2.2 Contact Mechanics

2.2.1 Introduction

While Classical Mechanics deals solely with bulk material properties Contact Mechanics deals with bulk properties that consider surface and geometrical constraints. It is in the nature of many rheological tools to probe the materials from "outside". For instance, a probe in the form of a pin in a pin-on-disk tester is brought into contact with the material of interest, measuring properties such as hardness, wear rates, etc.

Geometrical effects on local elastic deformation properties have been considered as early as 1880 with the Hertzian Theory of Elastic Deformation. This theory relates the circular contact area of a sphere with a plane (or more general between two spheres) to the elastic deformation properties of the materials. In the theory any surface interactions such as near contact Van der Waals interactions, or contact Adhesive interactions are neglected.

An improvement over the Hertzian theory was provided by Johnson et al. (around 1970) with the JKR (Johnson, Kendall, Roberts) Theory. In the JKR-Theory the contact is considered to be adhesive. Hence the theory correlates the contact area to the elastic material properties plus the interfacial interaction strength. Due to the adhesive contact, contacts can be formed during the unloading cycle also in the negative loading (pulling) regime. Such as the Hertzian theory, the JKR solution is also restricted to elastic sphere-sphere contacts.

A more involved theory (the DMT theory) also considers Van der Waals interactions outside the elastic contact regime, which give rise to an additional load. The theory simplifies to Bradley's Van der Waals model if the two surfaces are separated and significantly apart. In Bradley's model any elastic material deformations due to the effect of attractive interaction forces are neglected. Bradley's non-contact model and the JKR contact model are very special limits explained by the Tabor coefficient.

Contact Mechanical Models:

- Hertz: fully elastic model,
- JKR: fully elastic model considering adhesion in the contact zone,
- Bradley purely Van der Waals model with rigid spheres,
- DMT fully elastic, adhesive and Van der Waals model.

2.2.2 Hertz's Elastic Theory of Contact

Hertz analyzed the stresses at the contact of two elastic solids.

Consider two bodies with radii of curvature \( R' \) and \( R'' \).

1) Before deformation the bodies touch at \( O \), and the separation of points \( A_1 \) and \( A_2 \) is

\[
h = |A_2 - A_1| = \frac{1}{2R'} x^2 + \frac{1}{2R''} y^2
\]

2) Applying a normal load, the two bodies are compressed. The separation of \( A_1 \) and \( A_2 \) is now

\[
h' = h - (S_1 + S_2) + (w_1 + w_2)
\]

\( w_1 \) and \( w_2 \) are the normal elastic displacements of the surface at \( A_1 \) and \( A_2 \).

\( S_1 \) and \( S_2 \) correspond to the displacement of the distant points of each body.

see Figure next page
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(i) If points $A_1$ and $A_2$ are within the contact area, $h' = 0$

\[ \Rightarrow \begin{cases} w_1 + w_2 = (S_1 + S_2) - h \\ \text{elastic displacement of bodies } A_1 \text{ and } A_2 \\ \text{separation for weight pair contact} \end{cases} \]

\[ = \begin{cases} w_1 + w_2 = S - \frac{x^2}{2R'} - \frac{y^2}{2R''} \\ S = S_1 + S_2 = w_1(0) + w_2(0) \end{cases} \]

(corresponds to the approach of two distant points)
(ii) If \( A_1 \) and \( A_2 \) are outside the contact area
\[ \Rightarrow h^2 > 0 \]
\[ \Rightarrow w_1 + w_2 > 5 - \frac{x^2}{2R^1} - \frac{x^2}{2R^2} \]

3) The resultant force transmitted from one surface to the other through the contact area \( S \) is resolved into:
- a normal load \( L \)
- tangential force components \( Q_x \) and \( Q_y \)

It is:
\[ L = \int_S p \, dS \; ; \; p \text{ normal traction (pressure)} \]
\[ Q_x = \int_S q_x \, dS \; ; \; q_x, q_x \text{ lateral traction} \]
\[ Q_y = \int_S q_y \, dS \; ; \]

Hertz made following assumptions:

1) strains are small, and within elastic limit \( \Rightarrow a \ll R \)

2) each solid can be considered an elastic half-space
\( \Rightarrow a \ll R \)

3) the surfaces are continuous and non-conforming
\( \text{conform} \quad \text{non-conform} \Rightarrow a \ll R \)

4) The surfaces are frictionless: \( Q_x = Q_y = 0 \)
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**Point Force on an Elastic Half-Space**

The elastic displacement \( w(r) \) at distance \( r \) is:

\[
w(r) = \frac{1 - \nu^2}{\pi E} \frac{L}{r}
\]

where \( E \) is Young's modulus and \( \nu \) is Poisson's ratio.

