline{\beta} j$.

Now, given $U \in \mathbb{H}^{n\times n}$, we can write $M = jB$ for $A, B \in \mathbb{C}^{n\times n}$. Letting $z = \alpha + j\beta$ as above, we compute

$$\begin{aligned}
Mz &= (A + jB)(\alpha + j\beta) \\
&= (A\alpha + jBj\beta) + (Aj\beta + jB\alpha) \\
&= (A\alpha - \overline{B}\beta) + j(\overline{A}\beta + B\alpha) \\
&= \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.
\end{aligned}$$

Hence the map $\mathbb{H}^n \hookrightarrow \mathbb{C}^{2n}$ is given by

$$M \mapsto \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}.$$

107 Proposition

The map $\mathcal{C} \colon \mathbb{H}^n \hookrightarrow \mathbb{C}^{2n}$ above has the following properties:

(1) $\mathcal{C}(MN) = \mathcal{C}(M)\mathcal{C}(N)$.

(2) $\mathcal{C}(M^*) = \mathcal{C}(M)^*$.

(3) $\mathcal{C}(M) = 2\Re\operatorname{Tr}(A) = 2\Re\operatorname{Tr} M$.

PROOF Assemble the pieces yourself as an exercise; use the fact shown above that $(XY)^* = Y^* X^*$ for quaternionic matrices X, Y.

October 20th, 2014: Draft

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108 Remark

The following is helpful for problem # 5 on the homework due next Monday. We have a map $\mathcal{R}\colon \mathfrak{gl}(n,\mathbb{C}) \to \mathfrak{gl}(2n,\mathbb{R})$ given by sending $M = A + iB$ to

$$\mathcal{R}(A+iB) := \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

We wish to characterize the range more conceptually. Identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ by sending $z = x + iy$ to (x, y). Then: if $N \in \mathfrak{gl}(2n,\mathbb{R})$, N is in the range of $\mathcal{R}$ if and only if N commutes with multiplication by i.

PROOF If $z = x + iy$, then $iz = i(x+iy) = -y + ix$ corresponds to $(-y, x)$, which is given by

$$\begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Write J for the matrix on the right-hand side. N is in the rance of $\mathcal{R}$ if and only if $NJ = JN$.

109 Example

Examples of semisimple Lie algebras:

- Over $\mathbb{C}$: $\mathfrak{sl}(n,\mathbb{C})$ for $n \geq 2$; $\mathfrak{so}(n,\mathbb{C})$ for $n \geq 3$; $\mathfrak{sp}(n,\mathbb{C})$ for $n \geq 1$.

  110 Fact

  All of these are simple, except for $4,\mathbb{C})$.

  111 Theorem (Classification of Simple $\mathbb{C}$-Lie Algebras)

  With 5 exceptions, every simple $\mathbb{C}$-Lie algebra is isomorphic to one of the above.

  In light of the theorem, the above are called the classical $\mathbb{C}$-Lie algebras, and the remaining 5 are called exceptional $\mathbb{C}$-Lie algebras. They are called g_2, f_4, e_6, e_7, e_8. The following is a list in which each $\mathbb{C}$-simple Lie algebra occurs exactly once up to isomorphism:

  - A_ℓ; $\mathfrak{sl}(\ell+1,\mathbb{C})$; $\ell \geq 1$
  - B_ℓ; $\mathfrak{so}(2\ell+1,\mathbb{C})$; $\ell \geq 2$
  - C_ℓ; $\mathfrak{sp}(\ell,\mathbb{C})$; $\ell \geq 3$
  - D_ℓ; $\mathfrak{so}(2\ell,\mathbb{C})$; $\ell \geq 4$

  (We have $\mathfrak{sp}(1,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})$, $\mathfrak{sp}(2,\mathbb{C}) \cong \mathfrak{so}(5,\mathbb{C})$, and $\mathfrak{so}(6,\mathbb{C}) \cong \mathfrak{sl}(4,\mathbb{C})$.)

  Here ℓ is the "rank" of the Lie algebra, which is the dimension of a maximal abelian subalgebra.

- Over $\mathbb{R}$: recall that if $\mathfrak{g}$ is a $\mathbb{C}$-Lie algebra, then $\mathfrak{g}$ is simple if and only if $\mathfrak{g}_\mathbb{R}$ is simple (as a $\mathbb{R}$-Lie algebra). (We had shown this with "semisimple" replacing "simple"; this version was on homework.) Hence many examples come from simple $\mathbb{C}$-Lie algebras $\mathfrak{g}$.

  On the other hand, if $\mathfrak{g}_0$ is a $\mathbb{R}$-Lie algebra, complexifying it via $\mathfrak{g}_0 \otimes \mathbb{C}$ does not in general preserve simplicity. Recall that if $\mathfrak{g}_0 = \mathfrak{g}_\mathbb{R}$ where $\mathfrak{g}$ is a $\mathbb{C}$-Lie algebra, then $\mathfrak{g}_0 \otimes \mathbb{C} \cong \mathfrak{g} \otimes \overline{\mathfrak{g}}$. If $\mathfrak{g}_0 = \mathfrak{g}_\mathbb{R}$ and $\mathfrak{g}$ is simple, then $\mathfrak{g}_\mathbb{R} \otimes \mathbb{C} = \mathfrak{g} \oplus \mathfrak{g}$ is not $\mathbb{C}$-simple.

  112 Fact

  If $\mathfrak{g}_0$ is $\mathbb{R}$-simple, then either (a) $\mathfrak{g}_0 \otimes \mathbb{C}$ is $\mathbb{C}$-simple or (b) $\mathfrak{g}_0 \cong \mathfrak{g}_\mathbb{R}$ where $\mathfrak{g}$ is $\mathbb{C}$-simple.

  Hence to get all simple $\mathbb{R}$-Lie algebras, take $\mathfrak{g}_\mathbb{R}$ for $\mathfrak{g}$ $\mathbb{C}$-simple together with $\mathfrak{g}_0$ where $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$ $\mathbb{C}$-simple.

  113 Theorem (Classification of Simple $\mathbb{R}$-Lie Algebras)

  Every simple $\mathbb{R}$-Lie algebra is isomorphic to one of the following (with complexification in parens):

- $\mathfrak{g}_{\mathbb{R}}$ *for* $\mathfrak{g}$ $\mathbb{C}$*-simple;*
- $\mathfrak{so}(p,q)$*;* $(\mathfrak{g}_0 \otimes \mathbb{C} \cong \mathfrak{so}(p+q,\mathbb{C}))$
- $\mathfrak{su}(p,q)$*;* $(\mathfrak{sl}(p+q,\mathbb{C}))$
- $\mathfrak{u}_{\mathbb{H}}(p,q)$*, sometimes written* $\mathfrak{sp}(p,q)$*;* $(\mathfrak{sp}(p+q,\mathbb{C}))$
- $\mathfrak{sl}(n,\mathbb{R})$*;* $(\mathfrak{sl}(n,\mathbb{C}))$
- $\mathfrak{sl}(n,\mathbb{H})$*;* $(\mathfrak{sl}(2n,\mathbb{C}))$
- $\mathfrak{sp}(n,\mathbb{R})$*;* $(\mathfrak{sp}(n,\mathbb{C}))$
- $\mathfrak{sk}(n,\mathbb{H})$*;* $(\mathfrak{so}(2n,\mathbb{C}))$

(Maybe there's multiplicity; maybe some of these aren't even simple.)

The number of real forms of g_2 *is* 2*, of* f_4 *is* 3*, of* e_6 *is* 5*, of* e_7 *is* 4*, and of* e_8 *is* 3.

114 Fact

There is a one-to-one correspondence between simple $\mathbb{C}$-Lie algebras and compact simple $\mathbb{R}$-Lie algebras. Every semisimple $\mathbb{C}$-Lie algebra $\mathbb{C}$-Lie algebra has a unique compact real form (up to isomorphism).

115 Example

The simple $\mathbb{C}$-Lie algebras and corresponding compact $\mathbb{R}$-Lie algebras are:

- $\mathfrak{sl}(n,\mathbb{C})$ (compact Lie algebra: $\mathfrak{su}(n)$; compact Lie group: $\mathrm{SU}(n)$)
- $\mathfrak{so}(n,\mathbb{C})$ ($\mathfrak{so}(n)$, $\mathrm{SO}(n)$)
- $\mathfrak{sp}(n,\mathbb{C})$ ($\mathfrak{sp}(n)$, $\mathrm{Sp}(n)$)

The Lie algebras are all very similar: they are $X \in \mathfrak{sl}(n,K)$ for $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ with $X^* + X = 0$. The Lie groups $\mathrm{SU}(n), \mathrm{SO}(n), \mathrm{Sp}(n)$ (and frequently $U(n)$) are called the classical compact Lie groups.

116 Remark

The classification of $\mathbb{C}$-simple Lie algebras or compact simple $\mathbb{R}$-Lie algebras reduces to the classificatino of "root systems". We will discuss these eventually, though they will mostly be relegated to Sara's course in the Spring.

117 Remark

We may consider the representation theory of (compact) $\mathbb{C}$-Lie algebras or compact Lie groups. It turns out we get roughly the same thing either way, though each approach has its own advantages. "Weyl's unitary trick" allows us to go from (Cartan's) classification of the representation theory of compact $\mathbb{C}$-Lie groups to the representation theory of the compact $\mathbb{C}$-Lie algebras.

118 Remark

We next switch gears away from Lie algebras towards Lie groups. Hence we switch from Knapp to Sepanski.

Definition 119. Let G be a Lie group (that is, a manifold with smooth group operation and smooth inverse map). A $\boxed{\text{Lie subgroup}}$ is a subgroup which is the image of an injective immersion. The topology on the immersed manifold might not agree with the relative topology as a subset of the Lie group G.

The main example illustrating this is an "irrational line" on a torus. Precisely, let $G := \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ be a torus. We view $\mathbb{S}^1$ as $\mathbb{R}/\mathbb{Z}$. Let L be a line in $\mathbb{R}^2$ given by $\{y = mx\}$ where m is some fixed irrational constant. $\mathrm{im}(L) \subset \mathbb{T}^2$ is a Lie subgroup, but the topology of $\mathrm{im}(L)$ is that of L, whereas $\mathrm{im}(L)$ is dense in $\mathbb{T}$, so the subspace topology on $\mathrm{im}(L) \subset \mathbb{T}^2$ is very different from that of L.

120 Theorem (Closed Subgroup Theorem)

Let G be a Lie group, $H \subset G$ a closed subgroup. Then there is a unique smooth structure on H with respect to which it is an embedded Lie subgroup. Here the manifold topology is the relative topology; H is a Lie group with this manifold structure.

121 Example

Consider $\mathrm{GL}(n, \mathbb{F})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$. This is an open subset of $\mathbb{F}^{n^2}$ topologically. The conditions $M^*M = I$ or $\det M = 1$ are continuous conditions. All of our examples are closed subgroups of $\mathrm{GL}(n, \mathbb{F})$; we can even do the quaternionic ones by sticking them in $\mathrm{GL}(2n, \mathbb{C})$. Hence all are Lie groups.

October 22nd, 2014: Draft

Summary Today we will discuss Lie groups and their Lie algebras. It will mostly be review.

Definition 122. Let G be a Lie group. That is, G is a smooth manifold with smooth group multiplication and inversion. For fixed $g \in G$, we multiplication on the right or left by g gives diffeomorphisms of G: $r_g : G \to G$ is defined by $r_g(g') := g'g$ and likewise $l_g(g') := gg'$.

A vector field X on G is a smooth section of the tangent bundle to G. It is left-invariant if $(l_g)_*X = X$ for all $g \in G$. It follows that the vector space of left-invariant vector fields corresponds isomorphically to the tangent space of G at the identity, T_eG. The Lie bracket of vector fields is defined as usual; one must check the set of left-invariant vector fields is closed under the Lie bracket operation. One may then transport the Lie bracket to a Lie bracket of T_eG, which yields the Lie algebra $\mathfrak{g}$ of the Lie group G.

If $X \in \mathfrak{g}$, let $\gamma : \mathbb{R} \to G$ be a maximal integral curve of X starting at $\gamma(0) = e$. (Roughly, γ is a one-parameter curve whose derivatives agree X at all points.) Define $\exp X := \gamma(1)$. Indeed, $\gamma(1) \in G^0$, the identity component of G. Hence we have a map $\exp : \mathfrak{g} \to G^0$.

If $G = \mathrm{GL}(n, \mathbb{F})$, then $\mathfrak{g} \cong \mathfrak{gl}(n, \mathbb{F})$ and $\exp X = e^X$ using matrix exponentiation. (In this latter equation we implicitly use the identification of left-invariant smooth vector fields X and elements of T_eG.) We sometimes give $\mathfrak{g}$ the structure of a smooth manifold as a finite-dimensional real vector space.

123 Fact

1. exp is a local diffeomorphism at $0 \in \mathfrak{g}$.
2. Any neighborhood of $e \in G$ generates G^0 (in the group-theoretic sense), so $\exp \mathfrak{g}$ generates G^0.

Warning: exp need not be onto G^0. Homework: if $G = \mathrm{SL}(2, \mathbb{R})$ then $\exp : \mathfrak{sl}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R})$ is not surjective. However, we will show that if G is compact, then exp surjects onto G^0.

124 Remark

Sending a Lie group to its Lie algebra is functorial. More precisely, fi $\phi : H \to G$ is a homomorphism of Lie groups, then $d\phi_0 : \mathfrak{h} \to \mathfrak{g}$ is a homomorphism of Lie algebras. Here

$$d\phi_0(X) = \frac{d}{dt}\phi(\exp tX)\bigg|_{t=0}.$$

As a special case, if H is a Lie subgroup of G, then $d\phi_e$ is injective, so we can regard $\mathfrak{h} \subset \mathfrak{g}$ as a subalgebra. Indeed, $\exp_{\mathfrak{h}} = \exp_{\mathfrak{g}} |_{\mathfrak{h}}$.

125 Proposition

Given a Lie subgroup H with Lie algebra $\mathfrak{h}$ of a Lie group G with Lie algebra $\mathfrak{g}$,

$$\mathfrak{h} = \{X \in \mathfrak{g} : \exp(tX) \in H, \forall t \in \mathbb{R}\}.$$

Proof This follows quickly from the properties listed above.

Warning: We cannot simply take $t = 1$ in the above; the $\forall t \in \mathbb{R}$ is necessary. As a simple example, take $G = \mathrm{GL}(n, \mathbb{F})$ and $H = \mathrm{SL}(n, \mathbb{F})$. Take $X = 2\pi i I$. Now $\exp X = I \in H$, but $X \notin \mathfrak{sl}(n, \mathbb{F})$ since $\operatorname{Tr} X \neq 0$ (for $n > 1$).

126 Example

With the preceding proposition, we can verify many of the Lie subalgebras claims above. For instance, with $\mathrm{SL}(n, \mathbb{F}) \subset \mathrm{GL}(n, \mathbb{F})$, the defining condition for $A \in \mathrm{SL}(n, \mathbb{F})$ is that $\det A = 1$. If we have a one-parameter family of elements in $\mathrm{SL}(n, \mathbb{F})$, say A_t, we have $\det A_t = 1$. If $\frac{d}{dt} A_t\big|_{t=0} =: X$, then a standard fact is that $\frac{d}{dt} \det A_t\big|_{t=0} = \operatorname{Tr} X$. Hence an extra condition for Y to belong to $\mathfrak{sl}(n, \mathbb{F})$ and not just $\mathfrak{gl}(n, \mathbb{F})$ is that $\operatorname{Tr} Y = 0$. On the other hand, if $\operatorname{Tr} Y = 0$, then $\det e^{tX} = 1$, giving the other inclusion.

More generally, given a non-degenerate symmetric or skew-symmetric bilinear form Q on $\mathbb{F}^n$, let $H = O(\mathbb{F}^n, Q)$ or $\mathrm{Sp}(\mathbb{F}^n, Q)$. Then the defining condition to belong to $\mathrm{GL}(n, \mathbb{F})$ is $A^T Q A = Q$. We essentially differentiate this condition to get $X^T Q + QX = 0$ as being a necessary condition to belong to $\mathfrak{h}$, the Lie algebra of H. It is sufficient: given it, then $X^T = -QXQ^{-1}$, and the question is whether or not $(e^{tX})^T Q e^{tX} = Q$. The left side is $e^{-QtXQ^{-1}} Q e^{tX}$, which is $Qe^{-tX} e^{tX} = Q$.

127 Fact

If G is a Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra, then there is a unique connected Lie subgroup $H \subset G$ whose Lie algebra is $\mathfrak{h}$.

Warning: H need not be a *closed* subgroup; for instance, H could be the irrational line on the torus from last lecture. The torus $G = \mathbb{T}^2$ has abelian Lie algebra, so any one-dimensional subspace is a Lie subalgebra. These correspond bijectively to the lines on the torus, many of which are irrational.

Definition 128. Let G be a connected Lie group. We say G is reductive, semisimple, or simple if and only if $\mathfrak{g}$ is. (Sometimes we abuse notation and apply this to non-connected Lie groups like $O(n)$.)

129 Remark

Lie algebras have no (interesting) topology, but Lie groups definitely do. Our next goal is to consider the connectedness and fundamental group of a Lie group and see if they have any corresponding information in the Lie algebra.

130 Fact

Let G be a Lie group, $H \subset G$ a closed subgroup. We can form the quotient G/H. There is a natural smooth manifold structure on G/H such that the natural projection map $G \to G/H$ is a smooth submersion. G acts on G/H on the left via $g(g'H) := (gg')H$. This action is transitive (any point can be moved to any other point) and the isotropy group (stabilizer subgroup) of the identity coset of eH is H. A quotient G/H is called a $\boxed{\text{homogeneous space}}$.

Note that G/H need not be a group–we would have to require H normal for that.

131 Fact

If G is a Lie group and S is a set on which G acts (on the left) transitively. Suppose for some (hence, all) $s \in S$ that the isotropy group of s is a closed subgroup of G. (That is, $\{g \in G : g \cdot s = s\}$ is a closed subgroup of G.) Then S has a topology and a smooth structure so that the action of G on S is smooth and equivariantly diffeomorphic to G/H.

132 Proposition

Suppose G is a Lie group, $H \subset G$ a closed subgroup. If H is connected and G/H is connected, then G is connected.

PROOF Let G^0 be the identity component of G, so $H \subset G^0$. From manifold theory, the identity component is both open and closed, and G^0 is generated by a neighborhood of the identity, so G^0 is a closed subgroup. We get a smooth map

$$G/H \to G/G^0$$

(using the analogue of the universal property of quotient groups for Lie groups). This map is surjective and G/G^0 has the discrete topology. Since G/H is connected, its image is connected, so G/G^0, being discrete, must have a single point, i.e. $G = G^0$, so G is connected.

133 Proposition

$\mathrm{SO}(n)$ *is connected for* $n \geq 1$.

PROOF By induction on n. $\mathrm{SO}(1)$ has a single point. $\mathrm{SO}(2)$ is just a circle, so is connected. In general, $\mathrm{SO}(n)$ acts on S^{n-1} transitively with isotropy group of $e_1 \in S^{n-1}$ being matrices in $\mathrm{SO}(n)$ with two block diagonal matrices, the first being the 1×1 matrix (1), and the second being an element of $\mathrm{SO}(n-1)$. Hence $S^{n-1} \cong \mathrm{SO}(n)/\mathrm{SO}(n-1)$, which is connected. We may now apply the proposition directly.

October 24th, 2014: Draft

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134 Remark

Last time we briefly reviewed basic Lie group theory and ended the class with a discussion of connectedness in this setting. We begin today's lecture by discussing compactness. The following are all compact Lie groups: $O(n), \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n), \mathrm{Sp}(n)$; roughly, the reason is the same in each case: the columns are orthonormal in some sense, so the entries are each bounded, so matrices are a closed and bounded subset of Euclidean space, topologically. An example of a non-compact Lie group:

$$\begin{pmatrix} \cosh t & \sinh t \\ -\sinh t & \cosh t \end{pmatrix}.$$

Minor point: $\mathrm{SO}(p, q) = \mathrm{SO}(q, p)$, since a linear transformation preserves a quadratic form if and only if it preserves the negative of that quadratic form.

Recall we had shown that $\mathrm{SO}(1)$ and $\mathrm{SO}(2)$ are connected; that if G is a Lie group with closed subgroup H, then if H and G/H are connected, then G is connected. For the higher $\mathrm{SO}(n)$'s, we showed $\mathrm{SO}(n)/\mathrm{SO}(n-1) \cong S^{n-1}$ and inducted to show they are each connected.

135 Proposition

$\mathrm{SU}(n), U(n), \mathrm{Sp}(n)$ *are connected for* $n \geq 1$.

