8-7. Flow in a Convergent Channel (Hamel Flow)

Two-dimensional flow in channels bounded by nonparallel, planar walls was investigated by G. Hamel in 1916 [see Schlicting (1968, p. 102)]. We consider here the convergent case shown in Fig. P8-7(a), in which the flow is directed toward the narrow end of the channel, the fluid being withdrawn though a narrow slit at the intersection of the walls. The dynamic conditions are assumed to result in irrotational flow in the center of the channel and a boundary layer at each wall. This is one of the few problems in which analytical solutions can be obtained for both the irrotational and boundary layer regions.

(a) Show that in the irrotational region, where the flow is purely radial, the velocity is given by

$$v_r(r) = -\frac{q}{\alpha \pi r},$$

where q (>0) is the rate of fluid withdrawal per unit width. It is assumed here that the thickness of the boundary layers is negligible.

(b) Referring now to the boundary layer coordinates in Fig. P8-7(b), show that

$$\tilde{u}(\tilde{x}) = -\frac{1}{\tilde{x}}.$$

How must the velocity scale, U, be defined?

(c) Use the Falkner–Skan analysis to show that

$$f''' - (f')^2 + 1 = 0; \qquad f'(0) = 0, \quad f'(\infty) = 1, \quad f''(\infty) = 0,$$

where $f(\eta)$ is as defined in Eq. (8.4-24). Notice that $f(0) = 0$, which corresponds to the no-penetration condition, is replaced by $f''(\infty) = 0$ (see Example 8.4-3). Determine $g(\tilde{x})$.

(d) A first integration of the differential equation in part (c) is accomplished by multiplying by f'' and noting that $[(f'')^2]' = 2f''f'''$ and $[(f')^3]' = 3(f')^2 f''$. Show that

$$f'' = \sqrt{\frac{2}{3}}\,(1 - f')\sqrt{f' + 2}.$$

(e) Noticing that the differential equation in part (d) is separable, show that the final expression for the tangential velocity is

$$\frac{v_x}{u} = f'(\eta) = 3\tanh^2\left[\frac{\eta}{\sqrt{2}} + \tanh^{-1}\sqrt{\frac{2}{3}}\right] - 2$$

as given in Schlicting (1968, p. 153).

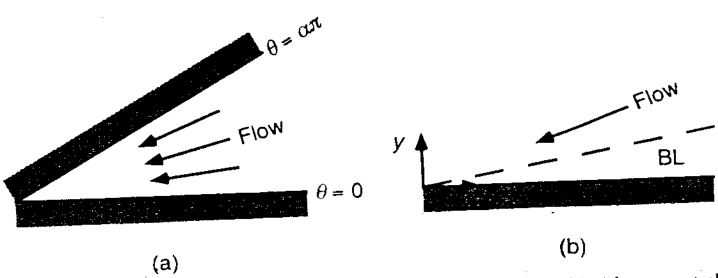

Figure P8-7. Flow in a converging channel. (a) Overall view; (b) enlargement showing boundary layer (BL) near one wall.

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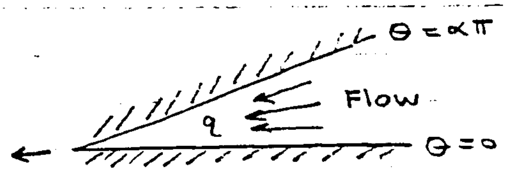

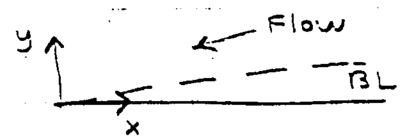

(a) Show that the velocity in the irrotational region is

$$v_r(r) = -\frac{q}{\alpha\pi r}, \quad v_\theta = 0, \quad -q = \text{flow rate per unit width}$$

Start with continuity (cylindrical coordinates):

check: $\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0$ with $v_r = \frac{-q}{\alpha\pi r}$ (satisfied)

$\quad\quad\quad 0 \quad\quad\quad\quad 0$

Check flow rate per unit width at radial position r:

$$\int_0^{\alpha\pi} v_r\, r\, d\theta = \int_0^{\alpha\pi}\left(-\frac{q}{\alpha\pi r}\right) r\, d\theta = -q \quad \text{(ok)}$$

$\uparrow$ $v_r = -q/\alpha\pi r$

Check conservation of momentum. For irrotational flow,

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0.$$

$$\frac{\partial\psi}{\partial\theta} = r v_r = -\frac{q}{\alpha\pi}, \quad \frac{\partial^2\psi}{\partial\theta^2} = 0$$

$$\frac{\partial\psi}{\partial r} = -v_\theta = 0$$

$\therefore \nabla^2\psi = 0$

Finally, $v_\theta = 0$ satisfies the no-penetration BC's at $\theta = 0$ and $\theta = \alpha\pi$. (The irrotational velocity field cannot be expected to satisfy no slip.)

