

Coordinate Transformation Formula Sheet

Table with the Del operator in rectangular, cylindrical, and spherical coordinates			
<u>Operation</u>	<u>Cartesian coordinates (x,y,z)</u>	<u>Cylindrical coordinates (ρ,φ,z)</u>	<u>Spherical coordinates (r,θ, φ)</u>
Definition of coordinates	$\rho = \sqrt{x^2 + y^2}$ $\phi = \arctan(y/x)$ $z = z$	$x = \rho \cos \phi$ $y = \rho \sin \phi$ $z = z$	$x = r \sin \theta \cos \phi$ $y = r \sin \theta \sin \phi$ $z = r \cos \theta$
	$r = \sqrt{x^2 + y^2 + z^2}$ $\theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$ $\phi = \arctan(y/x)$	$r = \sqrt{\rho^2 + z^2}$ $\theta = \arctan(\rho/z)$ $\phi = \phi$	$\rho = r \sin(\theta)$ $\phi = \phi$ $z = r \cos(\theta)$
Definition of unit vectors	$\hat{\rho} = \cos \phi \hat{x} + \sin \phi \hat{y}$ $\hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}$ $\hat{z} = \hat{z}$	$\hat{x} = \cos \phi \hat{\rho} - \sin \phi \hat{\phi}$ $\hat{y} = \sin \phi \hat{\rho} + \cos \phi \hat{\phi}$ $\hat{z} = \hat{z}$	$\hat{x} = \sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi}$ $\hat{y} = \sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi}$ $\hat{z} = \cos \theta \hat{r} - \sin \theta \hat{\theta}$
	$\hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}$ $\hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z}$ $\hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}$	$\hat{r} = \sin \theta \hat{\rho} + \cos \theta \hat{z}$ $\hat{\theta} = \cos \theta \hat{\rho} - \sin \theta \hat{z}$ $\hat{\phi} = \hat{\phi}$	$\hat{\rho} = \sin \theta \hat{r} + \cos \theta \hat{\theta}$ $\hat{\phi} = \hat{\phi}$ $\hat{z} = \cos \theta \hat{r} - \sin \theta \hat{\theta}$

A vector field $\mathbf{A}$	$A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}}$	$A_\rho \hat{\boldsymbol{\rho}} + A_\phi \hat{\boldsymbol{\phi}} + A_z \hat{\mathbf{z}}$	$A_r \hat{\mathbf{r}} + A_\theta \hat{\boldsymbol{\theta}} + A_\phi \hat{\boldsymbol{\phi}}$
Gradient ∇f	$\frac{\partial f}{\partial x} \hat{\mathbf{x}} + \frac{\partial f}{\partial y} \hat{\mathbf{y}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}}$	$\frac{\partial f}{\partial \rho} \hat{\boldsymbol{\rho}} + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}}$	$\frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}}$
Divergence $\nabla \cdot \mathbf{A}$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho} \frac{\partial (\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2} \frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$
Curl $\nabla \times \mathbf{A}$	$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{\mathbf{z}}$	$\left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \hat{\boldsymbol{\phi}} + \frac{1}{\rho} \left(\frac{\partial (\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi} \right) \hat{\mathbf{z}}$	$\frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\partial A_\theta}{\partial \phi} \right) \hat{\mathbf{r}} + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} (r A_\phi) \right) \hat{\boldsymbol{\theta}} + \frac{1}{r} \left(\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right) \hat{\boldsymbol{\phi}}$
Laplace operator $\Delta f = \nabla^2 f$	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$
Vector Laplacian $\Delta \mathbf{A} = \nabla^2 \mathbf{A}$	$\Delta A_x \hat{\mathbf{x}} + \Delta A_y \hat{\mathbf{y}} + \Delta A_z \hat{\mathbf{z}}$	$\left(\Delta A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2} \frac{\partial A_\phi}{\partial \phi} \right) \hat{\boldsymbol{\rho}} + \left(\Delta A_\phi - \frac{A_\phi}{\rho^2} + \frac{2}{\rho^2} \frac{\partial A_\rho}{\partial \phi} \right) \hat{\boldsymbol{\phi}} + (\Delta A_z) \hat{\mathbf{z}}$	$\left(\Delta A_r - \frac{2A_r}{r^2} - \frac{2}{r^2 \sin \theta} \frac{\partial (A_\theta \sin \theta)}{\partial \theta} - \frac{2}{r^2 \sin \theta} \frac{\partial A_\phi}{\partial \phi} \right) \hat{\mathbf{r}} + \left(\Delta A_\theta - \frac{A_\theta}{r^2 \sin^2 \theta} + \frac{2}{r^2} \frac{\partial A_r}{\partial \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial A_\phi}{\partial \phi} \right) \hat{\boldsymbol{\theta}} + \left(\Delta A_\phi - \frac{A_\phi}{r^2 \sin^2 \theta} + \frac{2}{r^2 \sin \theta} \frac{\partial A_r}{\partial \phi} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial A_\theta}{\partial \phi} \right) \hat{\boldsymbol{\phi}}$
Differential displacement	$d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}$	$d\mathbf{l} = d\rho \hat{\boldsymbol{\rho}} + \rho d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}$	$d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}}$
Differential normal area	$d\mathbf{S} = dy dz \hat{\mathbf{x}} + dx dz \hat{\mathbf{y}} + dx dy \hat{\mathbf{z}}$	$d\mathbf{S} = \rho d\phi dz \hat{\boldsymbol{\rho}} + d\rho dz \hat{\boldsymbol{\phi}} + \rho d\rho d\phi \hat{\mathbf{z}}$	$d\mathbf{S} = r^2 \sin \theta d\theta d\phi \hat{\mathbf{r}} + r \sin \theta dr d\phi \hat{\boldsymbol{\theta}} + r dr d\theta \hat{\boldsymbol{\phi}}$

Differential volume	$dV = dx dy dz$	$dV = \rho d\rho d\phi dz$	$dV = r^2 \sin \theta dr d\theta d\phi$
Non-trivial calculation rules: 1. $\text{div grad } f = \nabla \cdot (\nabla f) = \nabla^2 f = \Delta f$ 2. $\text{curl grad } f = \nabla \times (\nabla f) = \mathbf{0}$ 3. $\text{div curl } \mathbf{A} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$ 4. $\text{curl curl } \mathbf{A} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ 5. $\Delta fg = f\Delta g + 2\nabla f \cdot \nabla g + g\Delta f$			

Vector Transformations in Rectangular, Cylindrical, and Spherical Coordinates

From rectangular to cylindrical:

$$\begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

From cylindrical to rectangular:

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix}$$

From rectangular to spherical:

$$\begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

From spherical to rectangular:

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix}$$

From cylindrical to spherical:

$$\begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin \theta & 0 & \cos \theta \\ \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix}$$

From spherical to cylindrical:

$$\begin{pmatrix} A_\rho \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \\ \cos \theta & -\sin \theta & 0 \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix}$$