PROOF Same idea as for $\mathrm{SO}(n)$. $U(1) = S^1$ which is connected. $U(n)$ acts transitively on S^{2n-1} by $z \in \mathbb{C}^n$ and $A \in U(n)$ yields $A \cdot z := Az$ (and to pass to S^{2n-1}, take unit-length elements). Isotropy group of e_1 is block diagonal with 1 in row 1, column 1, and an element of $U(n-1)$ in the lower right, allowing us to induct. $\mathrm{SU}(n)$ also acts transitively on S^{2n-1} and $\mathrm{SU}(n)/\mathrm{SU}(n-1) \cong S^{2n-1}$ again. $\mathrm{Sp}(n)/\mathrm{Sp}(n-1) \cong S^{4n-1}$ by acting on $\mathbb{H}^n$ rather than $\mathbb{C}^n$; base case is $\mathrm{Sp}(1) = S^3$ (unit quaternions).

136 Example

$O(n)$ is not connected. Recall $A \in O(n)$ iff $A^T A = I$, so $\det A = \pm 1$. Hence $O(n) = \mathrm{SO}(n) \cup \{\det = -1\}$. Same for $O(n, \mathbb{C}) = \mathrm{SO}(n, \mathbb{C}) \cup \{\det = -1\}$.

137 Remark

Recall our map $\mathcal{R}\colon \mathbb{C}^{n\times n} \to \mathbb{R}^{2n\times 2n}$. Now $U(1) \cong \mathrm{SO}(2) \cong S^1$, and in fact $\mathcal{R}(U(1)) = \mathrm{SO}(2)$.

When we say “simply connected”, we will mean both connected and simply connected.

Definition 138. Let G be a connected Lie group. Then there exists a simply connected Lie group $\boxed{\widetilde{G}}$ called the $\boxed{\text{universal covering group}}$ of G. It comes with a map $\pi\colon \widetilde{G} \to G$ where π is a smooth covering map and a group homomorphism.

Moreover, $\widetilde{G}$ is unique up to a Lie group isomorphism which intertwines the covering maps in the following sense:

$$\begin{tikzcd} \widetilde{G}_1 \arrow[r, dashed, "\exists"] \arrow[d, "\pi_1"] & \widetilde{G}_2 \arrow[dl, "\pi_2"] \\ G & \end{tikzcd}$$

139 Theorem

There is a one-to-one correspondence between isomorphism classes of simply connected Lie groups and isomorphism classes of real Lie algebras. Precisely, given a simply connected Lie group, the corresponding Lie algebra is just the Lie group’s Lie algebra. The main content is that this is “invertible”: for each such Lie algebra, there is a simply connected Lie group giving it birth.

140 Proposition

Suppose G, H are Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$, G is simply connected and $\phi\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism. Then there is a unique Lie group homomorphism $\Phi\colon G \to H$ with $\Phi_* = \phi$.

Proof One important point is to apply the exponential map.

141 Example

$U(1) \cong \mathrm{SO}(2) \cong S^1$ have fundamental groups $\mathbb{Z}$, so are not simply connected. Since $\mathfrak{su}(2) \cong \mathfrak{so}(3) \cong \mathfrak{sp}(1)$, there is a unique simply connected Lie group with this Lie algebra. They come from $\mathrm{SU}(2), \mathrm{SO}(3), \mathrm{Sp}(1)$, respectively; are they all the same?

Explicitly,

$$\mathrm{SU}(2) := \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & -\overline{\alpha} \end{pmatrix} \qquad \text{where} \qquad |\alpha|^2 + |\beta|^2 = 1$$

Hence $\mathrm{SU}(2)$ is diffeomorphic to S^3, so is simply connected. $\mathrm{Sp}(1) \cong S^3$ likewise, so is simply connected. Abstractly, $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$ must then be isomorphic as Lie groups. Indeed, $\mathcal{C}\colon \mathbb{H}^{n\times n} \to \mathbb{C}^{2n\times 2n}$ above sends $\mathrm{Sp}(1)$ to $\mathrm{SU}(2)$ and is our isomorphism. What about $\mathrm{SO}(3)$?

142 Proposition

$\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$.

Proof We will construct a two-to-one covering homomorphism $\mathrm{Sp}(1) \to \mathrm{SO}(3)$, whence the fundamental group has order two by general covering space theory.

Let $z \in \mathrm{Sp}(1)$ be a unit quaternion. Consider the map $A_z\colon \mathbb{H} \to \mathbb{H}$ given by $q \mapsto zq\overline{z} = zqz^{-1}$. This is a $\mathbb{R}$-linear transformation $\mathbb{R}^4 \to \mathbb{R}^4$. $\mathbb{R}^4 = \mathbb{H}$ has a real inner product $\langle z, w\rangle = \Re z\overline{w}$ (which is just the standard Euclidean inner product when we view $\mathbb{H}$ as $\mathbb{R}^4$). In fact, A_z preserves this inner product, since it preserves lengths:

$$|A_z(q)| = |zq\overline{z}| = |z||q||\overline{z}| = |q|.$$

Hence we may consider $A_z \in O(\mathbb{R}^4)$. Also, $A_z(1) = 1$. Write $\Im\mathbb{H} := \{ai + bj + ck\}$; this is the orthogonal complement of $\mathbb{R}\cdot 1$, so A_z sends $\Im\mathbb{H}$ to $\Im\mathbb{H}$. In this sense, $A_z \in O(\mathbb{R}^3)$, so we have a map $A\colon \mathrm{Sp}(1) \to O(3)$. Since $\mathrm{Sp}(1)$ is connected, the image has to be connected, so it must land in the identity component, i.e. $A\colon \mathrm{Sp}(1) \to \mathrm{SO}(3)$. This is a Lie group homomorphism since

$$(A_{z_1} \circ A_{z_2})(q) = z_2 z_1 q \overline{z_1 z_2} = (A_{z_1 z_2})(q).$$

If $z \in \ker A$, then $A_z|_{\Im\mathbb{H}} = \mathrm{id}$, so $A_z = I$ on $\mathbb{H}$, so $zq\overline{z} = q$ for all $q \in \mathbb{H}$, i.e. z commutes with all quaternions, so $z \in \mathbb{R}$. Since $|z| = 1$, $z = \pm 1$, so the kernel is of size two. It follows group-theoretically that every non-empty fiber has size two, so we just need to show surjectivity. The induced map on Lie algebras ${}^{(}1) \to \mathfrak{so}(3)$ is injective since ${}^{(}1)$ is simple and the map is not identically zero. Since they have the same dimension, it is an isomorphsim. Now $\exp\colon {}^{(}1) \to \mathrm{Sp}(1)$ and $\exp\colon \mathfrak{so}(3) \to \mathrm{SO}(3)$ is a local diffeomorphism, from which it follows that the image of A contains a neighborhood of the identity, so is all of $\mathrm{SO}(3)$ since any neighborhood of the identity generates the identity component.

(One can also simply construct the inverse map, which is just the standard way of realizing a spatial rotation as quaternion multiplication; ah well.)

143 Corollary

(of proof). $\mathrm{SO}(3)$ is diffeomorphic to $\mathbb{R}P^3$: $\mathrm{SO}(3) = S^3/\{\pm 1\} = \mathbb{R}P^3$.

October 27th, 2014: Draft

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144 Remark

Last time, we constructed a 2-to-1 covering map $\mathrm{SU}(2) \cong \mathrm{Sp}(1) \to \mathrm{SO}(3)$. We realized $\mathbb{R}^3$ as $\Im\mathbb{H}$ and we defined for $z \in \mathrm{Sp}(1)$ a map $A_z\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $A_z(q) = zq\overline{z}$. We showed that $A_z \in \mathrm{SO}(\Im\mathbb{H})$. Let's interpret this map geometrically.

Consider $z = e^{i\theta} \in \mathrm{Sp}(1) \cap \mathbb{C}$. Then $A_z(i) = i$. If $q \in \mathrm{Span}\{j, k\}$, first note that $zq = q\overline{z}$ since i anti-commutes with j and k. Hence $A_z(q) = zq\overline{z} = z^2 q = e^{2i\theta} q$. This "2" is why it's 2-to-1. In particular, $A_z(j) = \cos(2\theta)j + \sin(2\theta)k$ and $A_z(k) = -\sin(2\theta)j + \cos(2\theta)k$. In the i, j, k basis, A_z has matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\theta) & -\sin(2\theta) \\ 0 & \sin(2\theta) & \cos(2\theta) \end{pmatrix}.$$

In particular, $A_{-1} = \mathrm{id}$ since $-1 = e^{i\pi}$. In particular, the curve $\gamma(t)$ given by $A_{e^{i\theta}} = \gamma(2\theta)$ for $0 \leq t \leq \pi$ generates $\pi_1(\mathrm{SO}(3))$.

More generally, if $z \in \mathrm{Sp}(1)$, we can write $z = \cos\theta + \sin\theta v$ for $v \in \Im\mathbb{H}$ with $|v| = 1$. Then $A_z(v) = v$, and in fact A_z is a rotation about the line v of angle 2θ with orientation determined by the right-hand rule. (There will be a basis in which the matrix is as written above.)

145 Proposition

Let $A \in \mathrm{SO}(n)$, $n \geq 3$ odd. Then A has $+1$ as an eigenvalue.

Proof Eigenvalues are either real or appear in complex conjugate pairs since the characteristic polynomial is real. Since n is odd, there must be at least one real eigenvalue. The only real eigenvalues are ± 1 since orthogonal matrices preserve length. Since the determinant, namely 1, is the product of the eigenvalues, it follows quickly by cases that at least one real eigenvalue is 1.

By the proposition, if $A \in \mathrm{SO}(3)$, then A fixes a line, and the complementary subspace is mutated by an element of $\mathrm{SO}(2)$, which is just a rotation. Hence all elements of $\mathrm{SO}(3)$ are rotation about some axis in $\mathbb{R}^3$ by some angle θ. This is an explicit way to get surjectivity in the proposition at the end of class.

146 Proposition

$\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}_2$ if $n \geq 3$; $\mathrm{SU}(n)$ is simply connected if $n \geq 2$; and $\mathrm{Sp}(n)$ is simply connected if $n \geq 1$.

PROOF (Sketch; advertisement for algebraic topology.) Use the homotopy exact sequence of a fibration. Consider $G/H = M$ for a Lie group G and a closed subgroup H; this is a special case of a fiber bundle. In our case, $G \to G/H$ has fibers (homeomorphic to) H (and there is a local triviality condition, so fibers "fit together" in some sense). There is an exact sequence of homotopy groups

$$\cdots \to \pi_2(G/H) \to \pi_1(H) \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) \to \pi_0(G) \to \pi_0(G/H) \to 0.$$

The π_0 terms are sets and the π_n for $n \geq 1$ terms are groups. Exactness means the kernel of one map is the image of the map to its left. (Formally, the π_0 terms are pointed sets, and the kernel of one map is the set of elements mapping to the fixed point; these points come from fixed base points chosen beforehand.) The maps $\pi_n(H) \to \pi_n(G)$ and $\pi_n(G) \to \pi_n(G/H)$ come from the inclusion $H \to G$ and the projection map $G \to G/H$; the "connecting" maps $\pi_{n+1}(G/H) \to \pi_n(H)$ are much less obvious.

147 Fact

$\pi_1(S^n) = 0$ if $n > 0$; $\pi_k(S^n) = 0$ if $n > k$; $\pi_n(S^n) = \mathbb{Z}$.

Computing $\pi_k(S^n)$ for $k > n$ is a huge motivating problem in algebraic topology. In any case, using $G/H = S^m$ for $m \geq 3$, the $\pi_0(G/H)$, $\pi_1(G/H)$, and $\pi_2(G/H)$ terms are each zero, so by exactness $\pi_i(H) \cong \pi_i(G)$ for $i = 0, 1$. We showed $\mathrm{SO}(n)/\mathrm{SO}(n-1) \cong S^{n-1}$ previously, and for $n \geq 4$ we get $\pi_i(\mathrm{SO}(n-1)) \cong \pi_i(\mathrm{SO}(n))$ for $i = 0, 1$. Inductively, $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ for $n \geq 3$ (using $\mathrm{SO}(3)$ as base case), and $\pi_0(\mathrm{SO}(n)) = 0$.

Definition 148. Since $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2$ for $n \geq 3$, we define the $\boxed{\text{spin group}}$ $\boxed{\text{spin}(n)}$ as the universal covering group of $\mathrm{SO}(n)$ for $n \geq 3$. Alternatively, $\mathrm{spin}(n)$ can be characterized as the unique simply connected Lie group with Lie algebra $\mathfrak{so}(n)$.

149 Example

By uniqueness, $\mathrm{spin}(3) \cong \mathrm{SU}(2) \cong \mathrm{Sp}(1)$. What is $\mathrm{spin}(4)$? We've seen $\mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(3, \mathbb{C})$ so $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Hence $\mathrm{spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. Likewise, one may show $\mathrm{spin}(5) \cong \mathrm{Sp}(2)$ (since $^(2) \cong \mathfrak{so}(5)$) and $\mathrm{spin}(6) \cong \mathrm{SU}(4)$ (since $\mathfrak{so}(6, \mathbb{C}) \cong \mathfrak{sl}(4, \mathbb{C})$, and $\mathfrak{sl}(n, \mathbb{C})$ has compact real form $\mathfrak{su}(n)$, $\mathfrak{so}(6) \cong \mathfrak{su}(4)$).

There is a general construction of $\mathrm{spin}(n)$ using Clifford algebras. We won't take the time to do so at the moment.

150 Remark

We next consider the topology of non-compact groups. By the Cartan decomposition, we can reduce the topology of non-compact groups to the topology of compact groups. This will roughly use the "polar form" of a complex matrix.

Recall that if $A \in \mathrm{GL}(n, \mathbb{C})$, then A can be uniquely written as $A = UP$ where $U \in U(n)$ and P is positive-definite Hermitian. (Sometimes this is called the PU-decomposition, with the order of the factors reversed.) This is similar to writing $z = re^{i\theta}$. From the spectral theorem, any positive-definite Hermitian matrix P can be written as $P = e^X$ for X Hermitian: roughly, diagonalize the matrix and take the logarithm of the eigenvectors.

Write $\mathcal{H}$ for the vector space of $n \times n$ Hermitian matrices. (It is not a subalgebra of $n \times n$ matrices.) There is a map $U(n) \times \mathcal{H} \to \mathrm{GL}(n, \mathbb{C})$ given by $(U, X) \mapsto Ue^X$. Fact: this map is a diffeomorphism. Since $\mathcal{H}$ can be contracted to a point, $U(n)$ is a strong deformation retract of $\mathrm{GL}(n, \mathbb{C})$.

151 Theorem

Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a closed subgroup such that (1) G is closed under conjugate transpose $A \mapsto A^*$; (2) G is the common zero locus of a set of real polynomials whose variables are the real and imaginary *parts of the matrix entries. Let $K := G \cap U(n)$ and let $\mathfrak{p} := \mathfrak{g} \cap \{\mathcal{H}\}$ where $\mathfrak{g}$ is the Lie algebra of G and $\mathcal{H}$ is the set $\mathbb{C}$-vector space of Hermitian matrices.*

Then the map $R \times \mathfrak{p} \to G$ given by $(U, X) \mapsto Ue^X$ is a diffeomorphism. In particular, G is homotopy equivalent to K. (Next time: K is a "maximal compact subgroup".)

Proof Sketch next time.

October 29th, 2014: Draft

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152 Remark
Robin will be out of town Monday and Wednesday. Monty McGovern will give the first couple of lectures on representation theory.

153 Remark
Recall the polar decomposition for $\mathrm{GL}(n, \mathbb{C})$, which we now review:

154 Theorem
If $A \in \mathrm{GL}(n, \mathbb{C})$, then there exists a unique unitary matrix U and a hermitian matrix X such that $A = Ue^X$. (This is sometimes written $A = UP$ where $P = e^X$ is positive-definite Hermitian.)

Proof If it exists, then $A^*A = e^X U^* U e^X = e^{2X}$, which we'll call P^2. For any A, A^*A is positive-definite Hermitian, and from the spectral theorem, every such matrix has a unique positive definite square root (since every positive real number has a unique positive square root). This determines P in terms of A, and since such P is invertible, it also determines U; this gives uniqueness.

As for existence, define P as the above unique positive-definite square root of A^*A and set $U := AP^{-1}$. Then U is unitary:

$$U^*U = P^{-1}A^*AP^{-1} = P^{-1}P^2P^{-1} = I.$$

(To get X, take the logarithm; use real logarithms of positive reals.)

155 Proposition
Let $\mathcal{H}$ be the vector space of Hermitian matrices (of size $n \times n$ for some fixed n). The map $U(n) \times \mathcal{H} \to \mathrm{GL}(n, \mathbb{C})$ given by $(U, X) \mapsto Ue^X$ is a diffeomorphism.

Proof Invertibility is the proposition above; smoothness is clear. Showing the inverse is smooth can't be done naively using the spectral theorem roughly since the eigenvectors of a smooth family of matrices do not vary smoothly. One can use the "resolvent" to finish the proof; we will not take the time.

Definition 156. A $\boxed{\text{linear (Lie) group}}$ is a Lie subgroup of $\mathrm{GL}(n, \mathbb{C})$.

157 Theorem
(Restatement from last time.) Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a closed linear group such that:

1) G is closed under $A \to A^*$ (so G^0 is reductive)

2) G is the common zero locus of a set of real polynomials in the real and imaginary parts of the matrix entries.

Set $K := G \cap U(n)$, $\mathfrak{p} := \mathfrak{g} \cap \mathcal{H}$ where $\mathfrak{g} := \mathrm{Lie}(G) \subset \mathfrak{gl}(n, \mathbb{C})$. Then K is a maximal compact subgroup of G, i.e. it is not contained in any strictly larger compact subgroup, and the map

$$K \times \mathfrak{p} \to G \qquad (U, X) \mapsto Ue^X$$

is a diffeomorphism. (Writing $G = Ue^X$ is called the $\boxed{\text{Cartan decomposition}}$ of G.)

Proof Claim: the polar decomposition in $\mathrm{GL}(n, \mathbb{C})$, i.e. $A = Ue^X$, already has $U \in G$ and $X \in \mathfrak{g}$. Given the claim, the diffeomorphism reduces to the previous diffeomorphism $U(n) \times \mathbb{H} \to \mathrm{GL}(n, \mathbb{C})$. For the claim, let $A \in G$ and consider $A^*A = e^{2X}$ for $A \in G$, with X obtained as above. X Hermitian implies there exists a unitary B so that $B^{-1}XB = D$ is diagonal. Claim: I can assume X is diagonal by letting $G' := B^{-1}GB$. Since B is unitary, G' also preserves $A \mapsto A^*$ and G' remains a common zero locus as above since conjugation will just correspond to a linear transformation on the variables. If we can show $D \in \mathfrak{g}'$, it will then follow that $X \in \mathfrak{g}$.

Hence we may as well assume X is diagonal. Say $2X = \mathrm{diag}(a_1, \ldots, a_n)$, and since this is Hermitian, each $a_j \in \mathbb{R}$. We know $e^{2X} \in G$ (since A and A^* are), and we want to show $X \in \mathfrak{g}$. Recall our criterion for showing a matrix is in a linear subgroup; we must show $e^{tX} \in G$ for all $t \in \mathbb{R}$. We know $(e^{2X})^k \in G$ for all $k \in \mathbb{Z}$.

158 Lemma

Let p be a polynomial $\mathbb{R}^n \to \mathbb{R}$. If $a_1, \ldots, a_n \in \mathbb{R}$ and $p(e^{ka_1}, \ldots, e^{ka_n}) = 0$ for all $k \in \mathbb{N}$, then $p(e^{ta_1}, \ldots, e^{ta_n}) = 0$ for all $t \in \mathbb{R}$.

Proof Let $p(x_1, \ldots, x_n) = \sum_{m \geq \ell_1, \ldots, \ell_n \geq 0} c_{\ell_1, \ldots, \ell_n} x_1^{\ell_1} \cdots x_n^{\ell_n}$. Substituting in $x_i = e^{ta_i}$, this gives $\sum c_\ell e^{t \sum a_j \ell_j}$, which we can collect as $\sum_{i=1}^N d_i e^{tb_i}$ for $b_1 < \cdots < b_N$. If this is 0 for $t = k \in \mathbb{N}$, this is 0 for all $t \in \mathbb{R}$. A simple argument for that uses asymptotics and induction; details left to reader.

From the lemma, $e^{tX} = \mathrm{diag}(e^{ta_1}, \ldots, e^{ta_n})$ satisfies the defining polynomials of G, so is in G, completing the proof of the diffeomorphism claim. Now suppose $K' \supset K$ is compact and $K' \subset G$. Pick $A \in K' - K$, so $A = UP$ for $U \in K$ and P positive definite Hermitian. Then $P = U^{-1}A \in K'$, and $P \notin K$ since otherwise $A \in K$. The eigenvalues of P are positive real numbers, not all 1, since otherwise P is the identity, which is in K. Let λ be an eigenvalue of P, where $\lambda > 0$ is $\neq 1$. Then $P^k \in K'$ has eigenvalue λ^k; taking $k \to \infty$ or $k \to -\infty$, $\lambda^k \to \infty$. But this contradicts the fact that K' is compact, since $\{P^k\} \subset K'$.