**Distributed Pressure**

Elemental force at \( B \):

\[
L = P(\xi, \eta) \, d\xi \, d\eta ; \quad P \text{ pressure}
\]

Surface (elastically deformed)

Top view of contact \( S \) (\( x-y \) plane):

Consider an elemental force at \( B \):

\[
L = P(\xi, \eta) \, d\xi
\]

The displacement due to \( L \) at any other point \( A(x, y) \) is:

\[
w(x, y) = \frac{1 - \nu^2}{\pi E} \iint_S \frac{P(\xi, \eta) \, d\xi \, d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} \quad (\star)
\]

\( S \) corresponds to \( \frac{1}{r} \) at \( (x, y) \).
we found before that inside the contact area
\[ w(x, y) = \frac{x^2}{2R_1} - \frac{y^2}{2R_2} \quad (*) \]

It follows from (*) and (**) that an ellipsoidal distribution of pressure would satisfy the equations:
\[ p(x, y) = p_0 \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}; \]

**Circular Point Contact**

\[ R_1 = R_2, \quad a = b \]

\[ p(r) = p_0 \sqrt{1 - \frac{r^2}{a^2}}; \quad r^2 = x^2 + y^2 \]

\[ w(r) = \frac{1 - \nu^2}{E} \frac{\pi p_0}{4a} (2a^2 - r^2) \]

using eq. (**)\n
\[ \Rightarrow \left[ \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right] \frac{\pi p_0}{4a} (2a^2 - r^2) = S - \frac{r^2}{2R} \]

to be true for all \( r \)

\[ \Rightarrow S = \frac{\pi a p_0}{2E^*} \quad \text{and} \quad \alpha = \frac{\pi p_0 R}{2E^*} \]
The equilibrium load $L = \int p(r) \, dS = \int p_0 \sqrt{1 - \frac{a^2}{r^2}} \, r \, dr = \frac{2}{3} p_0 \pi a^2$

$\Rightarrow$ Hertz Equations for Elastic Contact:

- radius of contact $a = \left[ \frac{3LR}{4E^*} \right]^{1/3}$
- mutual approach of dislocated points $S = \frac{a^2}{R} = \left[ \frac{9L^2}{16RE^{*2}} \right]^{1/3}$
- max. pressure $p_0 = \frac{3L}{2\pi a^2} = \frac{S}{2} P_m = \left[ \frac{6LE^{*2}}{\pi^3 R^2} \right]^{1/3}$
2.2.3 Adhesive Corrected Hertzian Theory: The JKR Theory

No attractive adhesion forces are considered in the Hertzian theory. Adhesion forces, as depicted below, generate "negative" loading forces.

Johnson, Kendall and Roberts considered adhesion within the contact regime in the Hertz model (→ Adhesive-elastic contact of non-conforming surfaces). The assumptions in the Hertz theory apply.

The normal elastic deformation is (as above)

\[ w_1 + w_2 = S - \frac{r^2}{2R} \quad (R_- = R_0, c = 0) \]

\[ \rho(r) = \rho_0 \left(1 - \frac{r^2}{a^2}\right)^{1/2}; \quad \rho_0 = \frac{2aE^*}{R} \]
Two components of energy have to be considered which define the total free energy, \( U_T \), of the system:

\[
U_T = U_E + U_s
\]

\( U_E \): stored elastic strain energy
\( U_s \): surface energy due to adhesive forces.

