8-4. Boundary Layer Separation on a Cylinder

Consider flow past a circular cylinder, as shown in Fig. 8-5. Assume that the pressure is given by the solution for irrotational flow.

(a) Use the results of Example 8.3-1 to determine $u(x)$, where $x = 0$ at the leading stagnation point.

(b) State the differential equation obtained for this geometry using the Kármán–Pohlhausen method.

(c) Use the equation from part (b) to compute $\delta(0)$.

(d) Solve the differential equation for $h(x)$ numerically and determine the location of the separation point. You should obtain an angle of 107°; the exact value for this idealized pressure field is reported to be 104.5° (Schlicting, 1968, p. 203). (*Hint:* The fourth-order Runge–Kutta method is satisfactory for the integration. To avoid the singularity at $x = 0$, start the integration at a small, but nonzero, value of x.)

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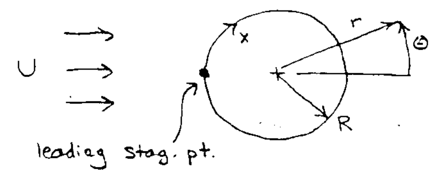


(a) Determine $u(x)$ from the results of Ex. 8.3-1, where $x=0$ is the leading stagnation point. From eq. (8.3-12),

$$u = -v_\theta(R,\theta) = 2U\sin\theta.$$

Note that changes in x are proportional to changes in θ, and that $x=0$ for $\theta=\pi$ and $x=\pi R$ for $\theta=0$. Thus,

$$x(\theta) = a + b\theta$$

$$x(0) = \pi R \Rightarrow a = \pi R, \quad x(\pi) = 0 \Rightarrow b = R.$$

$$x(\theta) = R(\pi - \theta), \quad \theta = \pi - \frac{x}{R}$$

$$\sin\theta = \sin\left(\pi - \frac{x}{R}\right) = \sin\frac{x}{R}$$

$$\boxed{u(x) = 2U\sin\frac{x}{R}}$$

(b) State the DE for the cylinder obtained using Kármán–Pohlhausen. From eq. (8.4-52), the DE is of the form

$$\frac{dh}{d\tilde{x}} = \frac{2}{\tilde{u}} f(\Lambda, \Omega)$$

Setting $L = R$ and using eqs (8.4-51),

$$\tilde{x} = \frac{x}{R}$$

$$h(\tilde{x}) = \left(\frac{\delta}{R}\right)^2 Re = \left(\frac{\delta}{R}\right)^2 \left(\frac{UR}{\nu}\right)$$

$$\tilde{u}(\tilde{x}) = \frac{u}{U} = 2\sin\tilde{x}$$

$$\Lambda(\tilde{x}) = h\frac{d\tilde{u}}{d\tilde{x}} = 2h\cos\tilde{x}$$

$$\Omega(\tilde{x}) = h^2\tilde{u}\frac{d^2\tilde{u}}{d\tilde{x}^2} = h^2(2\sin\tilde{x})(-2\sin\tilde{x}) = -4h^2\sin^2\tilde{x}$$

Substitute these expressions into eq. (8.4-52):

$$\frac{dh}{d\tilde{x}} = \frac{1}{\sin\tilde{x}} f(\Lambda, \Omega)$$

$$f(\Lambda,\Omega) = \frac{\left[2 + \left(\frac{1}{6} - 2I_1 - I_4\right)\Lambda - (2I_2 - I_5)\Lambda^2 + 2I_3\Lambda^3 - (I_2 - 2I_3\Lambda)\Omega\right]}{\left[I_1 + 3I_2\Lambda - 5I_3\Lambda^2\right]}$$

The constants I_n $(n = 1, 2, \ldots 5)$ are given by eqs. (8.4-46) - (8.4-50). To obtain a more workable form, relate Λ and Ω to h and combine some of the constants.