159 Corollary

In the notation of the theorem, the isomorphism $K \times \mathfrak{p} \to G$ yields a homotopy equivalence between K and G since $\mathfrak{p}$ is just a real vector space, hence is contractible.

160 Theorem (Lie Algebra version of Cartan decomposition)

Suppose G is a linear Lie algebra closed under $X \mapsto X^*$. Let $\mathfrak{k} := \mathfrak{g} \cap \mathfrak{u}(n)$, $\mathfrak{p} := \mathfrak{g} \cap \mathcal{H}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as vector spaces.

Proof $A = \frac{A - A^*}{2} + \frac{A + A^*}{2}$.

161 Remark

Let $\Theta \colon \mathfrak{g} \to \mathfrak{g}$ be given by $\Theta(X) := -X^*$. Θ is an automorphism of $\mathfrak{g}$. $\mathfrak{g} = \{X : \Theta(X) = X\}$, $\mathfrak{p} = \{X : \Theta(X) = -X\}$. Θ is called the $\boxed{\text{Cartan involution}}$.

On a linear Lie group G, $\Theta(A) := (A^*)^{-1}$. Then $K = \{A \in G : \Theta(A) = A\}$. For not-necessarily-linear Lie groups, one tries to construct Θ out of the structure of $\mathfrak{g}$. Fact: any real semisimple Lie algebra is isomorphic to a linear Lie algebra closed under $X \mapsto X^*$.

October 31st, 2014: Draft

162 Remark

Last time in our Cartan decomposition, G was a closed linear group, i.e. a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$, G was closed under $A \mapsto A^*$, and G was defined by polynomial equations. We observed that $K := G \cap U(n)$ is a maximal compact subgroup of G. This is *it* unique, unlike the case of maximal solvable ideals in our Lie algebra theory. For instance, if $g \in G$, conjugating K, i.e. forming gKg^{-1}, is another maximal compact subgroup, which in general is distinct from K. It happens that any two maximal compact subgroups of a Lie group are conjugate, so in particular they are isomorphic.

Recall $U(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) : q(Ax, Ay) = q(x, y)\}$ where $q(x, y) = x^*y$, though we could choose another positive-definite Hermitian form q instead, which will give another unitary group which is just as good as $U(n)$. They would give equally good K's, and they would be conjugates of the above K.

163 Remark

About the topology of non-compact groups: recall the theorem from last time gave a homotopy equivalence between G and K, so their topologies are largely the same. Some consequences:

- If $G := \mathrm{GL}(n, \mathbb{C})$, $K := U(n)$, so both are connected.
- $G := \mathrm{GL}(n, \mathbb{R})$, $K := O(n)$, so both have two connected components.
- $G := \mathrm{SL}(n, \mathbb{C})$, $K := \mathrm{SU}(n)$, so both are connected and simply connected if $n \geq 2$.
- $G := \mathrm{SL}(n, \mathbb{R})$, $K := \mathrm{SO}(n)$, so both are connected. When $n = 2$, both have fundamental group $\mathbb{Z}$; when $n > 2$, both have fundamental group $\mathbb{Z}/2$.
- $G := \mathrm{SO}(n, \mathbb{C})$, $K := \mathrm{SO}(n)$, so see previous case.

  Definition 164. In light of this example, we let $\boxed{\mathrm{spin}(n, \mathbb{C})}$ be the simply connected covering space of $\mathrm{SO}(n, \mathbb{C})$. In fact, this is a complex Lie group.

  A homework problem is to show $\mathrm{spin}(3, \mathbb{C}) \cong \mathrm{SL}(2, \mathbb{C})$.

- $G := O(p, q)$ for $p \geq q > 0$, $K := O(p) \times O(q)$. Indeed, the defining condition is $A^T I_{p,q} A = I_{p,q}$ and $A^*A = I$, so $A^T A = I$, so $A^{-1} = A^T$, so $I_{p,q}A = AI_{p,q}$. Writing this out in block form gives $A \in O(p) \times O(q)$. Since $O(p)$ and $O(q)$ have two connected components, both G and K have four connected components.
- $G := \mathrm{SO}(p, q)$ for $p \geq q > 0$, $K := S(O(p) \times O(q))$, i.e. the determinant 1 matrices in $O(p) \times O(q)$. There are two connected components.
- $G := \mathrm{SO}^0(p, q)$ or $\mathrm{SO}_e(p, q)$ for $p \geq q > 0$, meaning the identity component of $\mathrm{SO}(p, q)$; $K := \mathrm{SO}(p) \times \mathrm{SO}(q)$; both are connected. If $p = q = 1$, the fundamental group is $\mathbb{Z}$. If $p > q = 1$, the fundamental group is $\mathbb{Z} \times \mathbb{Z}/2$. If $p \geq q > 1$, the fundamental group is $\mathbb{Z}/2 \times \mathbb{Z}/2$.

165 Remark

We next discuss integration on Lie groups. This turns out to be powerful in the representation theory of Lie groups. The ability to integrate is probably the biggest single difference between Lie group and Lie algebra representation theory. We next review some basic integration theory.

166 Remark

Let G be a Lie group. A $\boxed{\text{measure}}$ for us formally means a Borel measure which is finite on compact sets. On nice spaces, like a manifold, this is equivalent to a Radon measure. In particular, we will use the Riesz representation theorem to realize our measures.

Suppose $f\colon G \to \mathbb{R}$ satisfies $f \geq 0$ and $f \in C_c(G)$, i.e. f has compact suport, i.e. $f = 0$ outside of some compact set. Integrating with respect to a measure dg,

$$0 \leq \int_G f\, dg < \infty.$$

Hence we can view dg as a linear functional on $C_c(G)$. Indeed, the Riesz representation theorem says that every positive linear functional on $C_c(G)$ (say, for Lie groups G) is given by integration against a unique measure.

Definition 167. A measure dg on a Lie group G is a $\boxed{\text{left-invariant measure}}$ if for all $f\colon G \to \mathbb{R}$ smooth,

$$\int_G (\ell_h^* f)(g)\, dg = \int_G f(g)\, dg$$

where $\ell_h\colon G \to G$ is the left multiplication by $h \in G$ map and $h^* f$ is the pullback of f by h. Another way to write this is

$$\int_G f(hg)\, dg = \int_G f(g)\, dg.$$

A right-invariant measure is defined analogously. A non-zero left invariant measure is called a (left) $\boxed{\text{Haar measure}}$. If G is compact, a Haar measure is called a $\boxed{\text{normalized Haar measure}}$ if $\int_G 1\, dg = 1$, i.e. $dg(G) = 1$. Given a Haar measure on a compact group, one can always normalize it, so we will frequently tacitly assume Haar measures have been normalized.

168 Theorem

Every Lie group G has a left (also, a right) invariant Haar measure.

Definition 169. Let $\boxed{\Omega^k G}$ denote the space of all smooth k-forms on G. $\phi \in \Omega^k G$ is called $\boxed{\text{left-invariant}}$ if $\ell_g^* \phi = \phi$ for all $g \in G$. We write $\boxed{\Omega_\ell^k G}$ for the space of all smooth left-invariant k-forms on G.

170 Proposition

The map $\Omega_\ell^* G \to \Lambda^k T_e^* G$ given by $\phi\colon \phi(e)$ is an isomorphism of vector spaces.

Proof This is exactly analogous to the usual isomorphism between left-invariant vector fields and the tangent space at the identity.

Proof As a consequence of the proposition, if $k = n$, we have $\dim_{\mathbb{R}}(\Lambda^n T_e^* G) = 1$. Any non-zero left-invariant n-form ω on G spans this space under the above isomorphism. In particular, any Lie group G is orientable. Hence $\int_G f\,\omega$ makes sense, using the orientation ω gives on G.

In particular, if $f \in C_c(G)$, the functional $f \mapsto \int_G f\,\omega$ is positive, so by the Riesz representation theorem, there exists a measure dg such that $\int_G f\,\omega = \int_G f\, dg$. dg is a left invariant Haar measure. It has been determined up to multiplication by a positive real number.

171 Theorem

Let G be a compact Lie group. There is a unique normalized left Haar measure and a unique normalized right Haar measure, and in fact they are the same.

Proof By the previous theorem, left and right (normalized) Haar measures exist, call them δg and

dg, respectively. Consider

$$\begin{aligned}\int_G f(g)\,\delta g &= \int_G \left(\int_G f(g)\,\delta g\right)\,dh \\ &= \int_G \left(\int_G f(gh)\,\delta g\right)\,dh \\ &= \int_G \left(\int_G f(gh)\,dh\right)\,\delta g \\ &= \int_G \left(\int_G f(h)\,dh\right)\,\delta g \\ &= \int_G f(h)\,dh = \int_G f(g)\,dg.\end{aligned}$$

The result follows. We used Fubini's theorem to interchange the order of integration.

172 Remark

Given a compact Lie group G, we have

$$\int_G f(gh)\,dg = \int_G f(hg)\,dg = \int_G f(g)\,dg = \int_G f(g^{-1})\,dg.$$

The first two equalities come from the previous theorem. For the third equality, if the assignment $f \mapsto \int_G f(g^{-1})dg$ gives a left-invariant Haar measure, equality must hold by uniqueness. We see

$$\int_G (\ell_h^* f)(g^{-1})\,dg = \int_G f(hg^{-1})\,dg = \int_G (f \circ i)(gh^{-1})\,dg = \int_G (f \circ i)(g)\,dg = \int_G f(g^{-1})\,dg,$$

where $i\colon G \to G$ is given by $g \mapsto g^{-1}$.

173 Remark

If G is non-compact, then left invariant Haar measures are unique up to multiplication by a positive real number. However, left and right invariant Haar measures don't necessarily differ by multiplication by a positive real number.

174 Example

$U(1) = \mathrm{SO}(2) = \mathbb{S}^1$, viewed as $\{e^{i\theta}\} \subset \mathbb{C}$ has Haar measure $\frac{d\theta}{2\pi}$. Likewise the n-torus $\mathbb{S}^1 \times \cdots \mathbb{S}^1$ is $\frac{d\theta_1 \cdots d\theta_n}{(2\pi)^n}$. For $\mathbb{R}^n$, the result is just the Lebesgue measure. $\mathrm{Sp}(1) \cong \mathrm{SU}(2) \cong \mathbb{S}^3$ has the usual induced measure coming from Euclidean space, $\frac{d\sigma}{2\pi^2}$.

November 3rd, 2014: Draft

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175 Remark

Monty McGovern is lecturing today.

Definition 176. Given a Lie group G (real or complex), a $\boxed{\text{representation}}$ (π, V) of G is a group homomorphism $\pi\colon G \to \mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$ where $V = \mathbb{R}^n$ or $\mathbb{C}^n$ is the underlying vector space on which the matrix groups act. We use "representation" interchangeably in this context for either π or V.

We say π or V is $\boxed{\text{irreducible}}$ if V has no G-stable subspace. That is, the only subspaces W of V for which each $g \in G$ sends W into W are 0 and V. Note that this is automatic if $\dim V = 1$.

177 Example

(1) We've already seen many matrix groups which are Lie groups, e.g. $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$, $U(n, \mathbb{C})$, $U(p, q)$, $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{Sp}(2n, \mathbb{C})$, $O(n, \mathbb{R})$, $O(n, \mathbb{C})$, ... All of these act irreducibly on $\mathbb{R}^n$ or $\mathbb{C}^n$, i.e. are irreducible representations. These are frequently called the $\boxed{\text{defining representations}}$, where π is just the inclusion map.

(2) There are standard ways of forming new representations from old ones. We will describe these using V instead of π; a good exercise is to reformulate these in terms of π. First suppose V, W are representations (both real or both complex) of G. Then $V \otimes W$ (the tensor being over $\mathbb{R}$ or $\mathbb{C}$) is also a representation given by

$$g \cdot v \otimes w := g \cdot v \otimes g \cdot w.$$

(The more general construction underlying this is that we can tensor product Hopf algebras and get another Hopf algebra.) In particular, $V \otimes V$ and $V^{\otimes n}$ are representations of G.

(3) If V is a representation on G, then $\boxed{S^n V}$ the $\boxed{n\text{th symmetric power of } V}$, namely the quotient of $V^{\otimes n}$ by the submodule generated by permutations of the factors of simple tensors. That is, we identify $v_{i_1} \otimes \cdots \otimes v_{i_n}$ and $v_{\pi(i_1)} \otimes \cdots \otimes v_{\pi(i_n)}$ for all permutations π. Likewise $\boxed{\Lambda^n V}$, the $\boxed{n\text{th exterior power of } V}$, is the quotient where we enforce antisymmetry rather than symmetry: we identify things exactly as before except the right-hand side picks up a factor of $\mathrm{sgn}(\pi)$.

(4) If V is a representation on G, the dual space V^* is also a representation, where V^* as a vector space is the space of linear functions from V to the base field, and given such a function f, $g \cdot f$ is defined via $(g \cdot f)(v) := f(g^{-1} \cdot v)$ for all $v \in V$. Likewise we can give a representation structure to $\mathrm{Hom}(V, W) \cong V^* \otimes W$.

Definition 178. Given two representations V and W of G and a map $\alpha \colon V \to W$, we say that α is an $\boxed{\text{intertwining map}}$ or an $\boxed{\text{intertwiner}}$ if

$$\alpha g(v) = g(\alpha v)$$

for all $g \in G$, $v \in V$. (That is, intertwining maps are linear maps commuting with the G-action.)

179 Lemma (Schur's Lemma)

If V, W are irreducible representations and α is an intertwining map, then α is an isomorphism or 0. If $V = W$ is a complex representation, then α is a scalar. If $V = W$ is a real representation, the space of all intertwining maps as an algebra is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

Proof (Sketch.) That α is 0 or an isomorphism follows quickly since the kernel and image are G-stable subspaces. In the $V = W$ complex case, an eigenspace is G-invariant and non-trivial, so is the whole space. The $V = W$ real case is more involved.

180 Example

Here is our first example of a non-irreducible representation of a matrix group. Let B be the group of all invertible upper triangular matrices in $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$. Here it is clear that the real or complex span of the first basis vector e_1 of V in which we're writing these matrices is stable under the G-action. Hence G does not act irreducibly on $\mathbb{R}^n$ or $\mathbb{C}^n$ for $n > 1$.

This group is *not* compact (despite being closed, it is not bounded). For compact groups, it turns out that every representation is "close" to irreducible. That is, it is either $\boxed{\text{semisimple}}$ or $\boxed{\text{completely reducible}}$: it is a finite direct sum of irreducible representations.

Proof Let G be a compact group, V a representation over $\mathbb{R}$ or $\mathbb{C}$. Let $\langle -, - \rangle$ be an arbitrary positive-definite form on V. Over $\mathbb{R}$, this is a symmetric, positive-definite, bilinear form. Over $\mathbb{C}$, this is a sesquilinear (i.e. linear in the first coordinate, conjugate-linear in the second) Hermitian (i.e. $\langle v, w \rangle = \overline{\langle w, v \rangle}$) positive-definite (in either case, $\langle v, v \rangle \geq 0$ with equality if and only if $v = 0$).

We will now transform $\langle -, - \rangle$ to a G-invariant positive definite form on V by simply integrating over G with respect to Haar measure, which exists and is unique as in the last lecture, since G is compact. Specifically, define $(-,-)$ by

$$(v, w) := \int_G \langle gv, gw \rangle \, dg$$

where dg is Haar measure. One may check $(-,-)$ is still positive definite, but now is also G-invariant: $(gv, gw) = (v, w)$ for all $v, w \in V, g \in G$.

Now given any G-invariant subspace W, form orthogonal subspace $W^\perp$, namely $\{v \in V : (v, w) = 0 \text{ for all } w \in W\}$. $W^\perp$ is G-invariant and complementary to G in the sense that $V \cong W \oplus W^\perp$. If W and $W^\perp$ are irreducible, we're done; otherwise, we can use the same process to break W or $W^\perp$ into further direct sums of representations. This process strictly decreases the dimension of the pieces, so it eventually terminates.

In all, we can indeed decompose V as the (orthogonal) direct sum of irreducible G-submodules V_i, $V = \oplus_{i=1}^n V_i$ where G acts irreducibly on each V_i.

By contrast, the upper triangular matrices B are not completely reducible for $n > 1$: any invariant subspace happens to contain the G-invariant subspace we identified earlier, $\mathbb{R}e_1$ or $\mathbb{C}e_1$, so this invariant subspace contains no G-invariant complementary subspaces. This is essentially a jazzed up version of Maschke's theorem.

181 Remark

We will next develop an especially beautiful irreducible representation called the spin representation of an especially beautiful group, the spin group, over $\mathbb{R}$ or $\mathbb{C}$. To do this, we start with what Monty (but nobody else) calls the $\boxed{\text{Clifford Group}}$. In terms of generators and relations, it has presentation

$$\boxed{G_n} := \langle a_1, \ldots, a_n : a_i^2 = \epsilon, \epsilon^2 = 1, a_i a_j = \epsilon a_j a_i \text{ if } i \neq j \rangle.$$

For instance, G_2 is the quaternion group. What is $|G_n|$? The elemnts of G_n are $\epsilon^{x_0} a_1^{x_1} \cdots a_n^{x_n}$ where each x_i is 0 or 1, and these are distinct, so $|G_n| = 2^n$. What about the center $Z(G_n)$?

$$Z(G_n) = \begin{cases} \{1, \epsilon\} & n \text{ even} \\ \{1, \epsilon, a_1 \cdots a_n, \epsilon a_1 \cdots a_n\} & n \text{ odd} \end{cases}$$

What about the conjugacy classes? They are $\{x, \epsilon x\}$ if x is not central and $\{x\}$ if x is central. Hence there are $2^n + 1$ conjugacy claases if n is even and $2^n + 2$ if n is odd.

What about the irreducible representations of G_n (in the group-theoretic sense)? If ϵ acts by 1, then each of $a_1, \ldots, a_n$ acts by ± 1, so there are 2^n such representations. If n is even, only one irreducible representation is left, which must then have dimension $2^{n/2}$. If n is odd, there are two such representations left, namely of dimensions $2^{(n-1)/2}, 2^{(n-1)/2}$.

Definition 182. The $\boxed{\text{Clifford algebra}}$ is the quotient of the group algebra $\mathbb{C}G_n$ by the ideal $(1 + \epsilon)$. Decomposing $\mathbb{C}G_n$ into irreducible representations, the quotient cuts out all representations on which ϵ acts by 1, leaving just the degree $2^{n/2}$ or $2^{(n-1)/2}$ irreducible representation mentioned above. More precisely, $\mathbb{C}G_n/(1 + \epsilon) \cong M_{2^{n/2}}\mathbb{C}$, i.e. all $2^{n/2} \times 2^{n/2}$ matrices over $\mathbb{C}$ in the even case, and likewise in the odd case with $M_{2^{(n-1)/2}}\mathbb{C} \oplus M_{2^{(n-1)/2}}\mathbb{C}$. This has order 2 periodicity.

The definition for the real case is the same, $\mathbb{R}G_n/(1 + \epsilon)$, though there is order 8 periodicity, which is very strongly related to $\boxed{\text{Bott periodicity}}$.

November 5th, 2014: Draft

183 Remark

Monty is lecturing again today. Last time we defined what we called the Clifford groups; this seems to be in standard use, so we will rename it the Clifford unit group, namely

$$G_n := \langle a_1, \ldots, a_n : a_i^2 = \epsilon, \epsilon^2 = 1, a_i a_j = \epsilon a_j a_i \text{if } i \neq j \rangle.$$

Again, the Clifford algebra is

$$C_n := kG_n / \langle 1 + \epsilon \rangle$$

where $k = \mathbb{R}$ or $\mathbb{C}$. In terms of (algebraic now) generators and relations,

$$C_n = \langle a_1, \ldots, a_n : a_i^2 = -1, a_i a_j = -a_j a_i, i \neq j \rangle.$$

Note that

$$(\sum x_i a_i)^2 = -\sum x_i^2$$

for $x_i \in k$. Now let V be the k-vector space spanned by the a_i; make this into an inner product space by decreeing that the a_i are an othonormal basis. Write $(-,-)$ for this inner product.

Definition 184. Take $\boxed{\text{Pin}(n)}$ to be the group generated by all "unit vectors" $\sum x_i a_i \in V$ (using the above notation) with $\sum x_i^2 = 1$. On C_n, we have an anti-automorphism τ fixing the generators a_i given by $\tau(a_{i_1} \cdots a_{i_m}) = a_{i_m} \cdots a_{i_1}$. Define an action of $\text{Pin}(n)$ on V by

$$g \cdot v := g v \tau(g)$$

for all $g \in \text{Pin}(n), v \in V$. Then we have

$$vwv = \begin{cases} -v & w = v \\ v & (v, w) = 0 \end{cases}$$

Thus the generators of $\text{Pin}(n)$ act by reflections on V. But the reflections generate $O(n)$ (for $K = \mathbb{R}$ or $\mathbb{C}$), so we conclude that $\text{Pin}(n)$ is a double cover of $O(n, K)$, not $O(n)$ itself because the square of every reflection is -1 instead of 1 (though -1 acts trivially).

