Recap

Conservation of momentum

Cauchy: $$\boxed{\rho \frac{D\vec{v}}{Dt} = \rho\vec{g} - \vec{\nabla} P + \vec{\nabla}\cdot\underline{\underline{\tau}}} \qquad (1)$$

Newtonian Fluid using the constitutive eq $\underline{\underline{\tau}} = 2\mu\underline{\underline{\Gamma}} = \mu\left[\vec{\nabla}\vec{v} + (\vec{\nabla}\vec{v})^t\right]$

$$\boxed{\rho \frac{D\vec{v}}{Dt} = \rho\vec{g} - \vec{\nabla} P + \vec{\nabla}\cdot 2\mu\underline{\underline{\Gamma}}} \qquad (2)\ \mu \text{ const.}$$

Non-Newtonian Fluid: set $\mu = f(\Gamma)$

~~Generalized Newtonian Fluid Momentum Conservation Equation~~

$$\boxed{\rho \frac{D\vec{v}}{Dt} = \rho\vec{g} - \vec{\nabla} P + \vec{\nabla}\cdot\left\{2\mu(\Gamma)\underline{\underline{\Gamma}}\right\}} \qquad (3)$$

Strain rate $\underline{\underline{\Gamma}} \equiv \frac{1}{2}\left[\vec{\nabla}\vec{v} + (\vec{\nabla}\vec{v})^t\right]$

Dissipation fct

Magnitude of strain rate; $\Gamma \equiv \left[\frac{1}{2}\left(\underline{\underline{\Gamma}} : \underline{\underline{\Gamma}}\right)\right]^{1/2}$; $\Phi = (2\Gamma)^2$ (Table 6-10)

$\mu(\Gamma)$ empirical relationships (e.g. Bingham Model)

Newtonian Fluid with $\rho = \text{const}$ and $\mu = \text{const}$.

Navier Stokes Eq.

$$\boxed{\rho \frac{D\vec{v}}{Dt} = \rho\vec{g} - \vec{\nabla} P + \mu\nabla^2\vec{v}}$$

Continuity eq for Newtonian Fluid and $\rho = \text{const}$ and $\mu = \text{const}$

$$\vec{\nabla}\cdot\vec{v} = 0$$

~~(general: $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$)~~

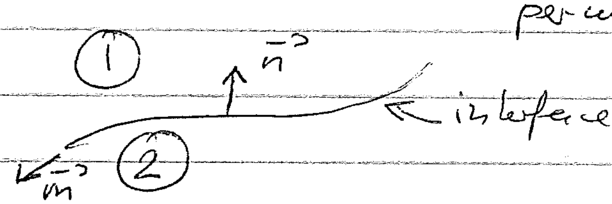
Surface Tension

Momentum Balance Eq. 6.6-6

Eq. 6.6-6 $$\int_{V(t)} \rho \frac{D\vec{v}}{Dt}\, dV = \underbrace{\int_{V(t)} \rho \vec{g}\, dV}_{\text{volume stresses}} + \underbrace{\int_{S(t)} \vec{s}\, dS}_{\text{surface stresses}} + \underbrace{\int_{C(t)} \mu \vec{m}\, dC}_{\text{line tension}}$$

line tension (surface energy per unit area)

$\Downarrow$



Stress Balance

Eq. 6.6-9 $$\boxed{\underbrace{\vec{s}(\vec{n})\big|_2 - \vec{s}(n)\big|_1}_{\text{stress difference at either side}} = \underbrace{-\vec{\nabla}_s \mu}_{\text{surface tension}} + \underbrace{2\varkappa \vec{n} \mu}_{\text{curvature (geometric) stress}}}$$

Special cases: $-\vec{\nabla}_s \mu = 0$ for μ uniform; $\varkappa = 0$ for planar interface

With $\vec{s}(\vec{n}) = \vec{n}\cdot\underline{\underline{\sigma}} = \vec{n}\cdot\left(-P\underline{\underline{\delta}} + \underline{\underline{\tau}}\right)$

stress balance resolved in:

normal stress component: $$\boxed{P_1 - P_2 = \underbrace{\tau_{nn}\big|_1 - \tau_{nn}\big|_2}_{\text{net normal shear stress}} - \underbrace{2\varkappa\mu}_{\text{geometric stress}}}$$

tangential stress component: $$\boxed{\tau_{nt}\big|_2 - \tau_{nt}\big|_1 = -\vec{t}\cdot\vec{\nabla}_s \mu}$$