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(b) For the BL on the lower wall, show that $\tilde{u}(\tilde{x}) = -1/\tilde{x}$.

Relate the BL and global coordinates: at $\Theta = 0$ (corresponding to $y = 0$), $x = r$. Thus

$$u(x) = v_r\big|_{\theta=0} \overset{\text{from (a)}}{=} -\frac{q}{\alpha\pi x}$$

If $\tilde{x} \equiv x/L$, where L is an arbitrary length, then

$$\tilde{u}(\tilde{x}) \equiv \frac{u}{U} = \left(-\frac{q}{\alpha\pi x}\right)\left(\frac{1}{U}\right) = -\left(\frac{q}{\alpha\pi U L}\right)\left(\frac{L}{x}\right) \equiv -\frac{1}{\tilde{x}} \quad \text{qed}$$

$$\therefore \quad U = \frac{q}{\alpha\pi L}$$

(c) Use the Falkner-Skan analysis to show that

$$\boxed{\begin{aligned} & f''' - (f')^2 + 1 = 0 \\ & f'(0) = 0, \ f'(\infty) = 1, \ f''(\infty) = 0. \end{aligned}}$$

From Ex. 8.4-2,

$$f(\eta) = \frac{\hat{\psi}(\tilde{x}, \hat{y})}{\tilde{u}(\tilde{x})\, g(\tilde{x})}, \quad \eta = \frac{\hat{y}}{g(\tilde{x})} \qquad (8.4-24)$$

$$f''' + ff'' \Big[\underbrace{g \frac{d}{d\tilde{x}}(\tilde{u} g)}_{C_1(\tilde{x})}\Big] + \underbrace{g^2 \frac{d\tilde{u}}{d\tilde{x}}}_{C_2(\tilde{x})} \left[1 - (f')^2\right] = 0 \qquad (8.4-25)$$

As with Falkner-Skan, assume that terms with $\tilde{x}$ must be constant (for a similarity solution to exist)

$$C_1 = g \frac{d}{d\tilde{x}}(\tilde{u} g) = \text{const.}$$

$$\Rightarrow$$

$$C_2 = g^2 \frac{d\tilde{u}}{d\tilde{x}} = \text{const.}$$

as $\tilde{u}(\tilde{x}) = -\frac{1}{\tilde{x}}$ (see (6))

$\Rightarrow$ $d\tilde{u}/d\tilde{x} = \tilde{x}^{-2}$, so

$$C_2 = \frac{g^2}{\tilde{x}^2} = \text{const.} \Rightarrow g \propto \tilde{x}.$$

Choosing $C_2 = 1$ gives

$$g(\tilde{x}) = \tilde{x}.$$

Also, since $\tilde{u} g = (-\tilde{x}^{-1})(\tilde{x}) = -1$, $C_1 = 0$. Thus, the DE for f reduces to that stated above,

$$\boxed{f''' - (f')^2 + 1 = 0.} \quad \textit{q.e.d.}$$

Check Boundary Condition

use: $\tilde{v}_x = \frac{\partial \hat{\psi}}{\partial \hat{y}} = f'$, $\hat{v}_y = -\frac{\partial \hat{\psi}}{\partial \tilde{x}} = g'(-\eta f' + f)$. (8.4-5)

Thus, the no-slip, no-penetration, and asymptotic matching conditions require that

no slip: no-penetration: outer edge:

$$f'(0) = 0, \quad f(0) = 0, \quad f'(\infty) = 1. \quad (8.4-13)$$

An alternative to the no-penetration condition $[f(0) = 0]$ is the requirement that $\partial \tilde{v}_x / \partial \hat{y} = 0$ at the outer edge of the BL ($\hat{y} \to \infty$). Thus,

$$\frac{\partial \tilde{v}_x}{\partial \hat{y}} = \frac{\partial}{\partial \hat{y}}(f') = \frac{d}{d\eta}(f') \frac{\partial \eta}{\partial \hat{y}} = \frac{f''}{g}$$

$$\frac{\partial \tilde{v}_x}{\partial \hat{y}}(\tilde{x}, \infty) = 0 \Rightarrow \boxed{f''(\infty) = 0}$$

$$\tilde{v}_x \equiv \frac{v_x}{u} = 0 \text{ for } x = 0 \Rightarrow \boxed{f''(0) = 0}$$

as $$\tilde{v}_x \equiv \frac{v_x}{u} = 1 \text{ for } x \to \infty \Rightarrow \boxed{f'(\infty) = 1}$$