We see now that $\text{Pin}(n)$ admits a faithful irreducible representation (over $\mathbb{C}$) of dimension $2^{n/2}$ if n is even or two such representations (over $\mathbb{C}$) each of dimension $2^{(n-1)/2}$ if n is odd.

Then $\text{Spin}(n, K)$ is the subgroup of $\text{Pin}(n, K)$ consisting of products of *evenly many* unit vectors. Over $\text{Spin}(n)$, for n even, the representation of dimension $2^{n/2}$ splits into two pieces, each of dimension $2^{n/2-1}$, called $\boxed{\text{half-spin}}$ or $\boxed{\text{chiral spin}}$ representations, while the two representations of dimension $2^{(n-1)/2}$ for n odd are isomorphic. Why? If n is odd and $K = \mathbb{R}$ or $\mathbb{C}$, we have $O(n, K) = \text{SO}(n, K) \otimes \mathbb{Z}/2$ with $\mathbb{Z}/2 = \{\pm 1\}$. Similarly we have $\text{Pin}(n, K) \cong \text{Spin}(n, K) \times \mathbb{Z}/2$ if n is odd, and the two representations of dimension $2^{(n-1)/2}$ of $\text{Pin}(n, K)$ for n odd are distinguished only by the action of $\mathbb{Z}/2$, which is either trivial or not, so they're isomorphic over $\text{Spin}(n, K)$. For n even, it turns out that $\text{Spin}(n, K)$ contains a copy of $\text{Pin}(n-1, K)$, and the action of the center of $\text{Pin}(n-1, K)$ distinguishes the two half-spin representations.

Moreover, note that $O(n, K)$ and $\text{SO}(n, K)$ do *not* act on these larger representations, since reflections act on them with square -1 rather than 1. (We then call these larger representations $\boxed{\text{genuine}}$ representations of Pin or Spin, i.e. they are representations which don't descend to representations of $O(n, K)$ or $\text{SO}(n, K)$.)

185 Remark

How do you get double covers of $\mathrm{SO}(p,q)$ or $O(p,q)$ from this? For this, we modify the definition of the Clifford algebras C_n. For fixed $p+q=n$, we use

$$\begin{aligned} C_n = C_{p+q} := \langle a_1, \ldots, a_p, a_{p+1}, \ldots, a_{p+q} : a_i^2 = 1, 1 \le i \le p, \\ a_j^2 = -1, p+1 \le j \le p+q, \\ a_u a_v = -a_v a_u, 1 \le u \ne v \le p+q \rangle. \end{aligned}$$

Now our vector space V is indefinite, having a form of signature (p,q). We can now define Pin and Spin as before.

186 Remark

For simplicity, take $p=0$. Then

$$C_n(\mathbb{R}) = \begin{cases} M_{2^{n/2}}\mathbb{R} & n \equiv_8 0, 6 \\ M_{2^{(n-1)/2}}\mathbb{C} & n \equiv_8 1, 5 \\ M_{2^{(n-2)/2}}\mathbb{H} & n \equiv_4 2, 4 \\ M_{2^{(n-3)/2}}\mathbb{H} \oplus M_{2^{(n-3)/2}}\mathbb{H} & n \equiv_8 3 \\ M_{2^{(n-1)/2}}\mathbb{R} \oplus M_{2^{(n-1)/2}}\mathbb{R} & n \equiv_8 7, \end{cases}$$

and so accordingly the spin representations are a bit more complicated.

187 Remark

On the other hand, the spin and half-spin representations *do* carry actions of $\mathfrak{so}(n,K)$, the Lie algebra of skew-symmetric matrices over K for $K=\mathbb{C}$ or $\mathbb{R}$. (Must do something more complicated, as above, to define $\mathfrak{so}(p,q)$.) We first define representations of Lie algebras.

Definition 188. If $\mathfrak{g}$ is a Lie algebra over $K=\mathbb{R}$ or $\mathbb{C}$, we say that a pair (π,V) is a $\boxed{\text{representation}}$ (or just π or V) if π is a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{gl}(V)$, so that

$$\pi[X,Y] = \pi(X)\pi(Y) - \pi(Y)\pi(X).$$

Irreducibility, submodules, Schur's lemma, etc. all carry over as before. Now, however, to get a Lie algebra action on $V \otimes W$ given such actions on V and W separately, we define

$$X \cdot (v \otimes w) := (X \cdot v) \otimes w + v \otimes (X \cdot w).$$

The Lie algebra also acts on the dual space V^* as follows: for $f\colon V \to K$,

$$(X \cdot f)(v) := -f(X \cdot v).$$

As before, $\mathrm{Hom}(V,W) \cong V^* \otimes W$ also carries a Lie algebra action given actions on V and W. For example, $\mathfrak{g}$ acts trivially on a particular $f \in \mathrm{Hom}(V,W)$ exactly when f is a $\mathfrak{g}$-module homomorphism from V to W.

More recently than Weyl's original proof that representations of a compact Lie group are completely reducible, using integration and Haar measure, is a purely algebraic proof due to Brouwer (see Humphreys' 1972 book) for compact Lie algebras that representations are completely reducible. Is there an intrinsic way to define compactness of a Lie algebra intrinsically? In fact, a Lie algebra is compact if and only if its Killing form is negative-definite.

Note also that any continuous homomorphism of Lie groups (which was assumed for representations) is automatically smooth.

189 Remark

Any representation of a compact Lie group comes equipped with a "road map": there will be a list of "weights" (living in a finite-dimensional vector space) and a multiplicity for each weight.

The weights of the spin representation are given by the set of vectors $\{(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2})\}$, each with multiplicity one, and the $\pm$'s are independent. The half-spin representations admit a very similar "weight decomposition" except the number of $+$ signs is always even for one of the half-spin representations and is always odd for the other; the multiplicities remain 1.

November 7th, 2014: Draft

190 Remark

Robin back lecturing today. He'll review some of what Monty discussed (perhaps at a gentler pace) and change some of our notation.

191 Remark

Let X be a set and let S_X denote the set of bijections $S \to S$. Indeed, this is a group under composition (the $\boxed{\text{symmetric group}}$). Given a group G, an action of G on X is just a group homomorphism $G \to S_X$.

Analogously, given a manifold M, we replace S_X with $\mathrm{Diff}(M)$, the group of diffeomorphisms from M to M. For instance, if M is just a real vector space, this is $\mathrm{GL}(M)$.

Definition 192. If G is a Lie group and V is a vector space (usually over $\mathbb{C}$) ($\dim V < \infty$) , a Lie group homomorphism $\rho\colon G \to \mathrm{GL}(V)$ is a $\boxed{\text{representation of } G \text{ on } V}$. Formally, the representation is the pair (V, ρ) (sometimes written (ρ, V)). Fact: if ρ is continuous, then ρ is smooth. We require $\dim V > 0$.

If $g \in G$, write $g \cdot v := \rho(g)(v)$. We can say "G acts on V" or "V is a G-module".

Definition 193. Let $\mathfrak{g}$ be a Lie algebra, V a vector space. A $\boxed{\text{representation of } \mathfrak{g} \text{ on } V}$ is a Lie algebra homomorphism

$$\tau\colon \mathfrak{g} \to \mathfrak{gl}(V).$$

If $X \in \mathfrak{g}$, $v \in V$, we write $X \cdot v := \tau(X)v \in V$. Explicitly, we require

$$\tau([X_1, X_2]) = \tau(X_1)\tau(X_2) - \tau(X_2)\tau(X_1).$$

Definition 194. If $\rho\colon G \to \mathrm{GL}(V)$ is a representation of G on V, then $d\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is the induced Lie algebra representation of $\mathfrak{g}$ on V. We alternatively write $\rho_* := d\rho$. Formally,

$$\rho_*(X) := \left.\frac{d}{dt}\right|_{t=0} \rho(\exp tX).$$

195 Remark

Recall we had a general theorem: if G, H are Lie groups, G is simply connected, and $\tau\colon \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, there is a unique $\rho\colon G \to H$ with $\rho_* = \tau$. Hence every Lie algebra representation $\tau\colon \mathfrak{g} \to \mathfrak{gl}(V)$ lifts to $\rho\colon G \to \mathrm{GL}(V)$ with G simply connected, $\mathrm{Lie}(G) = \mathfrak{g}$, and $\rho_* = \tau$.

More succinctly, there is a one-to-one correspondence between representations of a Lie algebra $\mathfrak{g}$ and Lie group representations of the correponding simply connected Lie group. This fails without the "simply connected" assumption, as we will see shortly.

Definition 196. Suppose $\rho\colon G \to \mathrm{GL}(V)$, $\rho'\colon G \to \mathrm{GL}(V')$ are representations of a Lie group G on vector spaces V, V'. An $\boxed{\text{intertwining operator}}$ is a linear transformation $T\colon V \to V'$ such that

$$\rho'(g) \circ T = T \circ \rho(g)$$

for all $g \in G$. We write $\boxed{\mathrm{Hom}(V, V')}$ to denote the vector space of linear transformations $V \to V'$, and we write $\boxed{\mathrm{Hom}_G(V, V')}$ to denote the subspace of intertwining operators.

Definition 197. (V, ρ) is $\boxed{\text{equivalent}}$ to (V', ρ') if there exists an intertwining operator $T\colon V \to V'$ which is an isomorphism of vector spaces.

198 Remark

Choose a basis for V. Identify $\mathrm{GL}(V)$ with $\mathrm{GL}(n,\mathbb{C})$ where $n := \dim_{\mathbb{C}}(V)$. Hence given a representation (V,ρ) of a Lie group G, $\rho(g)$ corresponds to a matrix $M_g \in \mathrm{GL}(n,\mathbb{C})$. The homomorphism property says $M_{g_1}M_{g_2} = M_{g_1g_2}$, so we are "representing" the group operation of G by using matrix multiplication.

Definition 199. (V,ρ) is $\boxed{\text{faithful}}$ if ρ is injective, i.e. one-to-one.

Definition 200. Given a representation (V,ρ) of a Lie group G, a subspace $W \subset V$ is $\boxed{\text{invariant}}$ if $\rho(g)W \subset W$ for all $g \in G$. (V,ρ) is $\boxed{\text{irreducible}}$ if the only invariant subspaces are $\{0\}$ and V. It is $\boxed{\text{reducible}}$ otherwise.

201 Lemma (Criterion for Irreducibility)

(V,ρ) is irreducible if and only if for all $0 \neq v \in V$, $V = \mathrm{Span}_{\mathbb{F}}\{g \cdot v : g \in G\}$.

Proof Suppose it's irreducible. The suggested span is an invariant subspace, and is non-zero since $v \neq 0$, so is the whole space. On the other hand, given any non-zero invariant subspace, it contains some non-zero v, so all $g \cdot v$ are in the invariant subspace, hence the invariant subspace is the whole space. $\{g \cdot v : g \in G\}$ is called the $\boxed{\text{orbit of } v}$.

202 Example

Some examples of representations follow. Let G be a Lie group, V a vector space.

(1) Let $\rho(g) := \mathrm{id}$ for all $g \in G$. This is called a $\boxed{\text{trivial representation}}$ or one says V has $\boxed{\text{trivial action}}$. *The* trivial representation occurs when $\dim V = 1$.

(2) If $\dim V = 1$, the representation is always irreducible.

(3) A Lie subgroup of $\mathrm{GL}(n,\mathbb{F})$ is essentially defined by an inclusion $G \hookrightarrow \mathrm{GL}(n,\mathbb{F})$ of Lie groups. It's a representation for $V := \mathbb{F}^n$. This is the $\boxed{\text{defining representation}}$ or $\boxed{\text{standard representation}}$. For all of our examples (being reductive), the standard representation is irreducible. For instance, use the criterion on $O(n)$ acting on $\mathbb{R}^n$. We need to check that for all $0 \neq v \in \mathbb{R}^n$, the orbit $\{g \cdot v : g \in O(n)\}$ spans $O(n)$. But the orbit is just the sphere of radius $|v|$, which evidently spans.

(4) Let $G := \mathbb{R}$ be the additive group of real numbers. Define

$$\rho(t) := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R}).$$

Since

$$\begin{pmatrix} 1 & t_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t_1 + t_2 \\ 0 & 1 \end{pmatrix},$$

this is a representation. However, $\mathrm{Span}\{e_1\}$ is a (non-trivial proper) invariant subspace, so this is not irreducible. On the other hand, let $c \in \mathbb{C}$. The map $t \mapsto e^{ct} \in \mathrm{GL}(1,\mathbb{C})$ is a (family of) irreducible representation(s) of $\mathbb{R}$. These representations are inequivalent if $c_1 \neq c_2$.

Let $S^1 := \mathbb{R}/2\pi\mathbb{Z}$, the map $t \mapsto e^{ikt}$ for $k \in \mathbb{Z}$ fixed is an irreducible representation of S^1. Thinking of S^1 instead as $\{e^{i\theta}\}$, this is simply the kth power map.

Note that if G is abelian, any irreducible representation is 1-dimensional. The above are in fact all the irreducible representations R and S^1.

(5) Let $G := \mathrm{SU}(2)$. The standard representation just means $\mathrm{SU}(2)$ acts on $\mathbb{C}^2$ by matrix multiplication. Fix $m \geq 0$. Define

$$V_m := \{\text{polynomials } p \colon \mathbb{C}^2 \to \mathbb{C} \text{ homogeneous of degree } m\}.$$

Note that $p \in V_m$ satisfies $p(\lambda z_1, \lambda z_2) = \lambda^m p(z_1, z_2)$ for all $\lambda \in \mathbb{C}$. Note that a basis for V_m is $\{z_1^m, z_1^{m-1}z_2, \ldots, z_2^m\}$, so $\dim_{\mathbb{C}}(V_m) = m+1$. For $g \in \mathrm{SU}(2)$, we define an a representation $\mathrm{SU}(2) \to \mathrm{GL}(V_m)$ by defining the action of $g \in \mathrm{SU}(2)$ on $p \in V_m$ by

$$(g \cdot p)(z) := \rho(g^{-1}z).$$

More concretely, given an arbitrary element of SU(2)

$$g = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}$$

with $|\alpha|^2 + |\beta|^2 = 1$, we set

$$(g \cdot p)(z) = p(\overline{\alpha}z_1 + \overline{\beta}z_2, -\beta z_1 + \alpha z_2).$$

Facts: (1) each V_m is irreducible; (2) every irreducible representation of SU(2) is equivalent to one of the V_m.

November 10th, 2014: Draft

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203 Proposition

Let $\Phi\colon G \to H$ be a homomorphism between Lie groups (i.e. a smooth group homomorphism). Let Φ_* or $d\Phi$ denote the induced homomorphism $\mathfrak{g} \to \mathfrak{h}$ of Lie algebras. If $X \in \mathfrak{g}$, then $\Phi(\exp_G X) = \exp_H(\Phi_*(X))$.

Proof Recall that $\exp tX$ is the one-parameter subgroup (i.e. an injective immersion of $\mathbb{R}$ by a group homomorphism) with $\frac{d}{dt}\big|_{t=0} \exp tX = X$. Since Φ is a homomorphism, $t \mapsto \Phi(\exp_G tX)$ is a one-parameter subgroup in H, so $\Phi(\exp_G tX)$ is $\exp_H(\frac{d}{dt}\big|_{t=0} \Phi(\exp_G tX)) = \exp_H(\Phi_* X)$.

204 Corollary

If $\rho\colon G \to \mathrm{GL}(V)$ is a representation of G, then $\rho(\exp X) = e^{\rho_* X}$ for all $X \in \mathfrak{g}$, where the right-hand side uses the matrix exponential.

Definition 205. If $\mathfrak{g}$ is a Lie algebra, $X \in \mathfrak{g}$, we defined $\operatorname{ad} X\colon \mathfrak{g} \to \mathfrak{g}$ given by $(\operatorname{ad} X)(Y) := [X, Y]$. Indeed, this defines $\boxed{\mathrm{ad}}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, which we long ago asserted was a Lie algebra homomorphism thanks to the Jacobi identity. In our current language, this is the $\boxed{\text{adjoint representation}}$ of the Lie algebra $\mathfrak{g}$ on $V = \mathfrak{g}$. Our next goal is to define and analyze the Lie group analogue of this representation.

Definition 206. If G is a Lie group, $g \in G$, define the conjugation-by-g map $C_g(h) := ghg^{-1}$. For every $g \in G$, C_g is in fact a Lie group automorphism. Hence we have a map $C\colon G \to \mathrm{Aut}(G)$ given by $g \mapsto C_g$. C is smooth in the sense that $G \times G \to G$ given by $(g, h) \to ghg^{-1}$ is smooth. C is also a group homomorphism since $C_{g_2} \circ C_{g_1} = C_{g_2 g_1}$.

G acts on G by conjugation. Fix $g \in G$. $C_{\mathfrak{g}}\colon G \to G$ is a diffeomorphism and a group automorphism, so $dC_g\colon \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra homomorphism. Being the image of an isomorphism under a functor, dC_g is in fact invertible, so $dC_g \in \mathrm{GL}(\mathfrak{g})$. Hence we may define $\mathrm{Ad}(g) := dC_g$, giving $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$. By the chain rule (equivalently, functoriality) $dC_{g_2} \circ dC_{g_1} = dC_{g_2 g_1}$. Hence In all, we have the $\boxed{\text{adjoint representation}}$ $\boxed{\mathrm{Ad}}\colon G \to \mathrm{GL}(\mathfrak{g})$, which is a smooth Lie group homomorphism.

207 Remark

If $G \subset \mathrm{GL}(n, \mathbb{C})$ and $X \in \mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$, then

$$\begin{aligned} \mathrm{Ad}(g)X = (dC_g)(X) &= \frac{d}{dt}\bigg|_{t=0} C_g(\exp tX) \\ &= \frac{d}{dt}\bigg|_{t=0} g\exp(tX)g^{-1} = gXg^{-1}. \end{aligned}$$

Hence in this context the adjoint representation is literally conjugation of matrices. More generally, for $C_g \colon G \to G$, from the proposition above we have $g\exp(X)g^{-1} = C_g(\exp X) = \exp(\mathrm{Ad}(g)X)$. For a matrix group, replace X by tX and take the derivative at $t = 0$ to get the statement above.

208 Theorem

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, Ad *the adjoint representation on G,* ad *the adjoint representation on $\mathfrak{g}$. Then:*

(1) $\mathrm{Ad}_* = \mathrm{ad}$

(2) $\mathrm{Ad}(\exp X) = e^{\mathrm{ad}\, X}$

Proof (2) follows from (1) and the previous corollary by taking $\rho := \mathrm{Ad} \colon gG \to \mathrm{GL}(\mathfrak{g})$. For (1), we compute

$$\begin{aligned} \mathrm{Ad}_*(X)(Y) &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}(\exp tX)Y \\ C_g &= R_{g^{-1}} \circ L_g \\ \mathrm{Ad}(g) &= dC_g = dR_{g^{-1})\circ dL_g}, \end{aligned}$$

so that

$$\mathrm{Ad}_*(X)Y = \left.\frac{d}{dt}\right|_{t=0} (dR_{\exp -tX} \circ dL_{\exp tX})(Y),$$

where L_g denotes left multiplication by g and $R_{g^{-1}}$ denotes right multiplication by g^{-1}. This last expression only depends on Y_e. X, Y are left invariant vector fields on G. Hence

$$\left.\frac{d}{dt}\right|_{t=0} (dR_{\exp -tX} \circ dL_{\exp tX})(Y) = \left.\frac{d}{dt}\right|_{t=0} dR_{\exp -tX}(Y_{\exp tX}).$$

Recall that the *flow* ϕ_t of X (see Jack Lee's smooth manifolds book for definition and discussion) satisfies $\phi_t(g) = g\exp(tX) = R_{\exp tX}(g)$. Hence

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} dR_{\exp -tX}(Y_{\exp tX}) &= \left.\frac{d}{dt}\right|_{t=0} (d\Phi_{-t})(Y_{\phi_t(e)}) = \mathcal{L}_X Y \\ &= [X, Y] = \mathrm{ad}(X)Y. \end{aligned}$$

In a matrix group, this computation is more straightforward:

$$\begin{aligned} (d\,\mathrm{Ad})(X)(Y) &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}(\exp tX)Y \\ &= \left.\frac{d}{dt}\right|_{t=0} (\exp tX)Y(\exp -tX) \\ &= XY - YX = [X, Y] \end{aligned}$$

using the Leibniz rule in the second to last equality.

209 Remark

Recall our assertion from last time involving the representation theory of $G := \mathrm{SU}(2)$. V_m was the space of homogeneous polynomials of degree m on $\mathbb{C}^2$ with basis $\{v_0 = z_1^0 z_2^m, \ldots, v_m = z_1^m z_2^0\}$. $\mathrm{SU}(2)$ acts on $p \in V_m$ by $\rho_m(g)(p) = (g \cdot p)(z) := p(g^{-1}z)$.

210 Theorem

For all $m \geq 0$, (ρ_m, V_m) is irreducible.

