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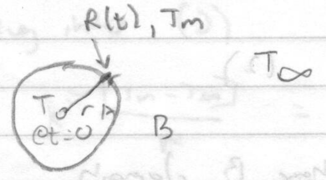
2. Deen Ex 3.5-4, melting of a Small Crystal



General Balance

$$\cancel{\frac{\partial T}{\partial t}} + \cancel{\vec{v}\cdot\nabla T}^{\,0} = \underbrace{\alpha\nabla^2 T}_{\text{no } \phi,\ \theta \text{ component}} + \cancel{\frac{H_v}{\rho C_p}}^{\,0}$$

ignore for pseudo steady approx $\quad v=0 \quad$ no v heat generation

· Use spherical coordinates for simplicity

$$0 = \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) \qquad (1)$$

BCs $\quad T(r,t) \quad$ for phase A domain

BC1 $\quad T(R(t), t) = T_m \quad$, $R(t) \equiv$ instantaneous R of sphere

BC2 $\quad \frac{\partial T}{\partial r}(0, t) = 0 \quad$ (symmetry)

Solve (1) w/ 2 BCs

Integrating gives

$$r^2 \frac{\partial T}{\partial r} = c_1$$

$$\frac{\partial T}{\partial r} = \frac{c_1}{r^2} \qquad \text{Apply BC 2}$$

$$0 = c_1$$

Integrate again

$$T = c_2 \qquad \text{apply BC1}$$

$$c_2 = T_m$$

$$T(r,t) = T_m \quad \text{for } 0 \le r \le R(t) \text{ and } t \text{ at steady state}$$

(an expected ss result)

Use balance on interface to find velocity, $\frac{\partial R(t)}{\partial t}$ (eq. 2.2-15)

$$\left[(\vec{F} - b\,\vec{v}_I)_B - \cancel{(\vec{F} - b\,\vec{v}_I)_A}^{\,0}\right]\cdot\vec{n}_I = \cancel{B_s}^{\,0}$$

no gradient in A $\qquad$ no heat generation at surface

$$\vec{F} = -k\frac{\partial T}{\partial r} \qquad b = -\rho\hat{\lambda} \qquad \vec{n}_I = e_r$$

$$\vec{v}_I = \frac{\partial R(t)}{\partial t}$$

$$-k\frac{\partial T}{\partial r} + \rho\hat{\lambda}\frac{\partial R(t)}{\partial t} = 0$$

at $r = R(t),\ t = t \longrightarrow$ $$k\frac{\partial T}{\partial r} = -\rho\hat{\lambda}\frac{\partial R(t)}{\partial t} \qquad (2)$$

Now lets apply general balance on phase B domain

(1) still holds $$\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = 0$$

BC 3 $\quad T(R(t), t) = T_m$

BC 4 $\quad T(\infty, t) = T_\infty$

Integrate (1) once $$\frac{\partial T}{\partial r} = \frac{C_3}{r^2}$$

Integrate again $$T = \frac{-C_3}{r} + C_4$$

Apply BC 4 $\quad T_\infty = 0 + C_4, \quad C_4 = T_\infty$

Apply BC3 $$T_m = \frac{-C_3}{R(t)} + T_\infty$$

$$R(t)\left[T_m - T_\infty\right] = -C_3$$

$$T(r,t) = \frac{R(t)\left[T_m - T_\infty\right]}{r} + T_\infty \qquad (3)$$

Use (3) to find $\frac{\partial T}{\partial r}$, then plug into (2)

$$\frac{\partial T}{\partial r} = \frac{-R(t)\left[T_m - T_\infty\right]}{r^2}$$

$$\frac{\partial T}{\partial r}(R(t), t) = \frac{-R(t)\left[T_m - T_\infty\right]}{R(t)^2}$$

$$= \frac{-\left[T_m - T_\infty\right]}{R}$$

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2. Contd.

plug into (2)

$$k\frac{[T_m - T_\infty]}{R} = -\rho\hat{\lambda}\frac{dR}{dt}$$

$$R\,dR = \frac{-k[T_\infty - T_m]}{\rho\hat{\lambda}}dt \quad , R(0) = R_0$$

Integrate

$$\frac{R^2}{2} = \frac{-k[T_\infty - T_m]t}{\rho\hat{\lambda}} + C_5$$

plug in IC: $2C_5 = R_0^2$

$$\therefore \quad R^2(t) = \frac{-2k[T_\infty - T_m]t}{\rho\hat{\lambda}} + R_0^2$$

what is melting time?