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(d) Show that $\boxed{f'' = \sqrt{\frac{2}{3}}\,(1-f')\sqrt{f'+2}}$

Multiply the DE in part (c) by f'':

$$f''f''' - f''(f')^2 + f'' = 0$$

use these identities
$$\begin{cases} \left[(f'')^2\right]' = 2f''f''' \\ \left[(f')^3\right]' = 3(f')^2 f'' \end{cases}$$

$$\stackrel{DE}{\Longrightarrow} \quad \frac{1}{2}\left[(f'')^2\right]' - \frac{1}{3}\left[(f')^3\right]' + (f')' = 0 \quad \Big| \text{ integrate}$$

$$\frac{1}{2}(f'')^2 - \frac{1}{3}(f')^3 + f' = C = \text{const.}$$

Apply the BC's at $\eta = \infty$ to determine C:

$$\text{DE:} \quad \frac{1}{2}\Big[\underbrace{f''(\infty)}_{0}\Big]^2 - \frac{1}{3}\Big[\underbrace{f'(\infty)}_{1}\Big]^3 + \underbrace{f'(\infty)}_{1} = \underline{\frac{2}{3} = C}.$$

Thus,

$$(f'')^2 - \frac{2}{3}(f')^3 + 2f' + \frac{4}{3} = 0$$

$$(f'')^2 - \frac{2}{3}\Big[\underbrace{(f')^3 - 3f' + 2}_{= (f'-1)^2(f'+2)}\Big] = 0$$

(i.e., $(x-1)^2(x+2) = (x^2-2x+1)(x+2) = x^3 - 2x^2 + x + 2x^2 - 4x + 2$
$= x^3 - 3x + 2$)

$$\Longrightarrow \quad f'' = \pm\sqrt{\frac{2}{3}}\,\underbrace{(f'-1)}_{\leq 0}\,\underbrace{\sqrt{f'+2}}_{>0 \text{ because } 0 \leq f' \leq 1.}$$

Because $\partial v_x/\partial y \geq 0$, $f'' \geq 0$, $\therefore$ Choose the "minus" root.

q.e.d

$$f'' = \sqrt{\tfrac{2}{3}}(1-f')\sqrt{f'+2}$$

(e) The DE for $f'(\eta)$ is separable. Integrate to determine f':

$$\frac{df'}{d\eta} = \sqrt{\tfrac{2}{3}}\left(1-f'\right)\sqrt{f'+2} \implies d\eta = \sqrt{\tfrac{3}{2}}\,\frac{df}{(1-f')\sqrt{f'+2}}$$

$$\eta = \sqrt{\tfrac{3}{2}}\int_0^{f'}\frac{ds}{(1-s)\sqrt{s+2}} \qquad \left(\text{satisfies } f'(0)=0\right)$$

From an integral table,

$$\int\frac{dx}{x\sqrt{a+bx}} = -\frac{2}{\sqrt{a}}\tanh^{-1}\sqrt{\frac{a+bx}{a}}$$

Let

$$x = 1-s, \quad dx = -ds, \quad s+2 = 3-x, \quad a=3, \quad b=-1.$$

$$s=0 \leftrightarrow x=1, \quad s=f' \leftrightarrow x=1-f'$$

Then

$$\int_0^{f'}\frac{ds}{(1-s)\sqrt{s+2}} = \int_1^{1-f'}\frac{(-dx)}{x\sqrt{3-x}} = \frac{2}{\sqrt{3}}\tanh^{-1}\sqrt{\frac{3-x}{3}}\,\Bigg|_{x=1}^{x=1-f'}$$

$$\eta = \sqrt{2}\left[\tanh^{-1}\sqrt{\frac{2+f'}{3}} - \tanh^{-1}\sqrt{\frac{2}{3}}\right]$$

$$\tanh^{-1}\sqrt{\frac{2+f'}{3}} = \frac{\eta}{\sqrt{2}} + \tanh^{-1}\sqrt{\frac{2}{3}}$$

$$\sqrt{\frac{2+f'}{3}} = \tanh\left[\frac{\eta}{\sqrt{2}} + \tanh^{-1}\sqrt{\frac{2}{3}}\right]$$

$$\frac{2+f'}{3} = \tanh^2\left[\frac{\eta}{\sqrt{2}} + \tanh^{-1}\sqrt{\frac{2}{3}}\right]$$

$$\boxed{f'(\eta) = \frac{v_x}{u} = 3\tanh^2\left[\frac{\eta}{\sqrt{2}} + \tanh^{-1}\sqrt{\frac{2}{3}}\right] - 2}$$