Proof Let $S^1 \cong T := \{t_\theta := \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : 0 \leq \theta < 2\pi\} \subset \mathrm{SU}(2)$. Then

$$\begin{aligned}(\rho_m(t_\theta)v_j)(z) &= v_j(t_\theta^{-1}z) = e^{-ij\theta}e^{i(m-j)\theta}z_1^j z_2^m - j \\ &= e^{i(m-2j)\theta)}v_j,\end{aligned}$$

i.e. $\rho_m(t_\theta)v_j = e^{i(m-2j)\theta}v_j$. Hence $\rho_m(t_\theta)$ is a diagonal linear transformation of V_m. Choose θ so that $e^{i(m-2j)\theta}$ are distinct for $0 \leq j \leq m$. A linear algebra fact is that if $L \in \mathrm{End}(V)$ is diagonalizable with distinct eigenvalues, then any invariant subspace for L must be a direct sum of eigenspaces. If $W \subset V_m$ is invariant, it must then be of the form $\mathrm{Span}\{v_{\sigma_1}, \ldots, v_{\sigma_d}\}$. To be continued next time. We will show that no such non-trivial proper span can be invariant under our action.

November 12th, 2014: Draft

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211 Remark

New homework has been posted. It's due next Wednesday.

212 Remark

Here we continue the proof from the end of last lecture.

Proof Recall $G := \mathrm{SU}(2)$, V_m was the space of homogeneous polynomials over $\mathbb{C}$ of degree m, $\rho_m \colon \mathrm{SU}(2) \to GL(V_m)$ is a representation of V_m defined by $\rho_m(g)(p(z)) := p(g^{-1}z)$. Let $\{p_0, \ldots, p_m\}$ be the obvious basis for V_m, namely $p_j(z) := z_1^j z_2^{m-j}$. We were showing (ρ_m, V_m) is irreducible for all $m \geq 0$. We had let T be a circle (see above) and computed the action of t_θ on our basis $\{p_j\}$ explicitly as

$$\rho_m(t_\theta)p_j = e^{i(m-2j)\theta}p_j.$$

Hence for each matrix $\rho_m(t_\theta)$, p_j is an eigenvector with eigenvalue $e^{i(m-2j)\theta}$.

We will find out later that T is a "maximal torus" in $\mathrm{SU}(2)$. The p_j's are called "weight vectors" for this maximal torus, meaning they're simultaneous eigenvectors for all matrices corresponding to that abelian subgroup. The functions from the torus to $\mathbb{C}$ giving the eigenvalues are called called the "weight" of that eigenvector.

If $W \subset V_m$ is an invariant subspace, we may choose θ such that the eigenvalues are distinct, so by the linear algebra fact mentioned last time, W is a direct sum of the weight vectors, i.e. $W = \mathrm{Span}\{p_{\sigma_1}, \ldots, p_{\sigma_d}\}$. Now consider

$$g_T := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} = \exp tX$$

where

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{su}(2).$$

(Note $X^2 = -I$.) A general element of $\mathrm{SU}(2)$ is of the form $g = (\alpha, -\overline{\beta}; \beta, \overline{\alpha}) \in \mathrm{SU}(2)$. Hence $g^{-1} = (\overline{\alpha}, \overline{\beta}; -\beta, \alpha)$. The action of g on p_j is then

$$(g \cdot p_j)(z) = (\overline{\alpha}z_1 + \overline{\beta}z_2)^j(-\beta z_1 + \alpha z_2)^{m-j}.$$

The induced action of the Lie algebra is

$$\begin{aligned}X_{p_j} &= \frac{d}{dt}\bigg|_{t=0} g_t \cdot p_j = \frac{d}{dt}\bigg|_{t=0} (\cos t z_1 + \sin t z_2)^j(-\sin t z_1 + \cos t z_2)^{m-j} \\ &= (jz_1^{j-1}z_2)z_2^{m-j} - (m-j)z_1^j z_2^{m-j-1}z_1 \\ &= jp_{j-1} - (m-j)p_{j+1}.\end{aligned}$$

If W contains any p_j, we may apply X to it and get linear combinations of p_{j-1} and p_{j+1} (setting $p_{-1} = p_{m+1} = 0$) with non-zero coefficients, forcing p_{j-1} and p_{j+1} in W. Hence W contains all p_j, so $W = V_m$. Hence the representation is indeed irreducible.

213 Remark

We may consider a different element of the Lie algebra in the end of the preceding proof. Consider

$$h_t = \begin{pmatrix} \cos t & i \sin t \\ i \sin t & \cos t \end{pmatrix} = \exp tY$$

where

$$Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \mathfrak{su}(2).$$

The same computations yield

$$Y \cdot p_j = -i(jp_{j-1} + (m-j)p_{j+1}).$$

Note that

$$\frac{1}{2}(X+iY)p_j = jp_{j-1}, \qquad \frac{1}{2}(X-iY)p_j = -(m-j)p_{j+1}.$$

This offers a different proof which may avoid the "linear algebra" fact we quoted. $\frac{1}{2}(X+iY) \in \mathfrak{su}(2) \otimes \mathbb{C} \cong \mathfrak{sl}(2, \mathbb{C})$ is called a $\boxed{\text{lowering operator}}$, and likewise $\frac{1}{2}(X-iY)$ is called a $\boxed{\text{raising operator}}$. Explicitly, these operators are

$$\frac{1}{2}(X+iY) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \qquad \frac{1}{2}(X-iY) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

214 Remark

We can also consider the analogous representations of $\mathrm{SU}(n)$ on V_m (i.e. replacing 2 with n), where V_m now consists of homogeneous polynomials over $\mathbb{C}$ of degree m in n variables. These turn out also to be irreducible representations of $\mathrm{SU}(2)$, but for $n > 0$ there are other "irreps", i.e. irreducible representations.

215 Remark

Given a representation $\tau \colon \mathfrak{g} \to \mathfrak{gl}(V)$ of a Lie algebra, we can talk about invariant subspaces and irreducible representations in exactly the same way as for representations of Lie groups.

216 Proposition

Let G be a connected Lie group. Suppose $\rho_i \colon G \to \mathrm{GL}(V)$ for $i = 1, 2$ are representations of G. If $(\rho_1)_* = (\rho_2)_*$, then $\rho_1 = \rho_2$.

Proof Recall that if $X \in \mathfrak{g}$, then $\rho(\exp X) = e^{\rho_* X}$. If $(\rho_1)_* = (\rho_2)_*$, then $\rho_1 = \rho_2$ on $\operatorname{im} \exp$. But this image generates the identity component, which is the whole group since G is connected.

217 Proposition

Let G be a connected Lie group, $\rho \colon G \to \mathrm{GL}(V)$ a representation with induced representation $\rho_* \colon \mathfrak{g} \to \mathfrak{gl}(V)$. If $W \subset V$ is a subspace of V, then W is invariant for ρ if and only if it is invariant for ρ_*. In particular, V is irreducible for ρ if and only if it is irreducible for ρ_*.

Proof If W is invariant for G and $X \in \mathfrak{g}$, then for each $w \in W$ we have

$$X \cdot w = \left.\frac{d}{dt}\right|_{t=0} (exptX)w \in W.$$

On the other hand, if W is invariant for $\mathfrak{g}$, pick $X \in \mathfrak{g}$ and $w \in W$. Then $\rho(\exp X)w = e^{\rho_* X} w$. Since W is invariant for $\mathfrak{g}$, one may check the exponential takes elements of W to W. Hence elements of the image of exp send W to itself, which amplifies to all elements of G as in the preceding proposition.

218 Remark

Suppose $\mathfrak{g}$ is an $\mathbb{R}$-Lie algebra. A $\boxed{\text{real representation}}$ $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is just what you'd expect: V is a $\mathbb{R}$-vector space, and ρ is $\mathbb{R}$-linear. A $\boxed{\text{complex representation}}$ uses a complex vector space V.

If ρ is a real representation on V, we can induce a representation of $\mathfrak{g}$ on $V \otimes_{\mathbb{R}} \mathbb{C}$. This is called the $\boxed{\text{complexification}}$ of (ρ, V). Explicitly, if $X \in \mathfrak{g}$, we define $\rho(X)(v + iw) := \rho(x)w + i\rho(X)w$.

If $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a complex representation, we can complexify ρ to get a representation $\rho_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, namely

$$\rho_{\mathbb{C}}(X + iY)v := \rho(X)v + i\rho(Y)v.$$

Note that complexifying the representation in this way does not change the underlying vector space, but rather changes the underlying Lie algebra.

219 Proposition

If $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a complex representation of a real Lie algebra $\mathfrak{g}$, then a subspace $W \subset V$ is invariant for ρ if and only if it is invariant for $\rho_{\mathbb{C}}$. In particular, ρ is irreducible if and only if $\rho_{\mathbb{C}}$ is irreducible.

PROOF Immediate.

220 Remark

Let G be a simply connected Lie group, $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}$. There exists a one-to-one correspondence between complex representations for G, complex representations for $\mathfrak{g}$, and complex representations for $\mathfrak{g}_{\mathbb{C}}$, which preserves irreducibility.

If we have a representation of $G \to \mathrm{GL}(V)$ and a homomorphism of Lie groups $H \to G$, the composite $H \to G \to \mathrm{GL}(V)$ is a representation of H. The reverse fails. For instance, we have our double cover $\mathrm{Spin}(n) \to \mathrm{SO}(n)$. Any $\mathrm{SO}(n)$-representation yields a $\mathrm{Spin}(n)$-representation. However, in $\mathrm{SU}(2) \cong \mathrm{Spin}(3) \to \mathrm{SO}(3)$, $\pm\,\mathrm{id}$ is sent to id. It turns out that any representation $\rho\colon \mathrm{SU}(2) \to \mathrm{GL}(V)$ passes to a representation $\mathrm{SO}(3) \to \mathrm{GL}(V)$ if and only if $\rho(-\,\mathrm{id}) = \mathrm{id}$. (Mod out by the kernel.) Which V_m pass to representations of SO(3)? We need $p_m(-z) = p_m(z)$, so this occurs if and only if m is even. Hence the most basic failure occurs for SU(2) on V_1, which we will see is equivalent to the standard representation of SU(2). This is called the $\boxed{\text{spin representation}}$ of SU(2) (or $\mathfrak{su}(2)$ or $\mathfrak{so}(3)$).

November 14th, 2014: Draft

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221 Remark

Henceforth, all representations will be assumed to be complex, unless we say otherwise. This in particular is true of the current homework on problems 4 and 5. In the definition of irreducible representations, we assume the representation is non-zero, i.e. not on the 0-dimensional vector space, by convention.

Today's main topic will be the construction of new representations from old ones.

Definition 222. Suppose V is a representation of a Lie group G, with $\rho\colon G \to \mathrm{GL}(V)$. Let V' denote the dual space of V, that is, the vector space of linear functionals $V \to \mathbb{C}$. The $\boxed{\text{dual representation}}$ on the dual space is defined via

$$\rho'\colon G \to \mathrm{GL}(V') \qquad \rho'(g)(\ell)v := \ell(\rho(g)^{-1}v)$$

for all $\ell \in V', v \in V$. That is, $g \cdot \ell)(v) = \ell(g^{-1} \cdot v)$. The usual fact that the double dual is naturally isomorphic to the original (finite dimensional) vector space generalizes to the corresponding double dual representation.

We can conjugate scalar multiplication on a given $\mathbb{C}$-vector space V to obtain the conjugate vector space $\overline{V}$. The action of G on $\overline{V}$ is literally the same on the level of sets. Of course, $\overline{\overline{V}} = V$.

We can combine these two operations to obtain four possible representations of G on $V, V', \overline{V}, \overline{V}'$. (Note that $'$ and $\bar{}$ commute.) It is easy to see that one of these is irreducible if and only if all of them are.

223 Remark

If we choose a basis $\{v_1, \ldots, v_n\}$ for V and consider $V \cong \mathbb{C}^n$, then $\rho(g)$ is represented by a matrix $M_g \in \mathrm{GL}(n, \mathbb{C})$. V' has dual basis $\{v_1', \ldots, v_n'\}$. If $v \in V$ has coordinates $(x_1, \ldots, x_n) \in \mathbb{C}^n$ and $\ell \in V'$ has coordinates $(\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, then $\ell(v) = \sum_{i=1}^n \xi_i x_i = \xi^T x$. Hence $(g \cdot \ell)(v) = \ell(g^{-1} v) = \xi^T (M_g^{-1} x) = ((M_g^{-1})^T \xi)^T x$. Hence $\rho'(g)$ is represented by the matrix $(M_g^{-1})^T$. When written explicitly in terms of matrices like this, the dual representation is called the $\boxed{\text{contragredient representation}}$.

The matrix representing $\rho(g)$ on $\overline{V}$ is similarly just $\overline{M_g}$.

Definition 224. If V, W are representations of G, the $\boxed{\text{direct sum of representations}}$ is a representation $V \oplus W$ of G given by $g \cdot (v, w) := (g \cdot v, g \cdot w)$. Likewise, the $\boxed{\text{tensor product of representations}}$ is a representation of $V \otimes W$ given by $g \cdot (v \otimes w) := (g \cdot v) \otimes (g \cdot w)$, extended by linearity. Likewise we can define representations on $\otimes^k V$, $S^k V$, and $\Lambda^k V$. Similarly we can define a representation of G on $\mathrm{Hom}(V, W) \cong V' \otimes W$ by $(g \cdot T)(v) := g \cdot (T(g^{-1} \cdot v))$.

225 Remark

Let $T \in \mathrm{Hom}(V, W)$. Then T is an intertwining operator from V to W if and only if $g \cdot T = T$ for all $g \in G$. For in this case $T(g \cdot v) = g \cdot (Tv)$, so $T(v) = g \cdot (T(g^{-1} \cdot v)) = (g \cdot T)(v)$. Recall we had written $\mathrm{Hom}_G(V, W)$ for the space of intertwining operators, which now makes sense: the intertwining operators are precisely the G-invariants of $\mathrm{Hom}(V, W)$.

If $q \colon V \times V \to \mathbb{C}$ is a non-degenerate bilinear form, then q induces an isomorphism $V \cong V'$ given by $v \mapsto q(v, -)$. If we view $q \in \otimes^2 V'$, then if q is fixed by the action of G on $\otimes^2 V'$, then the isomorphism $V \cong V'$ is an intertwining operator. One checks q is fixed by the action of G iff q is G-invariant iff $(g \cdot q)(v, w) := q(g^{-1} \cdot v, g^{-1} \cdot w) = q(v, w)$.

226 Proposition

Let V be the standard representation of $\mathrm{SU}(2)$. Then $V \cong V' \cong \overline{V} \cong \overline{V}'$, they're all irreducible, and they're all the irreducible representation V_1. The isomorphisms also hold for $\mathrm{SL}(2, \mathbb{R})$.

Proof For $V = \mathbb{C}^2$, define $q(v, w) := \det(vw) = v_1 w_2 - w_1 v_2$. This is very particular to having two dimensions. This is a skew-symmetric non-degenerate bilinear form on $\mathbb{C}^2$. If $A \in \mathrm{SU}(2)$, then

$$q(Av, Aw) = \det(Av Aw) = \det(A(vw)) = \det A \det(vw) = q(v, w).$$

Hence q induces $V \cong V'$ as representations.

A Hermitian inner product induces a $\mathbb{C}$-linear isomorphism $V \cong \overline{V'}$. If the inner product is G-invariant, i.e. $(g^{-1} \cdot v, g^{-1} \cdot w) = (v, w)$, then the isomorphism $V \cong \overline{V}'$ is a G-map, i.e. an intertwining operator. This applies to the present case. Apply this reasoning to V' instead of V to get $V' \cong \overline{V}$, finishing the result.

For $\mathrm{SL}(2, \mathbb{R})$, the standard representation (again, implicitly on $\mathbb{C}^2$) preserves the determinant used in the preceding proof, so $V \cong V'$, i.e. the dual of the standard representation is equivalent to the standard representation. Conjugation mapping $\mathbb{C}^n \to \mathbb{C}^n$ is an intertwining operator for any subgroup of $\mathrm{GL}(n, \mathbb{R})$ using the standard representation. Hence $V \cong \overline{V}$ in this case.

227 Fact

For $\mathrm{SL}(n, \mathbb{R})$, we have $V \cong \overline{V}$ where V is the standard representation. However, $V \not\cong V'$ if $n \geq 3$. For $\mathrm{SU}(n)$, we have $V \cong \overline{V'}$ where V is the standard representation, but $V \not\cong V' \cong \overline{V}$ if $n \geq 3$. On the next homework, we will show $V \not\cong \overline{V}$ in this latter case.

228 Lemma (Schur's Lemma)

Suppose V, W are complex finite dimensional irreducible representations of any Lie group G. Then $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W)$ is 1 if $V \cong W$ and is 0 if $V \ncong W$.

Proof Suppose $V \ncong W$. Let $T : V \to W$ be an intertwining operator. Then $\ker T \subset V$ is an invariant subspace by the intertwining condition. Since V is irreducible, $\ker T = 0$ or V. If $\ker T = V$, then $T = 0$. If $\ker T = 0$, then T is injective, an $\operatorname{im} T \subset W$ is invariant and is non-zero, so T is surjective, hence T is an isomorphism. This gives the second half of the assertion, and indeed it works over any field.

For the first half, suppose $V \cong W$ and let $T_0 : V \to W$ be an isomorphism which is also an intertwining operator. If $T \in \mathrm{Hom}_G(V, W)$, consider $T \circ T_0^{-1} : \mathrm{Hom}_G(W, W)$. Since $\mathbb{C}$ is algebraically closed, this composite has an eigenvalue λ. The corresponding eigenspace is a kernel, hence is an invariant subspace, so is all of W. It follows that $T = \lambda T_0$. The same argument works over any algebraically closed field.

229 Remark

Schur's lemma fails over $\mathbb{R}$. For instance, let $G = \mathrm{SO}(2)$ with the standard representation $V = \mathbb{R}^2$. If $G \in \mathrm{SO}(2)$, write

$$g_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Now $g_\theta \circ g_\phi = g_\phi \circ g_\theta$. It follows that $g_\phi \in \mathrm{Hom}_G(V, V)$ for all ϕ. These are not all simply multiples of the identity. In fact,

$$\mathrm{Hom}_{\mathrm{SO}(2)}(\mathbb{R}^2, \mathbb{R}^2) = \mathrm{Span}_{\mathbb{R}} \left\{ I, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Definition 230. Let V be a $\mathbb{C}$-vector space with Hermitian inner product $(V, (-,-))$. Given a Lie group G on $(V, (-,-))$, a $\boxed{\text{unitary representation}}$ of G is a homomorphism $G \to U(V, (-,-))$. That is, we require

$$(\rho(g) \cdot v, \rho(g) \cdot w) = (v, w).$$

We say a representation $\rho : G \to \mathrm{GL}(V)$ is $\boxed{\text{unitarizable}}$ if there exists a G-invariant inner product with respect to which ρ is unitary.

November 17th, 2014: Draft

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231 Remark

Last time we were talking about Schur's lemma. Given two irreducible complex representations of a Lie group G, it said $\dim \mathrm{Hom}_G(V, W)$ is 1 if V and W are equivalent and 0 otherwise.

1. If V is an irreducible complex representation of G, then any intertwining operator $T : V \to V$ is a multiple of the identity.

2. This result fails for real representations. For instance, $\mathrm{SO}(2)$ acting on $\mathbb{R}^2$ naturally is irreducble but $\mathrm{Hom}_G(\mathbb{R}^2, \mathbb{R}^2) = 2$. If we complexify this representation, Schur's lemma will then apply, so "something's gonna change". The complexification is just $\mathrm{SO}(2)$ acting on $\mathbb{C}^2$, and we still have $\dim \mathrm{Hom}_G(\mathbb{C}^2, \mathbb{C}^2) = 2$. Hence we must have a failure of irreducibility. The rotations

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

have eigenvalues $\pm e^{i\theta}$ with eigenvectors $(1; \pm i)$. It follows that $\mathbb{C}^2 = \mathbb{C}(1; i) \oplus \mathbb{C}(1; -i)$, so the representation indeed splits up into two pieces.

Recall there are one-to-one correspondences between complex representations of simply connected Lie groups G, complex representations of $\mathfrak{g}$ (a $\mathbb{R}$-Lie algebra), and complex representations of $\mathfrak{g}_{\mathbb{C}}$. These correspondences preserve irreducibility. On the other hand, a real representation of $\mathfrak{g}$ on a real vector space V can be turned into a complex representation of $\mathfrak{g}$ on $V \otimes_{\mathbb{R}} \mathbb{C}$, and this does not preserve irreducibility.

232 Remark

Let V be a $\mathbb{C}$-vector space, $(V, (-,-))$ an inner product. Recall that a unitary representation of G on $(V, (-,-))$ is a Lie group homomorphism $\rho\colon G \to U(V, (-,-))$. A representation $\rho\colon G \to \mathrm{GL}(V)$ is unitarizable if there is an inner product $(-,-)$ on V with respect to which ρ is unitary.

