INVISCID FLOW AND POTENTIAL FLOW

Streamlines

Let us describe the fluid flow by the flow velocity **v**=**v**(**x**,t). Following a fluid particle during the flow process a streamline is generated. At any particular time, t, the streamline has the same direction as **v**=**v**(**x**,t) (i.e., streamlines are tangent to **v**). Streamlines do not cross each other. Mathematically, streamlines can be described for two-dimensional flow by stream functions Ψ of constant value.

Gromeka-Lamb form of the Navier-Stokes equation

The Navier-Stokes equation

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{v} + \mathbf{g}$$

can be rewritten by using, $\Delta\mathbf{v} = \nabla(\nabla,\mathbf{v}) - \nabla\times(\nabla\times\mathbf{v})$, and the vorticity $\boldsymbol{\xi} = \nabla\times\mathbf{v}$, to

$$\frac{\partial\mathbf{v}}{\partial t} + \boldsymbol{\xi}\times\mathbf{v} = \mathbf{g} - \nabla\left(\frac{p}{\rho} + \frac{v^2}{2}\right) - \nu(\nabla\times\xi),$$

the Gromeka-Lamb equation. The Gromeka-Lamb equation is particularly suitable for working with curvilinear coordinates, or if something particular is known about the vorticity $\boldsymbol{\xi}$.

Special cases:

1. *Steady irrotational flow*: $$0 = \mathbf{g} - \nabla\left(\frac{p}{\rho} + \frac{v^2}{2}\right).$$

2. *Inviscid incompressible fluid flow*: $$\frac{\partial\mathbf{v}}{\partial t} + \boldsymbol{\xi}\times\mathbf{v} = \mathbf{g} - \nabla\left(\frac{p}{\rho} + \frac{v^2}{2}\right).$$

3. *Steady, inviscid and incompressible flow*:

$$\begin{aligned}\boldsymbol{\xi}\times\mathbf{v} &= \mathbf{g} - \nabla\left(\frac{p}{\rho} + \frac{v^2}{2}\right)\\ &= \nabla\left(\frac{p}{\rho} + \frac{v^2}{2} + \Phi_g\right)\end{aligned}$$

with $-\nabla\Phi_g = \mathbf{g}$. The equation for steady, inviscid and incompressible flow is also known as Lamb's equation.

From Lamb's equation follows that $\frac{p}{\rho} + \frac{v^2}{2} + \Phi_g =$ const. which corresponds to the Bernoulli‘s equation. It states that for a steady incompressible inviscid fluid, the quantity $\frac{p}{\rho} + \frac{v^2}{2} + \Phi_g$ is constant along the path of the fluid and along a vortex line. The vortex lines are tangent to $\boldsymbol{\xi}$. Streamlines and vortex lines must lie on the surface defined by $\boldsymbol{\xi}\times\mathbf{v}$ which is given by $\frac{p}{\rho}+\frac{v^2}{2}+\Phi_g$.

Stream function and velocity potential of 2D flow

Stream function Ψ and velocity potential Φ are defined for two-dimensional flow as:

$$v_x = -\frac{\partial\psi}{\partial y}; \quad v_y = \frac{\partial\psi}{\partial x}$$

$$v_x = -\frac{\partial\Phi}{\partial x}; \quad v_y = -\frac{\partial\Phi}{\partial y}$$

This equations provide the famous Cauchy Riemann equations:

$$\frac{\partial\Phi}{\partial x} = \frac{\partial\psi}{\partial y}; \quad \frac{\partial\Phi}{\partial y} = -\frac{\partial\psi}{\partial x}.$$

We can easily integrate these equations and obtain that Ψ = const. corresponds to the streamlines as mentioned above. The lines that are defined by ϕ=const. are called equipotential lines. Streamlines are everywhere perpendicular to the equipotential lines.

The governing equations (equations of change) for steady flow in two-dimension of an ideal fluid (ρ=const., μ=const.) is

continuity: $$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

motion: $$\frac{p}{\rho} + \frac{v_x^{\,2} + v_y^{\,2}}{2} + gz = \text{const}$$

Irrotational flow is mathematically described as $\boldsymbol{\xi}=\nabla\times\mathbf{v}=0$ which reduces for two-dimensional flow to:

$$\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} = 0.$$

The governing equations for steady two-dimensional potential flow (i.e., steady two-dimensional and irrotational flow of an ideal fluid) can be rewritten with the help of the definition of the stream function and the velocity potential. It is:

continuity: $\Delta\Phi = 0$, and
motion: $\Delta\psi = 0$

Complex Velocity Potential

With the velocity potential and the stream function, we can define a complex function

$$\Omega = \Phi + i\Psi \ ,$$

the complex velocity potential, which significantly simplifies stream visualizations, calculations and discussions. With Ω we can introduce a *complex velocity* d Ω /dz, which leads to

$$\frac{d\Omega}{dz} = v_x - iv_y \ .$$

Stagnation points in potential flow are found at zero complex velocity.