At equilibrium:
\[
\frac{\partial U_T}{\partial a} s = 0
\]

\[
\Rightarrow \frac{\partial U_E}{\partial a} s = -\frac{\partial U_s}{\partial a} s
\]

Case 1: Determination of \( U_E \) and \( \frac{\partial U_E}{\partial a} s \)

The general solution for the pressure distribution was found to be:

\[
P(r) = p_0 \left(1 - \frac{r^2}{a^2}\right)^{1/2} + p_0' \left(1 - \frac{r^2}{a^2}\right)^{-1/2}
\]

we neglected this term in the Hertzian theory

In the Hertzian theory a negative value of \( p_0' \) was rejected on the ground that tension could not be sustained for negative loads. In the presence of adhesive (attractive) forces, however, we cannot exclude the possibility of a negative \( p_0' \).
The elastic strain energy stored in the two bodies is calculated to:

\[ U_E = \frac{\pi^2 a^3}{E^*} \left( \frac{2}{15} \rho_0^2 + \frac{2}{3} \rho_0 \rho_0' + \rho_0'^2 \right) \]

The total energy compression is found to be:

\[ S = \left( \frac{\pi a}{2E^*} \right) (\rho_0 + 2\rho_0') \]

It follows for the variation, i.e., strain energy with the contact radius \( a \) (\( S \) is kept constant):

\[
\left[ \frac{\partial U_E}{\partial a} \right]_{S=\text{const}} = \frac{\pi^2 a^2}{E^*} \rho_0'^2
\]

b) Determination of \( U_S \) and \( \left[ \frac{\partial U_E}{\partial a} \right]_S \)

If \( y \) is the surface energy per unit area of each surface, the surface energy, \( U_S \), is:

\[ U_S = -2\pi a^2 y \]

\[ \left[ \frac{\partial U_S}{\partial a} \right]_S = -4\pi y a \]
Here it follows for equilibrium:

\[
\frac{11^2 \sigma^2}{2 \rho_0^2} \frac{d}{E^*} = 4 \pi \mu a
\]

(strain energy change)

adhesion energy change

\[\Rightarrow \rho_0' = -(4 \pi \mu E^*/\pi a)^{1/2}\]

(minus sign chosen)

The net contact force \( L \) is:

\[
L = \int_0^a 2\pi r \rho(r) \, dr = (\frac{2}{3} \rho_0 + 2 \rho_0') \pi a^2
\]

\[\Rightarrow \left( L - \frac{4E^*a^3}{3R} \right)^2 = 16 \pi \mu E^* a^3\]

or rewritten:

JKR Eq.

\[
a^3 = \frac{3R}{4E^*} \left( L + 3 \pi \mu R + \sqrt{6 \pi \mu R L + (3 \pi \mu R)^2} \right)
\]

Hertz

adhesive correction

or for the contact area \( A = \pi a^2 \)

\[
A = \pi (\frac{D R}{2})^{2/3} \left( L + 3 \pi \mu R + \sqrt{6 \pi \mu R L + (3 \pi \mu R)^2} \right)
\]

\[
D = \frac{3}{4} \frac{1}{E^*}; \quad E^* = \left[ \frac{(1-v_1^2)}{E_1} + \frac{(1-v_2^2)}{E_2} \right]^{-1}
\]

combined "plane stress modulus"
2.2.4 The Adhesive-Elastic Contact Formation

The adhesion force between two rigid spheres can be expressed as
\[
F_{\text{adh}} = -2\pi R^2 \Delta \gamma;
\]
where \( \Delta \gamma \) is called the "work of adhesion" per unit area. This corresponds to the Bradley model of adhesion. The elastic adhesion model (JKR) provides
\[
F_{\text{adh}}^{\text{JKR}} = -\frac{3}{2} \pi R^2 \Delta \gamma,
\]
which considers adhesion over the contact area, and an elastic response of the spheres. Considering that the JKR adhesion force equation is seemingly independent of any elastic modulus, there seems to be an inconsistency if compared to the Bradley model.