Note that there are many non-unitarizable representations if G is allowed to be non-compact. The homework considers representations of $\mathbb{R}$ and considers the map $x \mapsto e^{cx}$ for $c \in \mathbb{C}$. If c is pure imaginary, this is unitary relative to the usual norm-squared's corresponding inner product. If $c \notin i\mathbb{R}$, it turns out this representation is not unitarizable.

233 Theorem

If G is a compact Lie group, then every (finite dimensional) representation of G is unitarizable. (The result also holds for real representations if we replace "unitarizable" with "orthogonal" in the obvious way.)

Proof Choose any inner product $\langle -,- \rangle$. Define a new inner product by "averaging":

$$(v, w) := \int_G \langle g \cdot v, g \cdot w \rangle \, dg$$

where dg refers to Haar measure. It's easy to check this is indeed a positive sesquilinear nondegenerate (degeneracy would imply $g \cdot v = 0$ for almost all g, but each g is invertible, so $v = 0$). It's also G-invariant:

$$\begin{aligned}(h \cdot v, h \cdot w) &= \int_G \langle g \cdot h \cdot v, g \cdot h \cdot w \rangle \, dg \\ &= \int_G \langle gh \cdot v, gh \cdot w \rangle \, dg \\ &= \int_G \langle g \cdot v, g \cdot w \rangle \, dg \\ &= (v, w).\end{aligned}$$

Definition 234. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of a Lie group G. ρ is called $\boxed{\text{completely reducible}}$ (or $\boxed{\text{decomposable}}$ or $\boxed{\text{semi-simple}}$) if there are irreducible invariant subspaces $V_1, \ldots, V_N \subset V$ such that $V = V_1 \oplus \cdots \oplus V_N$.

In order to check that a given representation (ρ, V) is completely reducible, it suffices to show that every invariant subspace has an invariant complement. That is, there exists an invariant subspace $\widetilde{W} \subset V$ with $V = W \oplus \widetilde{W}$.

235 Example

Not all invariant subspaces have invariant complements. For instance, the upper triangular 2×2 matrices over $\mathbb{R}$ with 1's along the main diagonal acting on $\mathbb{C}^2$ have invariant subspace spanned by $(1; 0)$, but it has no invariant complement. Hence this representation is reducible but not completely reducible.

236 Theorem

Any unitary representation of a Lie group is completely reducible.

Proof If $W \subset V$ is an invariant subspace, then so is its orthogonal complement with respect to any inner product by an easy calculation.

237 Theorem

Any (finite dimensional) representation of a compact Lie group is completely reducible.

238 Theorem

If G is a compact Lie group, then $\mathfrak{g}$ is reductive.

Proof Recall that $\mathfrak{g}$ is reductive iff for all ideals $\mathfrak{a} \subset \mathfrak{g}$, there exists an ideal $\mathfrak{b} \subset \mathfrak{g}$ with $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$. Consider the adjoint representation of the Lie algebra, $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. A subspace $\mathfrak{a} \subset \mathfrak{g}$ is invariant for the adjoint representation if and only if $\mathfrak{a}$ is an ideal, directly from the definitions.

If $\rho\colon G \to U(V, (-,-))$ is a unitary representation, then $\rho_*\colon \mathfrak{g} \to \mathfrak{u}(V, (-,-))$ is a unitary representation of $\mathfrak{g}$. If $X \in \mathfrak{g}$, $(X \cdot v, w) = -(v, X \cdot w)$, so $\rho_*(X)$ is a skew-symmetric linear transformation of $(V, (-,-))$.

To prove the theorem, there exists an invariant inner product for the adjoint representation of G on $\mathfrak{g}$. Hence each $X \in \mathfrak{g}$ acts by skew transformations of $(-,-)$. Every invariant subspace, i.e. ideal, $\mathfrak{a} \subset \mathfrak{g}$ has an invariant complement exactly as in the proof of the preceding theorem.

239 Remark

The preceding discussion is one of the simplest examples of Weyl's unitary trick. We had to work significantly harder with the Killing form to produce similar results since it is not in general positive-definite, unlike the inner product we get from the unitary trick. Also, recall associativity of the Killing form, $([X, Y], Z) = B(X, [Y, Z])$, equivalently $B((\mathrm{ad}\, X)Y, Z) = -B(Y, (\mathrm{ad}\, X)(Z))$. This is essentially invariance under the adjoint representation.

240 Remark

If V is a finite dimensional representation of a compact Lie group, then $V \cong \oplus_{i=1}^n n_i V_i$ where the V_i are irreducible and pairwise inequivalent and $n_i \geq 1$. We call the n_i's "multiplicities". Indeed, this decomposition is unique up to reordering. We'll discuss this more next time.

241 Proposition

If V is a completely reducible representation of a Lie group G, then V is irreducible if and only if $\dim \mathrm{Hom}_G(V, V) = 1$.

Proof $\Rightarrow$ is Schur's lemma. $\Leftarrow$: if $V = V_1 \oplus V_2$ for non-trivial invariant subspaces V_1, V_2, the projections $\pi_i\colon V \to V_i$ for $i = 1, 2$ are intertwining operators.

242 Proposition

If (ρ, V) is a unitary representation of a Lie group G, then $V \cong \overline{V}'$ and $\overline{V} \cong V'$. In particular this is true of any representation of a compact Lie group.

Proof The second follows from the first by conjugating. For the first, an inner product gives a complex-linear isomorphism $V \cong \overline{V}'$. If the inner product is invariant, this isomorphism is an intertwining operator.

243 Proposition

Let V be an irreducible representation of a Lie group G. Any two invariant inner products differ by a positive multiple, $(-,-)_1 = c(-,-)_2$ for some $c > 0$.

Proof As in the preceding proposition, each $(-,-)$ induces an intertwining operator $V \cong \overline{V}'$. By Schur's lemma, they are the same up to a scalar. It must be positive from postive-definiteness.

244 Proposition

Let V be a unitary representation of a Lie group G. If $V_1, V_2 \subset V$ are invariant subspaces and $V_1 \not\cong V_2$, then V_1 is orthogonal to V_2 (relative to the underlying inner product of the unitary representation).

Proof Again, the inner product gives an intertwining operator $V \cong \overline{V}'$. Restriction of a linear functional to a subspace gives another intertwining operator $\overline{V}' \to \overline{V_2'}$. Similarly the inclusion of V_1 to V is an intertwining operator. The composite

$$\widetilde{T}\colon V_1 \to V \to \overline{V'} \to \overline{V_2'} \to V_2$$

is then an intertwining operator, so by Schur's lemma, it is zero. Indeed, unwinding these maps, $\widetilde{T}(v_1)(v_2) = (v_2, v_1) = 0$.

November 19th, 2014: Draft

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245 Remark

We begin today by discussing the isotypic decomposition or canonical decomposition of a compact Lie group. We will call a representation of G a G-module and will call an invariant subspace of a representation of G a G-submodule.

Let G be a compact Lie group, V a finite dimensional (complex) representation. From last time, V is completely reducible. Let $\widehat{G}$ denote the set of equivalence classes of irreducible finite dimensional (complex) representations of G, where we regard two such as the same if they are equivalent. Write $[-]$ to denote the equivalence class of $-$. For each equivalence class $[\rho] \in \widehat{G}$, we fix some representative (ρ, E_ρ).

246 Proposition (Isotypic or Canonical Decomposition)

Let V be a finite dimensional representation of a compact Lie group G. Choose $[p] \in \widehat{G}$. Then there is a unique maximal subspace of V which is equivalent to nE_ρ for some $n \in \{0, 1, \ldots, \}$.

Proof V is finite-dimensional, so maximal subspaces equivalent to some number of copies of E_ρ certainly exist. We must show there is precisely one maximal subspace. That is, it suffices to show if W_1, W_2 are G-submodules with $W_1 \cong n_1 E_\rho$, $W_2 \cong n_2 E_\rho$, then $W_1 + W_2 \cong NE_\rho$. For if this is true, V_ρ will be the sum of all subspaces equivalent to some nE_ρ for some n; finitely many summands suffice since V is finite dimensional.

247 Lemma

Let V be a completely reducible G-modul. If $U_1, U_2 \subset V$ are submodules with U_1 irreducible, then either $U_1 \subset U_2$ or $U_1 \cap U_2 = 0$.

Proof Consider $U_1 \cap U_2 \subset U_1$. The intersection is invariant, so since U_1 is irreducible, the intersection is either 0 or U_1, which gives the two stated cases.

For the claim, apply the lemma with $U_2 = W_2$ and U_1 a subspace of W_1 equivalent to E_ρ. Hence $U_1 \subset W_2$ or $U_1 \cap W_2 = 0$. Repeat this process on $W_2 + U_1$, which is either W_2 or $W_2 \oplus U_1$, so is either $n_2 E_\rho$ or $(n_2 + 1)E_\rho$.

Definition 248. The maximal subspace from the preceding proposition is called the $\boxed{\text{isotypic component}}$ of V and will be denoted $\boxed{V_{[\rho]}}$. We let $\boxed{n_\rho} := \dim V_{[\rho]} / \dim E_\rho$, which is the number of copies of E_ρ appearing in $V_{[\rho]}$.

249 Remark

While $V_{[\rho]}$ is canonically determined, we cannot decomposte it into a sum of E_ρ's canonically, $V_{[\rho]} \cong E_\rho \oplus \cdots \oplus E_\rho$. As an example, consider the trivial action on $\mathbb{R}^2$. Any two lines give a decomposition of this form, but they are not canonically determined. The number of copies of E_ρ, however, is the same in any decomposition, simply because the dimensions add to $\dim V_{[\rho]}$.

250 Proposition

$V = \oplus_{[\rho] \in \widehat{G}} V_{[\rho]}$.

Proof Since V is completely reducible, the internal sum of the $V_{[\rho]}$ is V. We must only show the pairwise intersections are trivial. For that, last time we showed that if V is a unitary representation with $V_1, V_2 \subset V$ invariant, irreducible, and inequivalent, then V_1 and V_2 are orthogonal relative to the invariant inner product. In our case, each $V_{[\rho]}$ is then orthogonal relative to an invariant inner product. But since inner products are positive-definite, it follows that their intersection is trivial, as required.

251 Example (Spherical Harmonics)
Let $P_m(\mathbb{R}^n)$ denote the set of all complex polynomials on $\mathbb{R}^n$ homogeneous of degree m. $O(n)$ acts on $P_m(\mathbb{R}^n)$ by $(g \cdot p)(x) := p(g^{-1}x)$ in analogy with our previous example. If $m = 1$, $P_1(\mathbb{R}^1) = V'$ where V is the standard representation. $P_m(\mathbb{R}^n)$ in general is the mth symmetric power of V'.

A basis for $P_m(\mathbb{R}^n)$ consists of monomials $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n}\}$ where $0 \leq \alpha_i$ and $\alpha_1 + \cdots + \alpha_n = m\}$. For convenience, we use multiindex notation: if $\alpha = (\alpha_1, \ldots, \alpha_n)$, then $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

If $m, n \geq 2$, then $P_m(\mathbb{R}^n)$ is *not* irreducible. Indeed, let $\Delta := \sum_{i=1}^n \partial_{x_i}^2$ be the Laplacian. Then $\Delta\colon \P_m(\mathbb{R}^n) \to P_{m-2}(\mathbb{R}^n)$. Indeed, $\Delta(g \cdot f) = g \cdot (\Delta f)$ for $g \in O(n)$, so Δ is G-invariant.

Let $H_m(\mathbb{R}^n) := \{p \in P_m(\mathbb{R}^n) : \Delta p = 0\}$. These are the $\boxed{\text{harmonic polynomials}}$. This is an invariant subspace, being the kernel of a G-invariant morphism.

Some base cases: $P_0(\mathbb{R}^n)$ consists of constants, i.e. $\mathbb{C}$. This is also $H_0(\mathbb{R}^n)$. $P_1(\mathbb{R}^n)$ is V', so is $\mathbb{C}^n$ dualized, and this is again also $H_1(\mathbb{R}^n)$.

252 Proposition
$P_m(\mathbb{R}^n) \cong H_m(\mathbb{R}^n) \oplus H_{m-2}(\mathbb{R}^n) \oplus \cdots \oplus X$ where the last term X is $H_1(\mathbb{R}^n)$ or $H_0(\mathbb{R}^n)$ depending on whether m is odd or even, respectively.

253 Theorem
If $n \geq 2$, then $H_m(\mathbb{R}^n)$ is irreducible for $O(n)$. If $n \geq 3$, then $H_m(\mathbb{R}^n)$ is irreducible for $\mathrm{SO}(n)$.

254 Remark
From the homework, if $n = 2, m \geq 1$, then $H_m(\mathbb{R}^n)$ is irreducible $O(n)$ but not for $\mathrm{SO}(n)$.

For $O(n)$ or $\mathrm{SO}(n)$, V the standard representation, $V \cong V' \cong \overline{V} \cong \overline{V}'$. Indeed, $V \cong \overline{V}$ essentially trivially, and for any compact group G, $V \cong \overline{V}'$; combine these. Alternatively, $V \cong V'$ since V preserves a bilinear form. This breaks down for $\mathrm{SU}(n)$, $n \geq 3$, namely $V \not\cong \overline{V}$ and $V \not\cong V'$, though $V \cong \overline{V}'$ and $\overline{V} \cong V'$.

255 Remark
To prove the theorem, we will introduce an inner product on $P_m(\mathbb{R}^n)$ and run through our general machinations explicitly. Let $p \in P_m(\mathbb{R}^n)$, $p(x) = \sum_{|\alpha|=m} c_\alpha x^\alpha$ homogeneous of degree m, $c_\alpha \in \mathbb{C}$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We define partial derivative operators for each multiindex α as

$$\partial^\alpha := \partial_{x_1}^{\alpha} \cdots \partial_{x_n}^{\alpha_n}.$$

Likewise we set $p(\partial) := \sum_{|\alpha|=m} c_\alpha \partial^\alpha$. For instance, if $p(x) = |x|^2 = x_1^2 + \cdots + x_n^2$, then $p(\partial) = \partial_1^2 + \cdots + \partial_n^2 = \Delta$.

Now if $p, q \in P_m(\mathbb{R}^n)$, define a bilinear operator

$$(p, q) := \overline{q}(\partial)(p).$$

Since p and q are both of degree m, the result is indeed in $\mathbb{C}$, and is is evidently sesquilinear. Note that $(x^\alpha, x^\beta) = \partial^\beta x^\alpha$, which is 0 if $\alpha \neq \beta$ and is $\alpha! := \alpha_1! \cdots \alpha_n!$ for $\alpha = \beta$. This shows that $\{x^\alpha\}$ is an orthogonal (though not orthonormal) basis, so $(-,-)$ is an inner product.

Again, $H_m(\mathbb{R}^n) = \ker \Delta$, $\Delta\colon P_m(\mathbb{R}^n) \to P_{m-2}(\mathbb{R}^n)$. Recall that if V, W are Hilbert spaces with corresponding forms $(-,-)_V$ and $(-,-)_W$. If $L\colon V \to W$ is a linear transformation, there is an adjoint $L^*\colon W \to V$ defined by $(v, L^*w)_v = (Lv, w)_w$. Basic fact: $\ker L = (\operatorname{im} L^*)^\perp$. In finite dimensions, this gives $(\ker L)^\perp = \operatorname{im} L^*$. Here, let $L = \Delta$, so that $H_m(\mathbb{R}^n)^\perp = \operatorname{im} L^*$. So, what is $L^* = \Delta^*$? In the preceding notation, we require

$$(p, \Delta^* q) = (\Delta p, q) = \overline{q}(\partial)(\Delta p) = (\Delta \overline{q}(\partial))(p) = (\overline{|x|^2 q})(\partial(p) = (p, |x|^2 q).$$

Hence $L^* q = |x|^2 q$.

November 21st, 2014: Draft

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256 Remark

Recall that $P_m(\mathbb{R}^n)$ was the polynomials over $\mathbb{C}$ on $\mathbb{R}^n$ homogeneous of degree m and that $H_m(\mathbb{R}^n)$ was the subspace of $P_m(\mathbb{R}^n)$ consisting of harmonic polynomials, i.e. those annihilated by the Laplacian Δ. $H_m(\mathbb{R}^n)$ is an $O(n)$-invariant subspace.

257 Theorem

With notation as above,

1. $P_m(\mathbb{R}^n) \cong H_m(\mathbb{R}^n) \oplus H_{m-2}(\mathbb{R}^n) \oplus \cdots \oplus X$ *where* $X = H_1(\mathbb{R}^n)$ *if* m *is odd and* $X = H_0(\mathbb{R}^n)$ *if* m *is even.*

2. *If* $n \geq 2$, $H_m(\mathbb{R}^n)$ *is irreducible under* $O(n)$ *and* $H_m(\mathbb{R}^n) \ncong H_{m'}(\mathbb{R}^n)$ *if* $m \neq m'$. *If* $n \geq 3$, $H_m(\mathbb{R}^n)$ *is irreducible under* $\mathrm{SO}(n)$.

Proof One of the homework problems deals with the case $n = 2$ in (2). If $n \geq 3$, the dimensions force $H_m(\mathbb{R}^n) \not\equiv H_{m'}(\mathbb{R}^n)$ for $m \neq m'$. Last time we were in the midst of proving (1). For that, we introduced an inner product on $P_m(\mathbb{R}^n)$ involving partial differential operators—see the remark at the end of last lecture. We had considered $\Delta \colon P_m(\mathbb{R}^n) \to P_{m-2})(\mathbb{R}^n)$ as a linear operator, so by standard Hilbert space theory there is an adjoint $\Delta^* \colon P_{m-2}(\mathbb{R}^n) \to P_m(\mathbb{R}^n)$. We had computed $\Delta^* p = |x|^2 p$. To prove (1), we have $P_m(\mathbb{R}^n) = H_m(\mathbb{R}^n) \oplus H_m(\mathbb{R})^n)^\perp$. We see

$$H_m(\mathbb{R}^n)^\perp = (\ker \Delta)^\perp = \operatorname{im} \Delta^* = |x|^2 P_{m-2}(\mathbb{R}^n).$$

(Here we've used some general Hilbert space facts, namely $\ker L = (\operatorname{im} L^*)^\perp$, so $(\ker L)^\perp = (\operatorname{im} L^*)^{\perp\perp} = \overline{\operatorname{im} L^*}$, but since we have only finitely dimensions, the closure does nothing.) So,

$$P_m(\mathbb{R}^n) = H_m(\mathbb{R}^n) \oplus |x|^2 P_{m-2}(\mathbb{R}^2) = H_m(\mathbb{R}^n) \oplus |x|^2 H_{m-2}(\mathbb{R}^n) \oplus |x|^4 P_{m-4}(\mathbb{R}^n) \oplus \ldots.$$

Iteratively, this gives

$$P_m(\mathbb{R}^n) = H_m(\mathbb{R}^n) \oplus |x|^2 H_{m-2}(\mathbb{R}^n) \oplus \cdots \oplus Y$$

where $Y = |x|^{m-1} H_1(\mathbb{R}^n)$ if m is odd and $|x|^m H_0(\mathbb{R}^n)$ if m is even. Since $|x|^2$ is invariant under $\mathrm{SO}(n)$, we can drop the $|x|^{2k}$'s from each factor. This proves (1).

258 Lemma

Let G *be a compact Lie group. Suppose* U, V, W *are finite dimensional representations. Suppose* $U \oplus V \cong U \oplus W$. *Then* $V \cong W$.

Proof Write $U \cong \oplus_{[\rho] \in \widehat{G}} m_\rho E_\rho$ and likewise for V, W with coefficients m'_ρ and m''_ρ, respectively. The decomposition for $U \oplus V$ has coefficients $m_\rho + m'_\rho$ and the decomposition for $U \oplus W$ has coefficients $m_\rho + m''_\rho$. The isomorphism forces $m_\rho + m'_\rho = m_\rho + m''_\rho$, so $m'_\rho = m''_\rho$, so $V \cong W$.

This essentially says the representation ring of a compact Lie group is a domain, while it fails in general for non-compact Lie groups.

We now must show $H_m(\mathbb{R}^n)$ is irreducible under $\mathrm{SO}(n)$ for $n \geq 3$. Our first claim is a (seemingly very) weak version of this, namely:

259 Lemma

For $n \geq 2$, there is a copy of a trivial representation in $H_m(\mathbb{R}^n)$ if and only if $m = 0$.

Proof Observation: if V is a representation of a compact Lie group G, then there is a copy of the trivial representation in V if and only if there is a non-zero vector $v \in V$ invariant under G. So, suppose $p \in P_m(\mathbb{R}^n)$ is invariant for $\mathrm{SO}(n)$. Since $\mathrm{SO}(n)$ acts transitively on the sphere, $p|_{S^{n-1}} = c$ for some constant c. Hence $p = c|x|^m$. Now $|x|^m$ is not harmonic unless $m = 0$.