Elementary Flow Building Blocks

Complex flow profiles can be simulated with the linear superposition of elementary flow “building blocks”, such as given for uniform flow, line and point sources and sinks, vortices and doublets (see Table 12.1). A famous example is the potential flow around a cylinder, which can be simulated as the linear superposition of uniform flow and a doublet, i.e.,

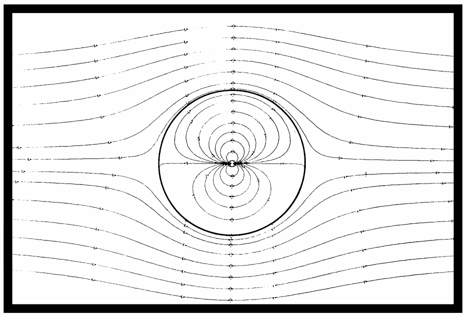
$$\Omega(z) = Uz + \frac{Ub^2}{z} \ .$$

One of the drawbacks of potential (irrotational) flow analysis is that it does not yield any acting drag (friction nor form drag) forces. In order to simulated drag, one has to “artificially” introduce a circulation term, i.e. a term that represents the *global* “swirl” of the fluid flow, while we still assume *local* irrotationality of the fluid particles. For our example of uniform potential flow around a cylinder, we would have to consider the following complex velocity potential:

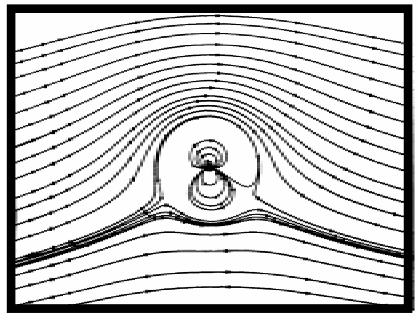
$$\Omega(z) = Uz + \frac{Ub^2}{z} + \frac{i\Gamma}{2\pi} ln(z) ,$$

with a third term, a vortex term, where Γ represents the potential circulation that depends on the set boundary condition.

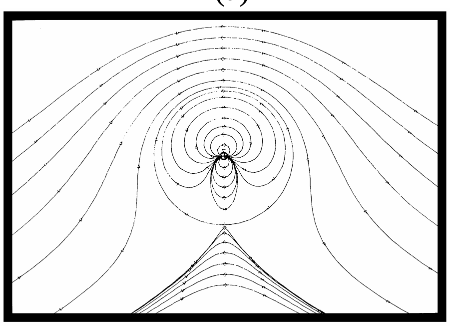
Potential Flow with Circulation Past a Cylinder



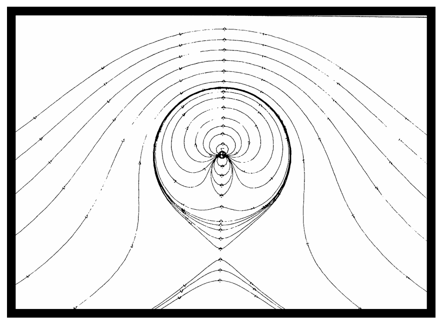
(a)



(b)



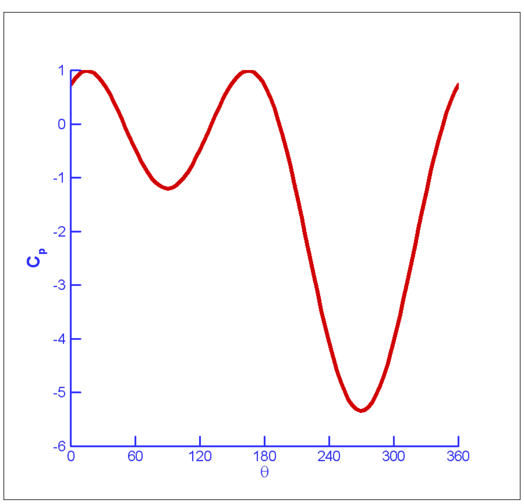
(c)

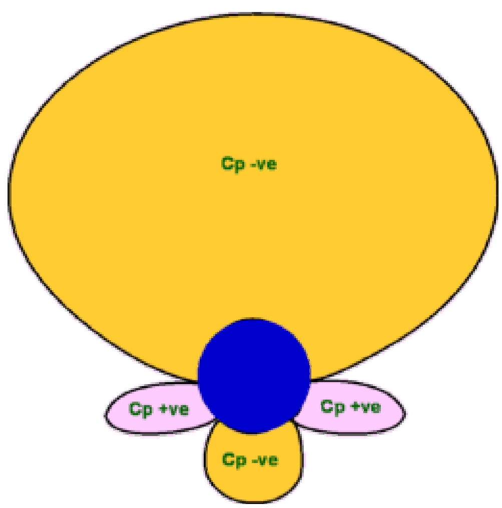


(d)

Source: United States Navel Academy Computing Facility.

(a) Flow about a cylinder with no circulation
(b) Flow net about a cylinder with circulation: $\Gamma/bU < 4\pi$
(c) “ $\Gamma/bU = 4\pi$
(d) “ $\Gamma/bU > 4\pi$





Source: http://instructional1.calstatela.edu/cwu/me408/PotentialFlow/PotentialFlow.htm