The apparent discrepancy was resolved by David Tabor (1977) who introduced the following parameter:
\[
\mu = \frac{(R^*)^{\frac{1}{3}} (\Delta \gamma)^{\frac{2}{3}}}{\sigma (E^*)^{\frac{1}{3}}},
\]
where \( E^* \) and \( R^* \) are the combined curvature and modulus, respectively, and \( \sigma \) the characteristic atom-atom distance. This coefficient, also called the Tabor Coefficient, determines whether or not the sphere may be treated as rigid.

\[
\text{Taber found that coefficient by analyzing the height of the neck in the JKR theory around the contact area:}
\]
\[
\text{JKR predicts } \frac{4E^*a_3^3}{3R} = L + 3\pi R \Delta \mu \frac{\sigma}{\sqrt{6\pi R^3 \Delta \mu}} + \sqrt{6\pi R^3 \Delta \mu} L + (3\pi R^3 \Delta \mu)
\]

\[
\text{the approach distance } S = \frac{a_3^3}{R}
\]

\[
\text{the contact becomes unstable at a critical contact of:}
\]
\[
a_3^3 = \left( \frac{9\Delta \mu R^2}{4E^*} \right)^{\frac{1}{3}} \quad \text{(see HW1)}
\]

\[
S_c = \frac{a_3^3}{R} \quad \text{"height of the neck around contact area}
\]

\[
S_c = \left( \frac{9}{4} \right)^{\frac{2}{3}} \Delta \mu \frac{2}{3} \cdot \frac{R^{\frac{1}{3}}}{E^*^{\frac{2}{3}}} \quad \text{(eq. 4)}
\]
Tabor compared the surface energy analysis in eq. with range of the surface forces provided by the atomic separation \( \delta \),

\[
\frac{S_c}{\delta} = \left( \frac{9}{4} \right)^{2/3} \Delta \mu^{2/3} \frac{R^{1/3}}{E^{* 2/3} \delta}
\]

and defined \( \mu \) as

\[
\mu = \frac{\Delta \mu^{2/3} R^{1/3}}{E^{* 2/3} \delta}
\]

Greenwood found numerically for a Lennard Jones potential, between elastic sphere contact:

(i) \( \mu = 0.005 \) (Bradley's equation, confirmed (Fig 16 Greenwood's paper))

(ii) \( \mu = 5 \) (JKR within \( \pm 2\% \) pull-off (jump of separation, 1%))

\[ \mu > 1 : \]

- As the surfaces are brought together the load follows the lower branch until point A.
- If the separation is increased for smaller \( \delta \) than at A the solution follows the upper curve as far as point C.
- An increase or decrease from A or C, respectively, will cause a jump to the other branch (to B or D respectively).

\[ \mu = 0.05 \quad (\rightarrow \text{Bradley}) \]

\[ \mu = 5 \quad (\rightarrow \text{JKR}) \]

• In an apparatus of finite stiffness (e.g., SFA, SFM) the jumps will be from A' to B' and from C' to D'.

For values of μ greater than about unity, the approach curves are S-shaped so that in a(a)

(i) fixed grips apparatus (e.g. indenter) there will be jumps in the load when the tangents to the curve are vertical

(ii) apparatus of finite stiffness jumps will occur even for lower values of μ.

The numerical solutions of Greenwood is based on the following equations:

(a) The pressure distribution \( p(r) \) is equal to \(-G(h)\) which is the tensile stress described by the Lenhard–Jones law as:

\[
G(h) = \frac{8 \Delta H}{3 \varepsilon} \left[ \left( \frac{\sigma}{h} \right)^3 - \left( \frac{\sigma}{h} \right)^9 \right]
\]

(integrated form of the 12-6 Potential for plane-plane geometry per unit area)

Attention: Do not confuse the tensile stress \( \sigma(h) \) with \( \sigma \) the LJ characteristic atom-atom distance.
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(b) The local deflection (elastic displacement) is given by:

\[ w(r) = \frac{1}{11} \frac{E^*}{r^3} \int \frac{P(x, y) \, dx \, dy}{\sqrt{(x-x_c)^2 + (y-y_c)^2}} \]

(c) Derjaguin approximation for the distance between sphere and plane:

\[ h = -S + G + \frac{r^2}{2R^*} + w(r) \]

- \(G\) atomic distance
- \(S\) displacement
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