Suppose G is a Lie group, $H \subset G$ is a Lie subgroup, $\rho\colon G \to \mathrm{GL}(V)$ is a representation. $\rho|_H\colon H \to \mathrm{GL}(V)$ is a representation of H. We usually denote this representation by $V|_H$, though note the vector space itself is unchanged. If the original representation is irreducible, it will often happen that the restricted representation is not irreducible. This leads to the $\boxed{\text{branching problem}}$, namely given an irreducible over G, how does that irreducible decompose into irreducibles over H? We'll consider this in the case $\mathrm{SO}(n-1) \subset \mathrm{SO}(n)$. We consider $H_m(\mathbb{R}^n)|_{\mathrm{SO}(n-1)}$.

260 Proposition

$H_m(\mathbb{R}^n)|_{\mathrm{SO}(n-1)} \cong H_m(\mathbb{R}^{n-1}) \oplus H_{m-1}(\mathbb{R}^{n-1}) \oplus \cdots \oplus H_0(\mathbb{R}^{n-1})$ (as $\mathrm{SO}(n-1)$ representations).

Proof Indeed,

$$P_m(\mathbb{R}^n)|_{\mathrm{SO}(n-1)} \cong P_m(\mathbb{R}^{n-1}) \oplus P_{m-1}(\mathbb{R}^{n-1}) \oplus \cdots \oplus P_0(\mathbb{R}^{n-1})$$

as follows. For $A \in \mathrm{SO}(n-1)$, we embed A in $\mathrm{SO}(n)$ as $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$. Letting $x = (x_1, x')$ for $x' \in \mathbb{R}^{n-1}$, we can write $p = p_m(x') + x_1 p_{m-1}(x') + \cdots + x_1^m p_0(x')$. Hence the suggested isomorphism is just $p \mapsto (p_m, \ldots, p_0)$. We now use the decomposition deduced above, $P_m(\mathbb{R}^n) \cong H_m(\mathbb{R}^n) \oplus P_{m-2}(\mathbb{R}^n)$. The left-hand side is

$$\begin{aligned} P_m(\mathbb{R}^n)|_{\mathrm{SO}(n-1)} &\cong H_m(\mathbb{R}^n)|_{\mathrm{SO}(n-1)} \oplus P_{m-2}(\mathbb{R}^n)|_{\mathrm{SO}(n-1)} \\ &\cong H_m(\mathbb{R}^n)|_{\mathrm{SO}(n-1)} \oplus [P_{m-2}(\mathbb{R}^{n-1}) \oplus P_{m-3}(\mathbb{R}^{n-1}) \oplus \cdots \oplus P_0(\mathbb{R}^{n-1})]. \end{aligned}$$

The right-hand side is

$$H_m(\mathbb{R}^{n-1}) \oplus \cdots \oplus H_0(\mathbb{R}^{n-1}) \oplus [P_{m-2}(\mathbb{R}^{n-1}) \oplus \cdots \oplus P_0(\mathbb{R}^{n-1})].$$

By the lemma, we may cancel the parts in brackets.

How many copies of the trivial representation occur in $H_m(\mathbb{R}^n)|_{\mathrm{SO}(n-1)}$? From the proposition combined with the lemma, the answer is exactly 1, again for $n \geq 3$. We've essentially just computed one very simple case of the branching rules for $(\mathrm{SO}(n), \mathrm{SO}(n-1))$. We're finally able to prove that $H_m(\mathbb{R}^n)$ is irreducible under $\mathrm{SO}(n)$ if $n \geq 3$. If not, $H_m(\mathbb{R}^n) = V_1 \oplus V_2$ for V_1, V_2 non-trivial $\mathrm{SO}(n)$-invariant subspaces. Then $H_m(\mathbb{R}^n)|_{\mathrm{SO}(n-1)} = V_1|_{\mathrm{SO}(n-1)} \oplus V_2|_{\mathrm{SO}(n-1)}$.

261 Proposition

If $V \subset P_m(\mathbb{R}^n)$ is any $\mathrm{SO}(n)$-invariant subspace, then V contains a non-zero vector fixed under $\mathrm{SO}(n-1)$.

Proof The map $P_m(\mathbb{R}^n) \to C(S^{n-1})$ given by $p \mapsto p|_{S^{n-1}}$, where $C(S^{n-1})$ denotes the continuous functions from the sphere to $\mathbb{C}$, is injective. It suffices to show that if

$W \subset C(S^{n-1})$ is a finite dimensional non-zero $\mathrm{SO}(n)$-invariant subsace, then W contains a non-zero p invariant under $\mathrm{SO}(n-1)$. Here we're simply letting W be the image of the suggested map, $\{p|_{S^{n-1}} : p \in V\}$. Since $W \neq 0$, pick $0 \neq p \in W$, so $p(x_0) \neq 0$ for some $x_0 \in S^{n-1}$. Since $\mathrm{SO}(n)$ acts transitively, we can assume $x_0 = (1, 0, \ldots, 0)$. Now define

$$\widetilde{p}(x) := \int_{\mathrm{SO}(n-1)} p(g \cdot x)\, dg.$$

Now $\widetilde{p}(x)$ is $\mathrm{SO}(n-1)$-invariant and $\widetilde{p}(e_1) \neq 0$. Why is $\widetilde{p} \in W$? We can express $\widetilde{p}$ as a limit of sums in W, and since W is finite-dimensional, the limit also belongs to W.

From the proposition, each V_1 and V_2 contains a distinct trivial representation, contradicting the fact that $H_m(\mathbb{R}^n)|_{\mathrm{SO}(n-1)}$ has precisely 1 copy of the trivial representation. The proposition above then gives the full branching rule for $n \geq 3$ in this case.

November 24th, 2014: Draft

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262 Remark

We briefly summarize the previous lecture. $P_n(\mathbb{R}^n)$ denotes the space of homogeneous $\mathbb{C}$-polynomials on $\mathbb{R}^n$ of degree m, $H_m(\mathbb{R}^n)$ is the subset of $P_n(\mathbb{R}^n)$ annihilated by the Laplacian Δ. If $n \geq 3$, we showed $H_m(\mathbb{R}^n)$ is irreducible for $\mathrm{SO}(n)$. One can check, using results on the homework, that $H_m(\mathbb{R}^2)$ is irreducible for $O(2)$.

For $n = 3$, we have the double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$, which is a group homomorphism. Every representation of $\mathrm{SO}(3)$ then "lifts" to a representation of $\mathrm{SU}(2)$ simply by precomposing ρ with this covering map. Since this map is surjective, it follows that irreducible representations of $\mathrm{SO}(3)$ lift to irreducible representations of $\mathrm{SU}(2)$, which is in particular true of $H_m(\mathbb{R}^3)$. We know from earlier that $V_m(\mathbb{C}^2)$ is an irreducible representation of $\mathrm{SU}(2)$ of dimension $m+1$, and this gives all irreducible representations of $\mathrm{SU}(2)$ up to equivalence. From homework, $\dim H_m(\mathbb{C}^3) = 2m+1$. Hence it must be that $H_m(\mathbb{R}^3) \cong V_{2m}(\mathbb{C}^2)$ as $\mathrm{SU}(2)$-representations. It is a very good exercise to write an intertwining operator realizing this equivalence.

Minor note: all representations discussed in this note are complex representations, so the dimensions are over $\mathbb{C}$.

263 Proposition

The restriction to S^{n-1} of any polynomial on $\mathbb{R}^n$ agrees with the restriction of a harmonic polynomial.

Proof Recall $P_m(\mathbb{R}^n) = \oplus_{j=0}^{\lfloor m/2 \rfloor} |x|^{2j} H_{m-2j}(\mathbb{R}^n)$. Hence $P_m(\mathbb{R}^n)|_{S^{n-1}} = \oplus_{j=0}^{\lfloor m/2 \rfloor} H_{m-2j}(\mathbb{R}^n)|_{S^{n-1}}$. Hence the restriction of any homogeneous polynomial to S^{n-1} can be written as a sum of restrictions of harmonic polynomials. We may apply this to each homogeneous component of a given not necessarily homogeneous polynomial.

264 Proposition

The restriction to S^{n-1} of harmonic polynomials is dense in $C(S^{n-1})$ and hence in $L^2(S^{n-1})$.

Proof The Weierstrass approximation theorem says that on any compact subset K of $\mathbb{R}^n$, the restriction of polynomials to K is dense in $C(S^{n-1})$. It is a basic result that $C(S^{n-1})$ is dense in $L^2(S^{n-1})$ under the L^2-norm. Here $C(S^{n-1})$ uses the supremum norm, $||u|| = \sup |u(x)|$. The L^2 inner product for S^{n-1} is given by $\int_{S^{n-1}} u\overline{v}\, d\sigma$ where $d\sigma$ refers to the surface measure of the sphere S^{n-1}, which is $O(n)$-invariant.

265 Proposition

$H_m(\mathbb{R}^n)|_{S^{n-1}} \perp H_{m'}(\mathbb{R}^n)|_{S^{n-1}}$ if $m \neq m'$ using the L^2 inner product on S^{n-1}. Indeed,

$$L^2(S^{n-1}) = \overline{\oplus_{m=0}^{\infty} H_m(\mathbb{R}^n)|_{S^{n-1}}}$$

where the right-hand side uses the closure in the L^2-direct sum and the summands are orthogonal. When $n = 2$, this gives Fourier series.

Proof Observe that $P_m(\mathbb{R}^n) \to C^\infty(S^{n-1})$ given by $p \mapsto p|_{S^{n-1}}$ is one-to-one. (This fails if p is not assumed homogeneous, eg. $1 - |x| \mapsto 0$.) Hence $H_m(\mathbb{R}^n) \to C^\infty(S^{n-1})$ is one-to-one, so is an isomorphism onto its image, $H_m(\mathbb{R}^m)|_{S^{n-1}}$. $O(n)$ acts on $C^\infty(S^{n-1})$ by $(g \cdot u)(s) := u(g^{-1}s)$. One checks $H_m(\mathbb{R}^n)|_{S^{n-1}}$ is $O(n)$-invariant.

Recall that if V is a finite dimensional $O(n)$ representation and V_1, V_2 are irreducible with $V_1 \not\cong V_2$, then $V_1 \perp V_2$ relative to any invariant inner product. (This was also used last lecture.) Recall that $H_m(\mathbb{R}^n) \not\cong H_{m'}(\mathbb{R}^n)$ if $m \neq m'$. It follows that $H_m(\mathbb{R}^n)|_{S^{n-1}}$ is inequivalent to $H_{m'}(\mathbb{R}^n)|_{S^{n-1}}$. Combining these two facts, the first statement follows. The second follows from the preceding density proposition.

266 Remark

Elements of $H_m(\mathbb{R}^n)$ are sometimes called $\boxed{\text{solid spherical harmonics}}$. Elements of $H_m(\mathbb{R}^n)|_{S^{n-1}}$ are sometimes called $\boxed{\text{surface spherical harmonics}}$. Indeed, $\dim H_m(\mathbb{R}^n)|_{S^{n-1}} = 2m+1$ and for $n = 3$ there is a basis for this space in terms of $\boxed{\text{Legendre functions}}$ which are roughly of the form $e^{\pm ik\theta}P(\cos\phi)$ where ϕ is the polar angle and θ is the angle in the xy-plane.

Last time, we showed that for all m, $H_m(\mathbb{R}^n)$ contains a unique (up to scale) SO$(n-1)$-invariant polynomial. Such a polynomial is called a $\boxed{\text{zonal harmonic}}$. It is a function of $\cos(\phi)$, something of the form $P(\cos\phi)$.

267 Proposition

Let $\boxed{\Delta_S}$ denote the Laplacian on S^{n-1}. (When $n = 2$, this is simply $\partial^2/\partial\theta^2$.) If $u \in H_m(\mathbb{R}^n)|_{S^{n-1}}$, then $\Delta_S u = m(2 - m - n)u$. Hence the preceding proposition more or less gives an orthogonal eigendecomposition of $L^2(S^{n-1})$ for Δ_S.

Proof We first write Δ on $\mathbb{R}^n$ in polar coordinates. This happens to give $\Delta = \partial_r^2 + (n-1)/r\partial_r + 1/r^2\partial_S$. One way to think of Δ_S is then the angular part of Δ. Now let $p \in H_m(\mathbb{R}^n)$. Let $x = r\sigma$ for $|\sigma| = 1$. Now $p(x) = |x|^m p(x/|x|) = r^m p(\sigma)$, so we can just let $u = p|_{S^{n-1}}$ be given by $u(\sigma) := p(\sigma)$. Now $\partial_r p = mr^{m-1}u$, $\partial_r^2 p = m(m-1)r^{m-2}u$. Then

$$0 = \Delta p = [m(m-1)u + m(n-1)u + \Delta_S u]\frac{1}{r^2} = m(m+n-2)u + \Delta_S u.$$

The result follows.

Definition 268. We next discuss functions on a compact Lie group G. Let (ρ, V) be a finite dimensional representation of G, i.e. $\rho\colon G \to \mathrm{GL}(V)$. As usual, if we fix a basis for V, say $v_1, \ldots, v_d$ where $d := \dim V$, we can write $\rho(g) \in \mathrm{GL}(V)$ as a matrix with respect to this basis, say $M(g) \in \mathrm{GL}(d, \mathbb{C})$. We may write d_ρ for the dimension of ρ, that is, the dimension of V.

Let $\boxed{\mathcal{MC}_\rho} := \mathrm{Span}_{\mathbb{C}}\{M(g)_{ij} : 1 \leq i, j \leq d\} \subset C^\infty(G)$ denote the $\boxed{\text{matrix coefficients}}$ of the representation ρ.

269 Proposition

$\mathcal{MC}_\rho$ is basis independent. Indeed, if $(\rho, V) \cong (\rho', V')$, then $\mathcal{MC}_\rho = \mathcal{MC}_{\rho'}$.

Proof If we have a new basis $v_1', \ldots, v_d'$, let S be the change of basis matrix so that $M'(g) = S^{-1}M(g)S$, that is,

$$M'(g)_{ij} = (S^{-1})_{ik}M(g)_{kl}S_{lj} = (S^{-1})_{ik}S_{lj}M(g)_{kl},$$

where we implicitly sum over k, l. But then the coefficients $(S^{-1})_{ik}S_{lj}$ are just in $\mathbb{C}$. Basis independence follows. For the second statement, there is an intertwining operator $T\colon V \to V'$, so that $\rho'(g) \circ T = T \circ \rho(g)$, i.e. $\rho'(g) = T \circ \rho(g) \circ T^{-1}$, or in basis form $M'(g) = SM(g)S^{-1}$ for a change of basis matrix S.

Definition 270. Let $\boxed{\mathcal{MC}} \subset C^\infty(G)$ denote the internal sum of all finite dimensional representations ρ of G of $\mathcal{MC}_\rho$ inside $C^\infty(G)$. We will show next time this is simply $\oplus_{[\rho]\in\widehat{G}}\mathcal{MC}_\rho$. Here $L^2(G)$ is constructed using the Haar measure.

271 Theorem (Peter-Weyl)
$\mathcal{MC}$ is dense in $C(G)$ and $L^2(G)$.

November 26th, 2014: Draft

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272 Remark
To be added.

December 1st, 2014: Draft

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273 Remark
The goal of this week is to prove the Peter-Weyl theorem. It's a cornerstone of representation theory of compact Lie groups. We'll spend the next couple of lectures building up some analytical results needed in our proof. We'll build up convolutions, left and right regular representations, and the spectral theorem for compact self-adjoint operators on a Hilbert space.

274 Remark
Let G be a compact Lie group, $\rho\colon G \to \mathrm{GL}(V)$ a unitary representation. Recall $\mathcal{MC}_\rho$ is defined to be the $\mathbb{C}$-linear span of $M(g)_{ij}$ where $1 \leq i, j \leq d\rho$ where $M(g)$ is the matrix of $\rho(g)$ in a fixed orthonormal basis for V. Recall that $\mathcal{MC}_\rho$ is independent of which basis we pick. Last time we showed that $\mathcal{MC} = \oplus_{[\rho]\in\widehat{G}}\mathcal{MC}_\rho \subset C^\infty(G)$. We also shows the Schur orthogonality relations, namely that

$$\{\sqrt{d\rho}M_\rho(g)_{ij} : [\rho] \in \widehat{G}, 1 \leq i, j \leq d\rho\}$$

is an orthonormal set in $L^2(G)$.

Recall the statement of the Peter-Weyl theorem. It has two parts, (1) $\mathcal{MC}$ is dense in $C(G)$, or equivalently every element of $C(G)$ can be uniformly approximated by elements in $\mathcal{MC}$; (2) $\mathcal{MC}$ is dense in $L^2(G)$. We'll see shortly that $C(G)$ is dense in $L^2(G)$. Recall that, on a compact group, $C(G) \subset L^1(G) \subset L^1(G)$ and $||u||_1 \leq ||u||_2 \leq ||u||_\infty$. Hence (2) will follow immediately from (1).

A corollary of the following characterization of when an orthonormal set is actually an orthonormal basis is that the set appearing in the Schur orthogonality relations is an orthonormal basis for $L^2(G)$.

275 Proposition
Let $\mathcal{S} = \{u_\alpha\}_{\alpha\in I}$ be an orthonormal set in a Hilbert space $\mathcal{H}$. The following are equivalent:

(1) $\mathcal{S}$ is an orthonormal basis.

(2) $\mathcal{S}$ is a maximal orthonormal set, meaning $u \in \mathcal{H}$ and $\langle u, u_\alpha\rangle = 0$ for all α implies $u = 0$.

(3) For all $u \in \mathcal{H}$, $||u||^2 = \sum_{\alpha\in I} |\langle u, u_\alpha\rangle|^2$.

(4) $\mathrm{Span}_{\mathbb{C}}\{u_\alpha : \alpha \in I\}$ *is dense in* $\mathcal{H}$*.*

276 Remark

The Peter-Weyl theorem (part (1)) is a consequence of the Stone-Weierstrass theorem if $G \subset \mathrm{GL}(n, \mathbb{C})$. That theorem says:

277 Theorem (Stone-Weierstrass)

Let X be a compact Hausdorff space. Let $A \subset C(X)$ be a subalgebra of $C(X)$ which contains constants, is closed under complex conjugation, and which separates points. Then A is dense in $C(X)$.

278 Remark

The classical Weierstrass approximation theorem is the special case where A consists of polynomial functions and $X = \mathbb{R}$, say. Recall that a subset A of $C(X)$ "separates points" if for all $x \neq y \in X$, there is some $f \in A$ such that $f(x) \neq f(y)$.

If $X = G$ is a compact Lie group and A is $\mathcal{MC}$, every hypothesis is satisfied immediately except that it is not clear whether or not $\mathcal{MC}$ separates points. If $G \subset \mathrm{GL}(n, \mathbb{C})$ then $\mathcal{MC}$ clearly separates points of G. Indeed, if $g \in \mathrm{GL}(n, \mathbb{C})$, each entry of g is in $\mathcal{MC}_\rho$ where ρ is the standard representation. One of the first consequences of the Peter-Weyl theorem will be that every compact Lie group is isomorphic to a subgroup of a general linear group, from which it follows that they separate points. This fails for non-compact groups.

Definition 279. Let G be a topological group. Suppose $u\colon G \to \mathbb{C}$. We call u $\boxed{\text{uniformly continuous}}$ if for all $\epsilon > 0$ there exists a neighborhood U of $e \in G$ so that $|u(gh) - u(g)| \leq \epsilon$ for all $g \in G$ and $h \in U$.

280 Remark

We're right-multiplying by h, so this is really "right" uniform continuity. If G is compact, right and left uniform continuity are equivalent, which is also equivalent to uniform continuity with respect to any metric defining the topology. In particular, a continuous function on a compact Lie group is uniformly continuous. Another way to state the key requirement for uniform continuity above is to say that $g_2^{-1} g_1 \in U$ implies $|u(g_1) - u(g_2)| < \epsilon$.

Definition 281. Let f, K be functions on a compact Lie group G. We'll define their $\boxed{\text{convolution}}$ $\boxed{f \star K}$ as

$$(f \star K)(g) := \int_G f(gh^{-1}) K(h)\, dh.$$

By the translation invariance of Haar measure, replacing h with $h^{-1}g$ this is

$$\int_G f(h) K(h^{-1}g)\, dh.$$

282 Remark

If g is commutative, convolution is too, though in general this need not be the case. Here we are assuming $f, K \in L^1(G)$. (It happens that the convolution of L^1 functions is L^1, though we won't need that.) For us, f will be in $L^1(G)$ (actually $L^2(G)$) and K will be in $C(G)$ (actually in $C^\infty(G)$).

Convolutions are used classically to construct approximate identities. On $\mathbb{R}^n$, suppose $\phi \in C_c(\mathbb{R}^n)$, $\int \phi = 1$, $\phi \geq 0$, and $\mathrm{Supp}\, \phi \subset B$. Then $\phi_\epsilon(x) = \epsilon^{-n} \phi(x/\epsilon)$ has $\mathrm{Supp}\, \phi_e \subset B_\epsilon$. For the group version of this, let $0 < \epsilon < 1$, choose $K_\epsilon \in C(G)$, suppose U_ϵ is a neighborhood of $e \in G$ with $\mathrm{Supp}\, K_\epsilon \subset U_\epsilon$, $K_\epsilon \geq 0$, $\int_G K_\epsilon(g)\, dg = 1$. Choose these so that $U_{\epsilon_1} \subset U_{\epsilon_2}$ if $\epsilon_1 < \epsilon_2$ and $\cap_{\epsilon > 0} U_\epsilon = \{e\}$. Then if $f \in C(G)$, $f \star K_\epsilon \to f$ uniformly as $\epsilon \to 0$. The proof is the same as in $\mathbb{R}^n$.

