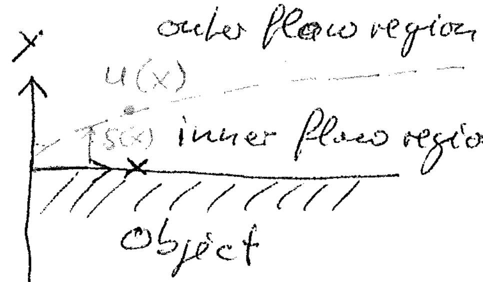
Boundary Layer

For flow around objects, we have to consider

- an outer potential flow region within which viscous boundary effects can be neglected, and
- an inner, boundary, flow region, within which the viscous boundary effects are most prominent.

The two flow regions have to be matched asympotically at their "interface", by introducing an "asymptotic" velocity $u(x)$ as

$$u(x) = \begin{cases} \lim\limits_{y \to 0} v_x^{\overset{\text{outer}}{o}}(x,y) \\ \lim\limits_{y \to \infty} v_x^{\overset{\text{inner}}{i}}(x,y) \end{cases}$$



A similar asymptotic expression is used for the asymptotic pressure at the "interface":

$$\mathbb{P}(x) = \lim_{y \to 0} \mathbb{P}^{o}(x,y) = \lim_{y \to \infty} \mathbb{P}^{i}(x,y)$$

Laminar Boundary Layer Theory of Newtonian Fluids

The x-momentum balance for a Newtonian fluid moving along an x-y surface, where $v_z = 0$ is

$$\underbrace{v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_y}{\partial y}}_{\text{advection term}} = \underbrace{-\frac{1}{\rho}\frac{d\mathcal{P}}{dx}}_{\text{pressure term}} + \underbrace{\nu \frac{\partial^2 v_x}{\partial y^2}}_{\text{viscous term}} \qquad \text{Eq. 9.2-17}$$

The flow has also to obey the continuity eq.

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad \text{Eq. 9.2-16}$$

We assumed $v_x \gg v_y \Rightarrow \mathcal{P} = \mathcal{P}(x)$ only

For $\delta(x) \ll L \Rightarrow v_z \ll v_x$ within BL and $v_x(x,y)$

Note: We do not assume $u(x) = U$, as we will later do for flat plates.

At the boundary (interface of inner and outer flow region), i.e., $y = \delta(x)$, the viscous and inertial terms (advection) have to balance, i.e., fulfill the order of magnitude comparison:

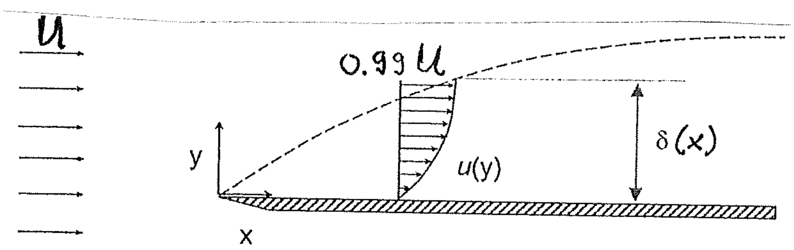
$$\frac{U^2}{L} \sim \frac{\nu U}{\delta^2}$$

which yields for the dimensionless scaling of boundary layer thickness:

Eq. 9.2-18 $$\boxed{\frac{\delta}{L} \propto \left(\frac{\nu}{UL}\right)^{\frac{1}{2}} = Re^{-1/2}}$$; used $Re = \frac{UL}{\nu}$ laminar regime

Boundary Layer Thickness: $\delta(x)$

Defined as the distance y from the surface at which $u(y)_x = 0.99\,U$, where $u(y)_x$ is the velocity of the stream within BL at x, and U is the outer stream velocity.



For a laminar BL

$$\boxed{\frac{\delta}{x} \cong \frac{5.0}{\sqrt{Re_x}} \; ; \; Re_x = \frac{Ux}{\nu}}$$

Laminar BL exist over a plate dimension with $L = x_{crit}$, whereby x_{crit} is defined via: $Re_{x_{crit}} \cong 5 \times 10^5$.

For a turbulent BL: $x > x_{crit}$ the BL thickness is given by

$$\boxed{\frac{\delta}{x} = \frac{0.385}{(Re_x)^{\frac{1}{5}}}}$$

We introduce the following dimensionless quantities:

$$\tilde{V}_x \equiv \frac{V_x}{U} \; ; \quad \hat{V}_y = \tilde{V}_y\, Re^{1/2} = \frac{V_y}{U}\, Re^{1/2}$$

$$\underbrace{\tilde{x} = \frac{x}{L}}_{\text{conventional}} \; ; \quad \underbrace{\hat{y} = \tilde{x}\, Re^{1/2} = \frac{x}{L}\, Re^{1/2}}_{\text{stretched by } Re^{1/2} \text{ to remove } Re \text{ from dimensionless governing eq}}$$

$$\tilde{\mathcal{P}} = \mathcal{P}/\rho U^2$$

With these dimensionless quantities the x-momentum eq reads:

$$\tilde{V}_x \frac{\partial \tilde{V}_x}{\partial \tilde{x}} + \hat{V}_y \frac{\partial \tilde{V}_x}{\partial \hat{y}} = -\frac{d\tilde{\mathcal{P}}}{d\tilde{x}} + \frac{\partial^2 \tilde{V}_x}{\partial \hat{y}^2} \qquad \text{Eq. 9.2-21}$$

