ChemE530

Vector Notation Problems for Recitation 09/27/19

T and **v** are defined as $\underline{\underline{T}} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$.

(i) Determine $\underline{\underline{T}} \cdot \vec{v}$

(ii) If $\underline{\underline{T}}$ is a stress tensor, find the force/area exerted on a fluid oriented such that its normal, $\vec{n}$, is collinear with $\vec{v}$.

(i)

$\underline{\underline{T}} \cdot \vec{v}$ = vector bc $\underset{\text{tensor}}{2} + \underset{\text{vector}}{1} - \overset{\text{dot}}{2} = 1$ $\qquad \underline{\underline{T}} = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

$$\underline{\underline{T}} \cdot \vec{v} = \sum_i \sum_j T_{ij} \delta_i \delta_j \cdot \sum_k v_k \delta_k = \sum_i \sum_j \sum_k T_{ij} v_k \delta_i \delta_j \cdot \delta_k \overset{*}{=} \sum_i \delta_i \sum_j T_{ij} v_j$$

*Recall $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$

$$\Rightarrow \underline{\underline{T}} \cdot \vec{v} = \begin{bmatrix} T_{11} v_1 + T_{12} v_2 + T_{13} v_3 \\ T_{21} v_1 + T_{22} v_2 + T_{23} v_3 \\ T_{31} v_1 + T_{32} v_2 + T_{33} v_3 \end{bmatrix} = \begin{bmatrix} 2 \times 2 + 3 \times 2 + 0 \\ 3 \times 2 + 4 \times 2 + 0 \\ 4 \times 2 + 5 \times 2 + 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 18 \end{bmatrix} \leftarrow$$

(ii)

$\hat{n}$ is colinear w/ $\vec{v}$ $\Rightarrow |v| = \sqrt{2^2 + 2^2 + 0} = \sqrt{8} = 2\sqrt{2}$

$$\frac{\vec{v}}{|v|} = \hat{n} = (1/\sqrt{2}, 1/\sqrt{2}, 0)$$

$\underline{\underline{T}}$ is symmetric thus $\underline{\underline{T}} \cdot \hat{n} = \hat{n} \cdot \underline{\underline{T}}$ = vector $\Rightarrow \underline{\underline{T}} \cdot \hat{n} = \begin{bmatrix} T_{11} n_1 + T_{12} n_2 \\ T_{21} n_1 + T_{22} n_2 \\ T_{31} n_1 + T_{32} n_2 \end{bmatrix} = \begin{bmatrix} 2(1/\sqrt{2}) + 3(1/\sqrt{2}) \\ 3(1/\sqrt{2}) + 4(1/\sqrt{2}) \\ 4(1/\sqrt{2}) + 5(1/\sqrt{2}) \end{bmatrix}$

$$\Rightarrow \boxed{\underline{\underline{T}} \cdot \hat{n} = (5/\sqrt{2}, 7/\sqrt{2}, 9/\sqrt{2})}$$

1-4 Flux Normal to a Surface (Deen Text, page 23)

Consider a point P = (x, y, z) = (2a/3, b/3, 2c/3), which is on the surface of a spheroid defined by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$$

Suppose that there is a flux, **f**, that has the components

$$\boldsymbol{f} = (f_x, f_y, f_z) = \left(\frac{2a}{3}, \frac{b}{3}, \frac{2c}{3}\right).$$

Compute f_n = **n.f** at point P, where n is the outward unit normal.

1-4. Flux Normal to a Surface

Given a point $P = (x, y, z) = (2a/3,\ b/3,\ 2c/3)$ on the surface of an ellipsoid defined by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

and a flux $\underline{f} = (f_x, f_y, f_z) = \left(2a/3,\ b/3,\ 2c/3\right)$, Compute $f_n = \underline{n} \cdot \underline{f}$.

There are two ways to compute the unit outward normal, $\underline{n}$:

<u>Method 1</u>: Cross Product of Tangent Vectors

$$\underline{r}_s = x\,\underline{e}_x + y\,\underline{e}_y + F(x,y)\,\underline{e}_z \qquad \text{(position on surface)}$$

$$\underline{A} = \frac{\partial \underline{r}_s}{\partial x}, \quad \underline{B} = \frac{\partial \underline{r}_s}{\partial y} \qquad \text{(tangent vectors, from Eq. (A.8-1))}$$

$$\underline{n} = \frac{\underline{A} \times \underline{B}}{|\underline{A} \times \underline{B}|} \qquad \text{Eq. (A.8-2)}$$