283 Proposition

Here we collect some analytic facts.

1. $C(G)$ *is dense in* $L^2(G)$*.*

2. *If $f \in L^1(G)$ and $K \in C(G)$, then $f \star K \in C(G)$. Indeed, $||f \star K||_\infty \leq ||f||_1 ||K||_\infty$.*

3. *$f \star K_\epsilon \to f$ uniformly as $\epsilon \to 0$ where K_ϵ is as in the preceding remark.*

Proof (1) One proof uses uniqueness in the Riesz representation theorem. In particular, suppose not. Then $\overline{C(G)}$ is a proper closed subspace, so there exists $0 \neq f \in L^2(G)$ such that $f \perp C(G)$, i.e. $\int_G f(g)u(g)\,dg = 0$ for all $u \in C(G)$. Now $f(g)\,dg$ is a complex Borel measure and the linear functional $u \mapsto \int_G u \cdot f\,dg$ is a continuous linear functional on $C(G)$. But the Riesz representation theorem says that every continouous linear fucntional on $C(G)$ has a unique representation as integration agains a measure, so that $f\,dg$ is the zero measure, so $f = 0$ almost everywhere, so $f = 0 \in L^2(G)$, a contradiction.

(2) We compute

$$\begin{aligned}
|(f \star K)(g)| &= |\int_G f(gh^{-1})K(h)\,dh| \\
&\leq \sup |K| \int_G |f(gh^{-1})|\,dh \\
&= ||K||_\infty ||f||_1.
\end{aligned}$$

For continuity, we compute

$$\begin{aligned}
|(f \star K)(g_1) - (f \star K)(g_2)| &= |\int_G f(h)(K(h^{-1}g_1) - K(h^{-1}g_2))dh| \\
&\leq \sum_{h \in G} |K(h^{-1}g_1) - K(h-1g_2)| ||f||_1.
\end{aligned}$$

Since K is continuous, by the above remark it is uniformly continuous, from which it now follows that $f \star K$ is (uniformly) continuous.

284 Remark
Proof idea of Peter-Weyl theorem (part 1): to prove $\mathcal{MC}$ is dense in $C(G)$, we'll choose $K_\epsilon \in C^\infty(G)$ satisfying the suggested properties, so that $f \star K_\epsilon$ is "close" to f. We'll then show $f \star K_\epsilon$ can be approximated uniformly by $\mathcal{MC}$. For that, we'll use the fact that $f \mapsto f \star K_\epsilon$ for each ϵ is compact on L^2. We'll then use the spectral theorem to decompose this operator via eigenfunctions of this convolution operator and the eigenfunctions will be matrix coefficients.

December 3rd, 2014: Draft

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285 Remark
There are two more ingredients before we can discuss the proof of the Peter-Weyl theorem. First up: left and right regular representations. We'll let the group act by translation on functions on the group; we'll pick L^2 functions since that's a nice Hilbert space equipped with an inner product.

Definition 286. Let G be a compact Lie group. For $g \in G$, define $L_g, R_g \colon L^2(G) \to L^2(G)$ by $(L_g f)(h) := f(g^{-1}h)$ and $(R_g f)(h) := f(hg)$. By the invariance of Haar measure, $||L_g f||_{L^2} = ||R_g f||_{L^2} = ||f||_{L^2}$. Now define $\boxed{\rho_L, \rho_R}\colon G \to U(L^2(G))$ by $\rho_L(g)f := L_g f$, $\rho_R(g)f := R_g f$. One may quickly check ρ_L, ρ_R are group homomorphisms. They are representations of G on $V := L^2(G)$. Note that V is typically infinite-dimensional here, in stark contrast to all of our previous representations.

287 Remark
Convolution operators commute with translations. For fixed K, consider the operator $T_K f := f \star K$. It's a fact that if $f \in L^2(G)$, $K \in L^1(G)$, then $f \star K \in L^2(G)$ and T_K is a bounded operator. Claim:

$T_K \circ L_{g_1} = L_{g_1} \circ T_K$. This just follows from the integral definition,

$$\begin{aligned}(f \star K)(g) &= \int_G f(gh^{-1})K(h)\,dh \\ &= \int_G f(h)K(h^{-1}g)\,dh \\ T_k(L_{g_1}f)(g) &= \int_G f(g_1^{-1}gh^{-1})K(h)\,dh \\ &= L_{g_1}(T_K f)(g).\end{aligned}$$

Note that T_K does not commute with right translations. There is still symmetry since we chose to convolve by K on the right when forming T_K. In summary, for all $K \in L^1(G)$, T_K is an intertwining operator for ρ_L.

288 Remark
The Peter-Weyl theorem again says $\mathcal{MC}$ is dense in $C(G)$, which again is our goal. Last time we said that if $f \in C(G)$ and $\epsilon > 0$, we can find $K \in C^\infty(G)$ such that $||f - f \star K||_\infty < \epsilon$. This time, we'll fix $K \in C^\infty(G)$ and approximate $f \star K = T_K f$ uniformly by $\mathcal{MC}$. Stringing these two operations together will give density of $\mathcal{MC}$ in $C(G)$.

We'll use the fact that T_K is a compact operator on $L^2(G)$ and some basic related facts, which we review now. Let $\mathcal{H}$ be a Hilbert space, B the closed unit ball. A linear transformation $T\colon \mathcal{H} \to \mathcal{H}$ is $\boxed{\text{bounded}}$ if $T(B)$ is bounded and is $\boxed{\text{compact}}$ if $\overline{T(B)}$ is a compact set. We write T^* for the Hilbert space adjoint and we call T self-adjoint if $T = T^*$.

289 Theorem (Spectral Theorem for Compact Self-Adjoint Operators)
Let T be a compact self-adjoint operator on a Hilbert space $\mathcal{H}$. If $\lambda \in \mathbb{R}$, let $E_\lambda := \{f \in \mathcal{H} : Tf = \lambda f\}$ be the eigenspace associated to λ. Then:

(1) $\{\lambda \in \mathbb{R} : E_\lambda \neq 0\}$ is finite or countable and $\lambda = 0$ is the only possible accumulation point.

(2) If $\lambda \neq 0$, then $\dim E_\lambda < \infty$.

(3) $\mathcal{H}$ has an orthonormal basis consisting of eigenvectors for T.

290 Example
If $\mathcal{H} = L^2(X, d\mu)$ and if $k \in L^2(X \times X)$, then $(Tf)(x) = \int_X k(x,y)f(y)\,d\mu(y)$ is a compact operator. This is called a $\boxed{\text{Hilbert-Schmidt operator}}$. The adjoint T^* uses $\overline{k(y,x)}$ as its kernel of integration. Hence T is self-adjoint if and only if $k(x,y) = \overline{k(y,x)}$ almost everywhere with respect to x, y.

For us, we'll use $X := G$, $d\mu := dg$, T_K with integral kernel $k(g,h) := K(h^{-1}g)$. We'll have $K \in C^\infty(G)$, though even if $K \in L^2(G)$ then this $k \in L^2(G \times G)$. T_K is self-adjoint if $K(h^{-1}g) = \overline{K(g^{-1}h)}$ (almost everywhere, though since K is smooth, everywhere). This is equivalent to saying $K(g) = \overline{K(g^{-1})}$. We'll call K $\boxed{\text{symmetric}}$ in this case. On $\mathbb{R}^n$, for instance, this is just saying the function is even. We can always arrange for this by replacing K with $\frac{1}{2}(K(g) + \overline{K(g^{-1})})$ and none of the important properties of K will be affected.

291 Proposition
Let $K \in C^\infty(G)$ be symmetric in the preceding sense. If $\lambda \neq 0$, then $E_\lambda \subset \mathcal{MC}$ where E_λ is the λ-eigenspace of the convolution operator T_K.

Proof First observe $E_\lambda \subset C^\infty(G)$, since convolving with a smooth function yields a smooth function, and K is smooth, i.e. $f = \frac{1}{\lambda} T_K f \in C^\infty$. Claim: for all $g \in G$, $\rho_L(g)\colon E_\lambda \to E_\lambda$. This is simply because T_K is an intertwining operator for ρ_L, i.e.

$$\lambda \rho_L(g) f = \rho_L(g) T_K f = T_K \rho_L(g) f.$$

Hence by the spectral theorem $\rho_L|_{E_\lambda}$ is a finite dimensional representation of G, and indeed is unitary since ρ_L is unitary on all of $L^2(G)$. Now choose a basis $f_1, \ldots, f_d$ for E_λ. Hence we have some expansion coefficients (matrix coefficients!) $M_{ji}(g)$ given by

$$\rho_L(g) f_i =: \sum_{j=1}^{d} M_{ji}(g) f_j.$$

If we evaluate this at h we get $f_i(g^{-1}h) = \sum_{j=1}^{d} M_{ji}(g) f_j(h)$. Let $h = e$ and replace g by g^{-1} to get $f_i(g) = \sum_{j=1}^{d} f_j(e) M_{ji}(g^{-1})$. Each $f_j(e) \in \mathbb{C}$, so we've expressed any $f_i(g)$ as a $\mathbb{C}$-linear combination of $M_{ji}(g^{-1})$. To deal with the inverses, we have

$$f_i(g) = \sum_{j=1}^{d} f_j(e)(M^{-1})_{ji}(g) = \sum_{j=1}^{d} f_j(e) M'_{ij}(g).$$

292 Theorem

$\mathcal{MC}$ is dense in $C(G)$.

Proof Given $f \in C(G)$ and $\epsilon > 0$, we can choose $K \in C^\infty(G)$ symmetric such that $||f - f \star K||_\infty \leq \epsilon$. We now want to find some $\psi \in \mathcal{MC}$ so that $||T_K f - \psi||_\infty \leq \epsilon$. Recall that $T_K \colon L^2(G) \to C(G)$ continuously and indeed that $||f \star K||_\infty \leq c||f||_{L^2(G)}$ (where $c = ||K||_\infty$), which will allow us to switch from the L^∞ norm to the L^2 norm. As before, T_K is a compact self adjoint operator on $L^2(G)$. let $\phi_1, \phi_2, \ldots$ be an orthonormal basis of eigenfunctions for T_K with non-zero eigenvalues $\lambda_1, \lambda_2, \ldots$ such that $L^2(G) = \overline{\mathrm{Span}\, \phi_i} \oplus \ker T_K$. We can write $f = \sum_{j=1}^\infty a_j \phi_j + \phi$ where $\phi \in \ker T_K$ and $\sum_{j=1}^\infty |a_j|^2 = ||f||_2 < \infty$. Choose N such that $\sum_{j>N} |a_j|^2 \leq (\epsilon/c)^2$ (where $c = ||K||_\infty$ as before). Now

$$\begin{aligned} T_K f &= T_K(\sum_{j=1}^{N} a_j \phi_j) + T_K(\sum_{j>N} a_j \phi_j) \\ &= \sum_{j=1}^{N} a_j \lambda_j \phi_j + T_K(\sum_{j>N} a_j \phi_j). \end{aligned}$$

Call $\psi := \sum_{j=1}^N a_j \lambda_j \phi_j$. From the proposition, $\psi \in \mathcal{MC}$. Hence

$$\begin{aligned} ||T_K f - \psi||_\infty &= ||T_K(\sum_{j>N} a_j \phi_j)||_\infty \leq c||\sum_{j>N} a_j \phi_j||_{L^2(G)} \\ &= c\sqrt{\sum_{j>N|a_j|^2}} \leq \epsilon. \end{aligned}$$

December 5th, 2014: Draft

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293 Remark

Today we'll get to see a couple of applications to the Peter-Weyl theorem. Recall (again) it says $\mathcal{MC}$ is dense in $C(G)$ and in $L^2(G)$. This is the last lecture of the quarter.

Definition 294. A representation (ρ, V) of a Lie group is $\boxed{\text{faithful}}$ if ρ is injective, i.e. $\ker \rho = 0$. Hence ρ is isomorphic onto its image, meaning G is isomorphic to a Lie subgroup of some $\mathrm{GL}(n, \mathbb{C})$. Conversely, if $G \subset \mathrm{GL}(n, \mathbb{C})$ is a subgroup, then the standard representation of G acting on $\mathbb{C}^n$ is faithful. A $\boxed{\text{linear group}}$ is by definition a subgroup of a general linear group $\mathrm{GL}(n, \mathbb{C})$. Hence G is isomorphic to a linear group if and only if G has a faithful representation.

295 Theorem

Every compact Lie group has a faithful finite-dimensional representation. In particular, it is isomorphic to a linear group.

296 Remark

This fails for general Lie groups. A simple example (more details in the first homework set next quarter) involves $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, the universal cover of $\mathrm{SL}(2,\mathbb{R})$. The maximal compact subgroup is $\mathrm{SL}(2,\mathbb{R}) \cap U(2) = \mathrm{SO}(2) \cong S^1$, so the fundamental group of $\mathrm{SL}(2,\mathbb{R})$ agrees with the fundamental group of S^1, namely is $\mathbb{Z}$, so $\widetilde{\mathrm{SL}(2,\mathbb{R})} \xrightarrow{\pi} \mathrm{SL}(2,\mathbb{R})$ is a $\mathbb{Z}$-fold covering. *Fact*: every finite dimensional representation of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ factors through π. So, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ has no faithful finite dimensional representation.

On the other hand, every real or complex Lie algebra has a faithful (i.e. injective) finite dimensional representation (Ado's Theorem). Hence it is isomorphic to a subalgebra of some $\mathfrak{gl}(n,\mathbb{C})$.

Proof The Peter-Weyl theorem implies that $\mathcal{MC}$ separates points of G, i.e. if $g_1 \neq g_2 \in G$, there exists $\psi \in \mathcal{MC}$ such that $\psi(g_1) \neq \psi(g_2)$. This implies there exists a representation (ρ, V) of G with $\rho(g_1) \neq \rho(g_2)$, since otherwise every matrix coefficient would send g_1 and g_2 to the same thing.

To construct our faithful representation ρ, we'll run an iterative process. Choose $e \neq g_1 \in G$ and a representation (ρ_1, V_1) with $\rho_1(g_1) \neq \rho_1(e) = \mathrm{id}_{V_1}$. If ρ_1 is injective, great, we're done. Otherwise, consider $\ker \rho_1$, which is a closed (normal) proper subgroup of G.

297 Fact

If G is a compact Lie group and $H \subset G$ is a closed proper subgroup, then either $\dim H < \dim G$ or H has fewer connected components than G.

Proof $\dim H \leq \dim G$ is clear. If equality holds, H contains a neighborhood of the identity, so H contains the identity component. G is the union of copies of the identity component, so H is a union of some subset of the components of G, which by assumption is proper.

From the fact, $\ker \rho_1$ either has smaller dimension or fewer components than G, so at each step of our iteration, at least one of these (finite) numbers will decrease. For the next step, we choose $e \neq g_2 \in \ker \rho_1$ and we choose a representation (ρ_2, V_2) such that $\rho_2(g_2) \neq \rho_2(e) = I_{V_2}$. Now replace $\ker \rho_1$ with $\ker \rho_2 \cap \ker \rho_1$, which again is a closed proper subgroup, and iterate. This process eventually terminates since the dimension will eventually decrease to zero (which requires a small argument, since the number of connected components may increase when the dimension drops). That is, $\ker \rho_1 \cap \cdots \cap \ker \rho_k = 0$. Hence $V_1 \oplus \cdots V_k$ with $\rho_1 \oplus \cdots \oplus \rho_k$ is faithful.

298 Remark

Recall that $\mathcal{MC}$ is dense in $L^2(G)$, so $\oplus_{[\rho]\in\widehat{G}} \mathcal{MC}_{[\rho]}$ is dense in $L^2(G)$. By the Schur orthogonality relations, $\mathcal{MC}_{[\rho_1]} \perp \mathcal{MC}_{[\rho_2]}$ for $[\rho_1] \neq [\rho_2]$ with respect to the L^2-norm. Hence

$$L^2(G) = \widehat{\bigoplus}_{[\rho]\in\widehat{G}} \mathcal{MC}_{[\rho]},$$

where $\widehat{\oplus}$ indicates we're using a Hilbert space direct sum, meaning the closure of the "algebraic" direct sum. That is, every element in $L^2(G)$ can be written as a convergent infinite sum using matrix coefficients. In our earlier notation, $\dim \mathcal{MC}_{[\rho]} = d\rho^2$, where $d\rho = \dim E_\rho$.

Since $L^2(G)$ is a separable Hilbert space, $L^2(G)$ is countable, and each summand in the preceding decomposition is finite dimensional, there must be (at most) countably many summands appearing in the preceding decomposition. That is, $\widehat{G}$ is (at most) countable. On the other hand, $G = \mathbb{R}$ has uncountably many irreducible representations (even unitary ones), namely $x \mapsto e^{itx}$ for all $t \in \mathbb{R}$.

299 Theorem

Recall $\rho_L, \rho_R \colon G \to U(L^2(V))$ were the left and right regular representations of the compact Lie group G. Let $[\rho] \in \widehat{G}$.

(1) $\mathcal{MC}_{[\rho]}$ is invariant under ρ_L and ρ_R.

(2) $\rho_L|_{\mathcal{MC}_{[\rho]}} \cong d\rho\, E'_\rho$ where E'_ρ is the dual representation of E_ρ.

(3) $\rho_R|_{\mathcal{MC}_{[\rho]}} \cong d\rho\, E_\rho$.

In particular, every E_ρ arises in ρ_R.

Proof (1) Choose a basis $v_1, \ldots, v_{d\rho}$ for E_ρ. Let $M(g)$ be the matrix of $\rho(g)$, so since ρ is a homomorphism, $M(gh) = M(g)M(h)$. Hence $M_{ij}(g^{-1}h) = \sum_{k=1}^{d\rho} M_{ik}(g^{-1})M_{kj}(h)$. But $M_{ij}(g^{-1}h) = (L_g M_{ij})(h)$, so $\rho_L(g)M_{ij} = \sum_{k=1}^{d\rho} M_{ik}(g^{-1})M_{kj} \in \mathcal{MC}_{[\rho]}$, giving invariance. Indeed, this argument shows that we may fix j and divide up $\mathcal{MC}_{[\rho]}$ into $d\rho$ invariant subspaces—see below.

Similarly, for ρ_R we have $(R_g M_{ij})(h) = M_{ij}(hg) = \sum_{k=1}^{d\rho} M_{ik}(h)M_{kj}(g)$, so $\rho_R(g)M_{ij} = \sum_{k=1}^{d\rho} M_{kj}(g)M_{ik}$.

The idea for (2) and (3), the idea is to view all $d \times d$ matrices as a representation space for $\mathrm{GL}(d)$ by left multiplication. This gives d copies of the corresponding columns. Representing $\mathrm{GL}(d)$ by right multiplication gives d copies of the rows, which are dual to the columns.

More precisely, using our basis above, by definition of matrix multiplication, $\rho(g)v_j = \sum_{k=1}^{d\rho} M_{kj}(g)v_k$. Compare this to the formula we obtained for ρ_R, namely

$$\rho_R(g)M_{ij} = \sum_{k=1}^{d\rho} M_{kj}(g)M_{ik}.$$

Fix $1 \leq i \leq d\rho$. We now see $\{M_{ij} : 1 \leq j \leq d\} \subset \mathcal{MC}_{[\rho]}$ is an *invariant* subspace of dimension d. Indeed, $M_{ik}(g) \mapsto v_k$ is an intertwining operator which yields (3). For (2), fix $1 \leq j \leq d\rho$ and consider $\{M_{ij} : 1 \leq i \leq d\}$. We have

$$\begin{aligned}\rho_L(g)M_{ij} &= \sum_{k=1}^{d\rho} M_{ik}(g^{-1})M_{kj} = \sum_{k=1}^{d\rho} (M^{-1})_{ik}(g)M_{kj}\\ &= \sum_{k=1}^{d\rho} (M')_{ki}(g)M_{ij}.\end{aligned}$$

Hence the map $M_{kj} \mapsto v_k$ is an intertwining operator involving the dual representation.

300 Remark

One can consider $L^2(G)$ as a representation of $G \times G$ by $\sigma(g_1, g_2)f := \rho_L(g_1)\rho_R(g_2)f$. Evaluated at h, this is just $f(g_1^{-1}hg_2)$. Doing so, the $\mathcal{MC}_\rho$'s are irreducible, so $L^2(G) = \widehat{\oplus}\mathcal{MC}_{[\rho]}$ is a decomposition into irreducibles of multiplicity 1. Indeed, $\mathcal{MC}_{[\rho]} \cong E'_\rho \otimes E_\rho$. It's all very cute.

List of Symbols

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