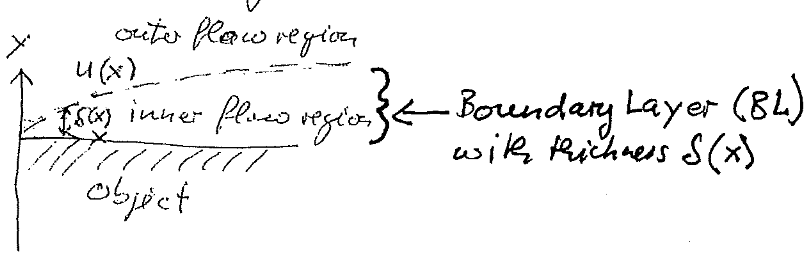
Recap: Boundary Layer Introduction

Consider for flow around objects two flow regions
- outer flow region (potential flow)
- inner flow region (viscous boundary layer)

The two flow regions have to be matched asymptotically at their "interface", by introducing an "asymptotic" velocity $u(x)$ as

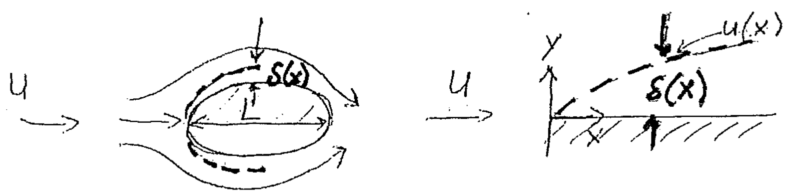
$$u(x) = \begin{cases} \lim\limits_{y \to 0} v_x^{o}(x,y) & (o = \text{outer}) \\ \lim\limits_{y \to \infty} v_x^{i}(x,y) & (i = \text{inner}) \end{cases}$$



A similar asymptotic expression is used for the asymptotic pressure at the "interface":

$$\mathcal{P}(x) = \lim_{y \to 0} \mathcal{P}^{o}(x,y) = \lim_{y \to \infty} \mathcal{P}^{i}(x,y)$$

We assume $\delta(x) \ll L \Rightarrow v_z \ll v_x$ within BL, $v_x(x,y)$



Note: We do not assume $u(x) = u$, as we will later do for flat plates.

Laminar Boundary Layer Theory of Newtonian Fluids

The x-momentum balance for a Newtonian fluid moving along an x-y surface, where $v_z = 0$ is

$$\underbrace{v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_y}{\partial y}}_{\text{advection term}} = \underbrace{-\frac{1}{\rho}\frac{d\mathcal{P}}{dx}}_{\text{pressure term}} + \underbrace{\nu \frac{\partial^2 v_x}{\partial y^2}}_{\text{viscous term}} \qquad \text{Eq. 9.2-17}$$

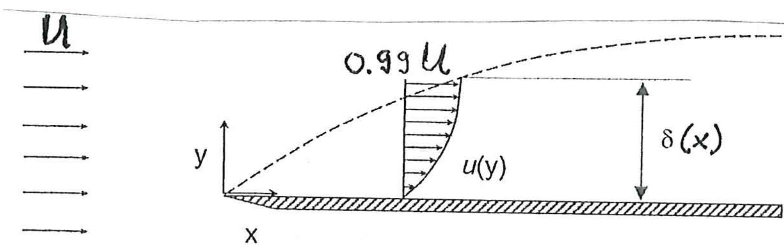
The flow has also to obey the continuity eq.

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad \text{Eq. 9.2-16}$$

We assume that $v_x > v_y$, so that $\mathcal{P} = \mathcal{P}(x)$ only.

Boundary Layer Thickness: $\delta(x)$

Defined as the distance y from the surface at which $u(y)_x = 0.99\,U$, where $u(y)_x$ is the velocity of the stream within BL at x, and U is the outer stream velocity.



For a laminar BL

$$\boxed{\frac{\delta}{x} \cong \frac{5.0}{\sqrt{Re_x}} \;;\quad Re_x = \frac{Ux}{\nu}}$$

Laminar BL exist over a plate dimension with $L = x_{crit}$, whereby x_{crit} is defined via: $Re_{x_{crit}} \cong 5\times 10^5$.

For a turbulent BL: $x > x_{crit}$ the BL thickness is given by

$$\boxed{\frac{\delta}{x} = \frac{0.385}{(Re_x)^{\frac{1}{5}}}}$$

We introduce the following dimensionless quantities:

$$\tilde{V}_x \equiv \quad \equiv \frac{V_x}{U} \;;\quad \hat{V}_y = \tilde{V}_y\, Re^{1/2} = \frac{V_y}{U}\, Re^{1/2}$$

$$\underbrace{\tilde{x} = \frac{x}{L}}_{\text{conventional}} \;;\quad \underbrace{\hat{y} = \tilde{x}\, Re^{1/2} = \frac{x}{L}\, Re^{1/2}}_{\text{stretched by } Re^{1/2} \text{ to remove Re from dimensionless governing eq}}$$

$$\tilde{P} = P/\rho U^2$$

With these dimensionless quantities the x-momentum eq reads:

$$\tilde{V}_x \frac{\partial \tilde{V}_x}{\partial \tilde{x}} + \hat{V}_y \frac{\partial \tilde{V}_x}{\partial \hat{y}} = -\frac{d\tilde{P}}{d\tilde{x}} + \frac{\partial^2 \tilde{V}_x}{\partial \hat{y}^2} \qquad \text{Eq. 9.2-21}$$

To connect the boundary layer eq. to the outer flow region, we set $\frac{dP}{dx} = -\rho u \frac{du}{dx} \iff \frac{d\tilde{P}}{d\tilde{x}} = \tilde{u}\frac{d\tilde{u}}{d\tilde{x}}$

with $\tilde{u}(\tilde{x}) = \frac{u(x)}{U}$