Apply to the ellipsoid:

$$\left(\frac{z}{c}\right)^2 = 1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 \qquad \text{on surface}$$

$$z = c\left[1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2\right]^{1/2} \equiv F(x,y)$$

$$\frac{\partial F}{\partial x} = c\left(\frac{1}{2}\right)\left(-\frac{2x}{a^2}\right)\left[1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2\right]^{-1/2}$$

$$= -\frac{cx}{a^2}\left[1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2\right]^{-1/2}$$

$$\frac{\partial F}{\partial x}\left(2a/3,\ b/3\right) = -\frac{c}{a^2}\left(\frac{2a}{3}\right)\left[1-\left(\frac{2}{3}\right)^2-\left(\frac{1}{3}\right)^2\right]^{-1/2}$$

$$= -\frac{2c}{3a}\left(\frac{4}{9}\right)^{-1/2} = -\frac{c}{a}$$

$$\frac{\partial F}{\partial y} = -\frac{cy}{b^2}\left[1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right]^{-1/2}$$

$$\frac{\partial F}{\partial y}\left(2a/3,\ b/3\right) = -\frac{c}{3b}\left(\frac{4}{9}\right)^{-1/2} = -\frac{c}{2b}$$

$$\underline{A} = (1)\ \underline{e}_x + (0)\ \underline{e}_y - (c/a)\ \underline{e}_z$$

$$\underline{B} = (0)\ \underline{e}_x + (1)\ \underline{e}_y - (c/2b)\ \underline{e}_z$$

$$\underline{A}\times\underline{B} = \begin{vmatrix} \underline{e}_x & \underline{e}_y & \underline{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \underline{e}_x\left(A_yB_z - A_zB_y\right) + \underline{e}_y\left(A_zB_x - A_xB_z\right) + \underline{e}_z\left(A_xB_y - A_yB_x\right)$$

$$= \underline{e}_x\left(\frac{c}{a}\right) + \underline{e}_y\left(\frac{c}{2b}\right) + \underline{e}_z\ (1)$$

$$\left|\underline{A}\times\underline{B}\right| = \left[(c/a)^2 + (c/2b)^2 + 1\right]^{1/2}$$

$$\boxed{\begin{aligned} \underline{n} &= \frac{(c/a)\ \underline{e}_x + (c/2b)\ \underline{e}_y + \underline{e}_z}{\left[(c/a)^2 + (c/2b)^2 + 1\right]^{1/2}} \\ &= \frac{2bc\ \underline{e}_x + ac\ \underline{e}_y + 2ab\ \underline{e}_z}{\left(4b^2c^2 + a^2c^2 + 4a^2b^2\right)^{1/2}} \end{aligned}}$$

Method 2: Gradient

$$G(x,y,z) \equiv z - F(x,y)$$

$$\underline{n} = \frac{\nabla G}{|\nabla G|} \qquad \text{Eq. (A.8-7)}$$

Apply to the ellipsoid:

$$G(x,y,z) = z - c\left[1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2\right]^{1/2}$$

$$\underline{\nabla} G = \frac{\partial G}{\partial x}\,\underline{e}_x + \frac{\partial G}{\partial y}\,\underline{e}_y + \frac{\partial G}{\partial z}\,\underline{e}_z$$

$$\frac{\partial G}{\partial x} = -\frac{\partial F}{\partial x} = \frac{c}{a} \qquad \text{at point } P$$

$$\frac{\partial G}{\partial y} = -\frac{\partial F}{\partial y} = \frac{c}{2b} \qquad \text{at point } P$$

$$\frac{\partial G}{\partial z} = 1$$

$$\underline{\nabla} G = \left(\frac{c}{a}\right)\underline{e}_x + \left(\frac{c}{2b}\right)\underline{e}_y + \underline{e}_z$$

$$|\underline{\nabla} G| = \left[\left(\frac{c}{a}\right)^2 + \left(\frac{c}{2b}\right)^2 + 1\right]^{1/2}$$

$$\boxed{\underline{n} = \frac{2bc\,\underline{e}_x + ac\,\underline{e}_y + 2ab\,\underline{e}_z}{\left(4b^2c^2 + a^2c^2 + 4a^2b^2\right)^{1/2}}}$$

Same result as method 1.

Now calculate the normal component of the flux at point P:

$$f_n = \underline{n} \cdot \underline{f} = \left[\frac{2bc\,\underline{e}_x + ac\,\underline{e}_y + 2ab\,\underline{e}_z}{(4b^2c^2 + a^2c^2 + 4a^2b^2)^{1/2}}\right] \cdot \left[\frac{2a}{3}\underline{e}_x + \frac{b}{3}\underline{e}_y + \frac{2c}{3}\underline{e}_z\right]$$

$$= \frac{abc}{(4b^2c^2 + a^2c^2 + 4a^2b^2)^{1/2}} \left(\frac{4}{3} + \frac{1}{3} + \frac{4}{3}\right)$$

$$\boxed{f_n = \frac{3abc}{(4b^2c^2 + a^2c^2 + 4a^2b^2)^{1/2}}}$$