To connect the boundary layer eq. to the outer flow region, we set $\frac{d\mathcal{P}}{dx} = -\rho u \frac{du}{dx} \iff \frac{d\tilde{\mathcal{P}}}{d\tilde{x}} = \tilde{u}\frac{d\tilde{u}}{d\tilde{x}}$

with $\tilde{u}(\tilde{x}) = \frac{u(x)}{U}$

which yields for the scaled, dimensionless boundary equation

$$\tilde{v}_x \frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \hat{v}_y \frac{\partial \tilde{v}_x}{\partial \hat{y}} = \tilde{u}\frac{d\tilde{u}}{d\tilde{x}} + \frac{\partial^2 \tilde{v}_x}{\partial \hat{y}^2} \quad \text{Eq. 9.2-26}$$

pressure term

This eq. is valid for curved surfaces as long as the prior assumptions are met, e.g., $\delta(x) \ll L$, $v_z = 0$

Parallel Flow past a flat plate

For parallel flow past a flat plate, we set

$$\tilde{u}\frac{d\tilde{u}}{d\tilde{x}} = 0 \quad \text{(no pressure term)}$$

as $u(x) = U$ (constant)

$\Rightarrow$ x-momentum balance:

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial x} = \nu \frac{\partial^2 v_x}{\partial y^2} \quad \text{Eq. 9.4-1}$$

which is equivalent to the stream fct representation that also includes the continuity eq:

$$\frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial^2 \hat{\psi}}{\partial \tilde{x}\,\partial \hat{y}} - \frac{\partial \hat{\psi}}{\partial \tilde{x}} \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} = \frac{\partial^3 \hat{\psi}}{\partial \hat{y}^3} \quad \text{Eq. 9.4-4}$$

for which we scaled the dimensionless stream fct as

in y

$$\tilde{v}_x = \frac{\partial \hat{\psi}}{\partial \hat{y}} \;; \quad \hat{v}_y = -\frac{\partial \hat{\psi}}{\partial \hat{x}}$$

$$\hat{\psi} = \frac{\psi}{UL} Re^{1/2}$$

The PDE is solved using a similarity analysis this time not in time t but in x.
We can show that the stream function is self-similar regarding ~~xx~~ the boundary layer thickness δ, which in turn is a function of x. (see textbook Deen p. 379)

$\psi \sim u\delta$ scales with boundary layer thickness δ and u.
As $u = U$ is constant.
ψ scales with thickness $\delta(x)$

$$\Rightarrow \boxed{f(\eta) \equiv \frac{\hat{\psi}(\tilde{x},\hat{y})}{g(\tilde{x})}}$$ modified stream fct.

with the similarity parameter $\eta = \dfrac{\hat{y}}{g(\tilde{x})}$

The PDE converts into the ODE

$$f''' + (g g')\, f f'' = 0 \qquad \text{Eq. } (9.4.7)$$

which yields

Eq 9.4-8: $$\begin{cases} g(x) = (2\tilde{x})^{1/2} \\ \eta(\tilde{x},\hat{y}) = \dfrac{\hat{y}}{(2\tilde{x})^{1/2}} \end{cases}$$

BL thickness: $\Rightarrow \boxed{\delta(x) \sim x^{1/2}}$

on flat plate numerically: $\boxed{\delta(x) \cong \dfrac{5.0}{\sqrt{Re_x}}}$ Derived with Kármán Polhausen

With BCs: no-slip, no penetration and asymptotic matching $\tilde{v}_x(\hat{y} \to \infty) = 1$

Blasius eq. $\Rightarrow \boxed{f''' + f f'' = 0}$ $\quad f(0) = 0,\ f'(0) = 0,\ f'(\infty) = 1$

Eq. 9.4.10

The numerical solution of Blasius eq yields:

Velocity distribution:

$$\boxed{\frac{V_x}{U(x)}(\eta)} \quad \text{See } \boxed{\text{Fig } 9\text{-}11} \text{ for } \beta = 0$$

as $f'(\eta)\,U = v_x$

Local Shear Stress: Viscous Drag

$\Rightarrow \frac{\partial v_x}{\partial y} = f''(\eta)$

$$\boxed{\tau_0 = \mu \left.\frac{\partial v_x}{\partial y}\right|_{y=0}} = \frac{\mu U}{L} Re^{1/2} \frac{f''(0)}{g} \quad ; \quad \text{from Table } f''(0) = 0.33201$$

$$= \boxed{0.332 \frac{\mu U}{L} Re^{1/2} \tilde{x}^{-1/2}} \quad ; \quad \tilde{x} = \frac{x}{L}$$

Eq 9.4-12 (used for $f'(0) = 0.46960$)

Drag on both sides of Flat Plate

plate width W

$$\boxed{F_D = 2WL \int_0^1 \tau_0 \, d\tilde{x} = 1.328 \mu U W Re^{1/2}}$$

Eq. 9.4-13

or drag coefficient:

$$\boxed{C_D = \frac{2F_D}{\rho U^2 A}} \quad ; \quad \text{for a flat plate } A = 2WL \text{ (total wetted area)}$$

$$= \boxed{\frac{1.328}{Re^{1/2}}}$$