$$\left(\frac{1}{6} - 2I_1 - I_4\right)\Lambda = 2\left(\frac{1}{6} - 2I_1 - I_4\right)h\cos\tilde{x} \equiv A_1 h\cos\tilde{x}$$

$$(I_5 - 2I_2)\Lambda^2 = 4(I_5 - 2I_2)h^2\cos^2\tilde{x} \equiv A_2 h^2\cos^2\tilde{x}$$

$$2I_3\Lambda^3 = 16 I_3 h^3\cos^3\tilde{x} \equiv A_3 h^3\cos^3\tilde{x}$$

$$-I_2\Omega = -I_2(-4h^2\sin^2\tilde{x}) = 4I_2h^2\sin^2\tilde{x} \equiv B_1 h^2\sin^2\tilde{x}$$

$$2I_3\Lambda\Omega = 2I_3(2h\cos\tilde{x})(-4h^2\sin^2\tilde{x}) = -16I_3h^3\sin^2\tilde{x}\cos\tilde{x}$$

$$\equiv B_2 h^3\sin^2\tilde{x}\cos\tilde{x}$$

$$3I_2\Lambda = 3I_2(2h\cos\tilde{x}) = 6I_2h\cos\tilde{x} \equiv C_1 h\cos\tilde{x}$$

$$-5I_3\Lambda^2 = -5I_3(4h^2\cos^2\tilde{x}) = -20I_3h^2\cos^2\tilde{x} \equiv C_2h^2\cos^2\tilde{x}$$

The DE is now

$$\boxed{\frac{dh}{dx} = \frac{2 + A_1h\cos\tilde{x} + A_2h^2\cos^2\tilde{x} + A_3h^3\cos^3\tilde{x} + B_1h^2\sin^2\tilde{x} + B_2h^3\sin^2\tilde{x}\cos\tilde{x}}{\sin\tilde{x}\left(I_1 + C_1h\cos\tilde{x} + C_2h^2\cos^2\tilde{x}\right)}}$$

(c) Use the result of part (b) to compute $\delta(0)$.

For a symmetric object which is rounded at the leading stagnation point (e.g., a cylinder), $dh/dx = 0$ at $x = 0$. Thus, the numerator on the right-hand side of the DE must vanish at $x = 0$. Letting $h(0) = h_0$,

$$0 = 2 + A_1h_0 + A_2h_0^2 + A_3h_0^3$$

The physically significant root, calculated numerically, is

$$h_0 = 3.5262.$$

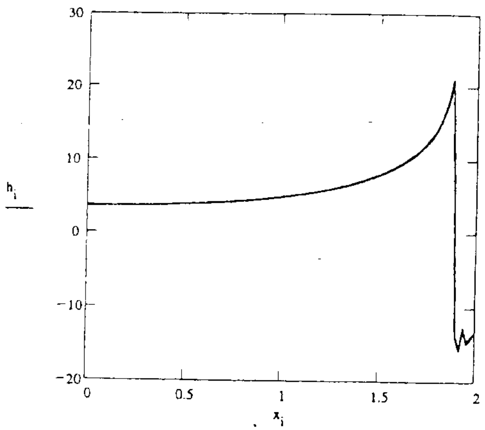
(d) Determine $h(x)$ numerically and find the separation pt.

Plots are shown on the following pages. The separation point $(\tilde{x}_s)$ is defined by $\Lambda(\tilde{x}_s) = -12$. It is found that

$$\tilde{x}_s = \frac{x_s}{R} = 1.873$$

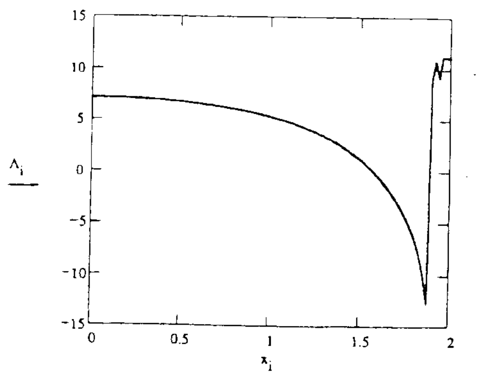
$$\text{Separation angle} = \left(\frac{1.873}{\pi}\right)(180°) = 107.3°$$

Dimensionless boundary layer thickness:

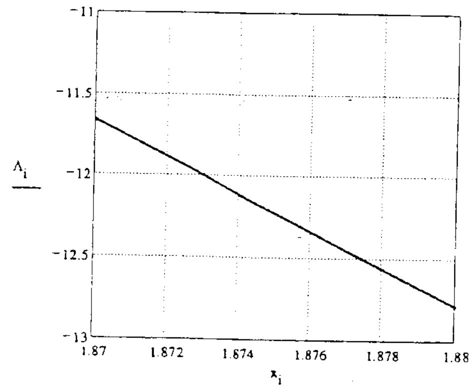


Note that the solution breaks down at about x = 1.9.

Shape factor:



Shape factor near the separation point:



Boundary layer separation (Λ_i = -12) occurs at x = 1.873.