Bi-Directional Flow
Navier-Stokes Eq.

TABLE 6-12
Stream Function Equations

	Geometry	Velocity components	Form of Navier–Stokes equation[a]	Differential operators
planar flow	Cartesian (x, y)	$v_x = \frac{\partial \psi}{\partial y}$ $v_y = -\frac{\partial \psi}{\partial x}$	$\frac{\partial}{\partial t}(\nabla^2 \psi) - \frac{\partial(\psi, \nabla^2 \psi)}{\partial(x, y)} = \nu \nabla^4 \psi$	$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ $\nabla^4 = \nabla^2(\nabla^2)$
	Cylindrical (r, θ)	$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$ $v_\theta = -\frac{\partial \psi}{\partial r}$	$\frac{\partial}{\partial t}(\nabla^2 \psi) - \frac{1}{r}\frac{\partial(\psi, \nabla^2 \psi)}{\partial(r, \theta)} = \nu \nabla^4 \psi$	$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ $\nabla^4 = \nabla^2(\nabla^2)$
axi-symm. flow	Cylindrical (z, r)	$v_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$ $v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}$	$\frac{\partial}{\partial t}(E^2 \psi) + \frac{1}{r}\frac{\partial(\psi, E^2 \psi)}{\partial(z, r)} + \frac{2}{r^2}\frac{\partial \psi}{\partial z} E^2 \psi$ $= \nu E^4 \psi$	$E^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$ $E^4 = E^2(E^2)$
	Spherical (r, θ)	$v_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}$ $v_\theta = -\frac{1}{r \sin\theta}\frac{\partial \psi}{\partial r}$	$\frac{\partial}{\partial t}(E^2 \psi) - \frac{1}{r^2 \sin\theta}\frac{\partial(\psi, E^2 \psi)}{\partial(r, \theta)}$ $+ \frac{2E^2 \psi}{r^2 \sin^2\theta}\left(\frac{\partial \psi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial \psi}{\partial \theta}\sin\theta\right)$ $= \nu E^4 \psi$	$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\right)$ $E^4 = E^2(E^2)$

[a]The Jacobian determinants are given by

$$\frac{\partial(f, g)}{\partial(x, y)} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{vmatrix}.$$

Vorticity Eq : Eq. 6.8-7

in terms of vorticity: $\vec{w} \equiv \vec{\nabla} \times \vec{V} \Rightarrow \frac{D\vec{w}}{Dt} = \vec{w} \cdot \vec{\nabla}\vec{V} + \nu \nabla^2 \vec{w}$

Eqs 6.8
- for planar flow: $w_z = -\nabla^2 \psi$ for rect. (x, y) and cyl. (r, θ)
- for axisymm. flow: $w_\theta = -\frac{1}{r} E^2 \psi$ cyl. (z, r)
  $w_\phi = -\frac{1}{r \sin\theta} E^2 \psi$ sph. (r, θ)

for irrot. flow: $\vec{w} = 0$,

Laplace Eq.: $\nabla^2 \psi = 0$ for planar flow
$E^2 \psi = 0$ for axisymm. flow

$\Big\}$ applicable to steady and unsteady flow

(lacks to describe viscous boundary)

$\Rightarrow$ irrot. flow is pseudo steady

Navier Stokes Eq. for planar flow: $\frac{Dw_z}{Dt} = \nu \nabla^2 w_z$; $w_z = -\nabla^2 \psi$