Stress Balance:

normal component: $P_1 - P_2 = -\tau_{nn}|_2 + \tau_{nn}|_1 - 2\mathcal{H}\gamma$

tangential component: $\tau_{nt}|_2 - \tau_{nt}|_1 = -\vec{t} \cdot \vec{\nabla}_s \gamma$

TABLE A-5 (p. 636)

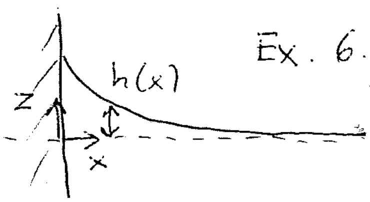
Unit Normal, Surface Gradient, and Mean Curvature[a]

Surface given by $z = F(x,y)$	Surface given by $z = F(r)$
$\mathbf{n} = \frac{1}{H}\left(-\frac{\partial F}{\partial x}\mathbf{e}_x - \frac{\partial F}{\partial y}\mathbf{e}_y + \mathbf{e}_z\right)$	$\mathbf{n} = \frac{1}{H}\left(-\frac{dF}{dr}\mathbf{e}_r + \mathbf{e}_z\right)$
$H = \left[\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + 1\right]^{1/2}$	$H = \left[\left(\frac{dF}{dr}\right)^2 + 1\right]^{1/2}$
$\nabla_S = \frac{1}{H^2}\left\{\left[\left(\frac{\partial F}{\partial y}\right)^2 + 1\right]\mathbf{e}_x - \frac{\partial F}{\partial x}\frac{\partial F}{\partial y}\mathbf{e}_y + \frac{\partial F}{\partial x}\mathbf{e}_z\right]\frac{\partial}{\partial x} + \frac{1}{H^2}\left[-\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}\mathbf{e}_x + \left[\left(\frac{\partial F}{\partial x}\right)^2 + 1\right]\mathbf{e}_y + \frac{\partial F}{\partial y}\mathbf{e}_z\right]\frac{\partial}{\partial y} + \frac{1}{H^2}\left[\frac{\partial F}{\partial x}\mathbf{e}_x + \frac{\partial F}{\partial y}\mathbf{e}_y + \left[\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2\right]\mathbf{e}_z\right]\frac{\partial}{\partial z}$	$\nabla_S = \frac{1}{H^2}\left(\mathbf{e}_r + \frac{dF}{dr}\mathbf{e}_z\right)\frac{\partial}{\partial r} + \left(\frac{1}{r}\mathbf{e}_\theta\right)\frac{\partial}{\partial \theta} + \frac{1}{H^2}\frac{dF}{dr}\left(\mathbf{e}_r + \frac{dF}{dr}\mathbf{e}_z\right)\frac{\partial}{\partial z}$
$2\mathcal{H} = \frac{1}{H^3}\left\{\left[\left(\frac{\partial F}{\partial y}\right)^2 + 1\right]\frac{\partial^2 F}{\partial x^2} - 2\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}\frac{\partial^2 F}{\partial x \partial y} + \left[\left(\frac{\partial F}{\partial x}\right)^2 + 1\right]\frac{\partial^2 F}{\partial y^2}\right]$	$2\mathcal{H} = \frac{1}{H^3}\left(\frac{d^2F}{dr^2} + \frac{H^2}{r}\frac{dF}{dr}\right)$

[a] In these expressions the unit normal $\mathbf{n}$ points toward positive z. Reversing the direction of $\mathbf{n}$ will change the sign of $\mathcal{H}$.

Example

$z = h(x)$



Ex. 6.6-2 : Wetted wall

from Table A-5:

$$\vec{n} = \frac{1}{H}\begin{pmatrix} -\frac{\partial h}{\partial x} \\ 0 \\ 1 \end{pmatrix} \quad ; \quad H = \left[\left(\frac{\partial h}{\partial x}\right)^2 + 1\right]^{\frac{1}{2}}$$

$$\vec{\nabla}_s = \frac{1}{H^2}\begin{pmatrix} \frac{\partial}{\partial x} + \frac{\partial h}{\partial x}\frac{\partial}{\partial z} \\ \left\{\left(\frac{\partial h}{\partial x}\right)^2 + 1\right\}\frac{\partial}{\partial y} \\ \frac{\partial h}{\partial x}\frac{\partial}{\partial x} + \left(\frac{\partial h}{\partial x}\right)^2\frac{\partial}{\partial z} \end{pmatrix}$$

$$2\mathcal{H} = \frac{\left(\frac{\partial h}{\partial x}\right)^2}{H^3} = \frac{\frac{d^2h}{dx^2}}{\left[1 + \left(\frac{dh}{dx}\right)^2\right]^{3/2}}$$

Eq 6.6-16