$R^2(t) \to 0$, solve for t_p

$$0 = \frac{-2k(T_\infty - T_m)t_p}{\rho\hat{\lambda}} + R_0^2$$

$$\boxed{t_p = \frac{R_0^2\rho\hat{\lambda}}{2k(T_\infty - T_m)}}$$

when is pseudo steady valid? t_h must be much faster than t_p

$$t_h = \frac{\rho\hat{C}_p R_0^2}{k} \quad [=] \quad \frac{\frac{g}{m^3}\cdot\frac{J}{g\cdot K}\cdot m^2}{\frac{J}{s\cdot m\cdot K}} \quad [=] \quad s$$

$$t_h = \frac{L^2}{\alpha}$$ conduction char. time

$$t_d = \frac{L^2}{D}$$ diff. char time

$$\frac{t_h}{t_p} \ll 1$$

or

$$\frac{\frac{\cancel{\rho}\hat{C}_p\cancel{R_0^2}}{\cancel{k}}}{\frac{\cancel{R_0^2}\cancel{\rho}\hat{\lambda}}{2\cancel{k}(T_\infty - T_m)}} = \boxed{\frac{2\hat{C}_p(T_\infty - T_m)}{\hat{\lambda}} \ll 1}$$

(see attached plots)

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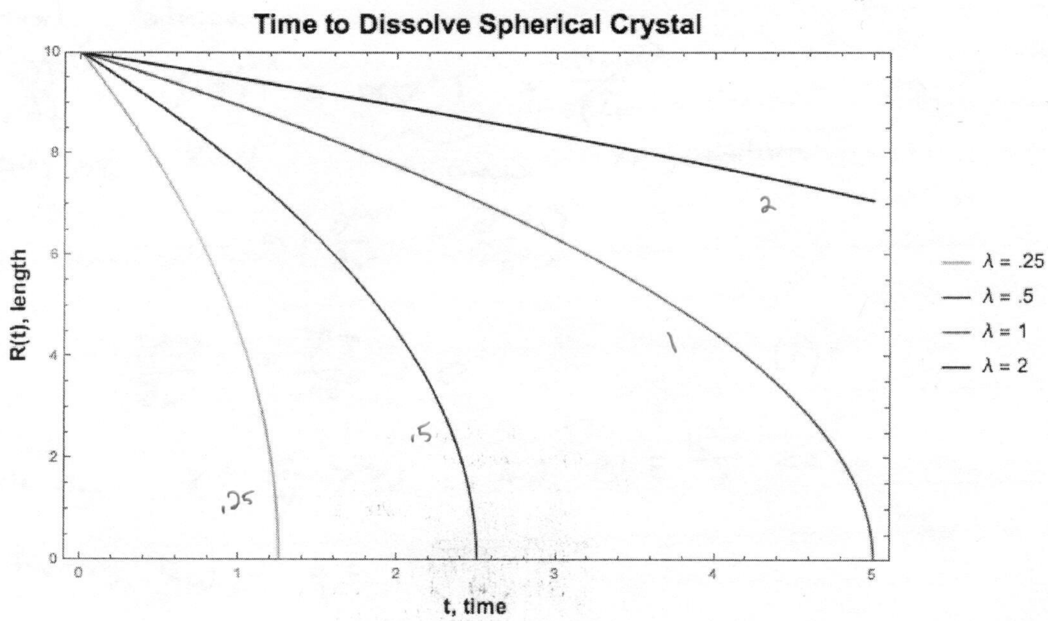
2.

```
In[79]:= R0 = 10
         ρ = 1
         k = 1
         DT = 10

In[97]:= Plot[
          {f3[k, ρ, .25, DT, R0, t], f3[k, ρ, .5, DT, R0, t], f3[k, ρ, 1, DT, R0, t],
           f3[k, ρ, 2, DT, R0, t]}, {t, 0, 5}, PlotRange → {0, 10},
          PlotLegends → {"λ = .25", "λ = .5", "λ = 1", "λ = 2"},
          PlotStyle → {Green, Blue, Red, Black},
          Frame → True,
          FrameLabel → {
            {Style["R(t), length", Bold, Black , 14], ""},
            {Style["t, time", Bold, Black, 14],
             Style["Time to Dissolve Spherical Crystal", Bold, Black, 18]}},
          ImageSize → 600
         ]
```

Out[97]=



note: λ changing shows characteristic curve shifts for any constant change $\frac{k \Delta T}{\rho \lambda}$ ratio value.

3.

```
In[38]:= DSolve[{θ''[η] + 2 η θ'[η] - 2 θ[η] == 0, θ[Infinity] == 0, θ'[0] == -2}, θ[η], η]
```

Out[38]= $\left\{\left\{\theta[\eta] \to \frac{2\, e^{-\eta^2}\left(1 - e^{\eta^2}\sqrt{\pi}\,\eta + e^{\eta^2}\sqrt{\pi}\,\eta\,\mathrm{Erf}[\eta]\right)}{\sqrt{\pi}}\right\}\right\}$

```
In[40]:= Simplify[%38]
```

Out[40]= $\left\{\left\{\theta[\eta] \to \frac{2\, e^{-\eta^2}}{\sqrt{\pi}} - 2\,\eta + 2\,\eta\,\mathrm{Erf}[\eta]\right\}\right\}$