Example 1:

In a flow field, ***v*** = (u, v, w), the velocity components were found to be

$$u = a(x^2 - y^2)$$

$$v = -2axy$$

$$w = 0$$

With the third component w = 0, the flow is two-dimensional.

a) Determine if the flow is incompressible
b) Define the scalar stream function ψ for the given problem (only exists if flow is incompressible)

Example 2:

Given are the velocity components of a 2D velocity flow field of an incompressible fluid

$$u = \frac{y^3}{3} + 2x - x^2 y$$

$$u = \frac{x^3}{3} - 2y + xy^2$$

Show that the flow is irrotational and incompressible.

① $\vec{V} = (u, v, w) \quad u = a(x^2-y^2) \quad v = -2axy \quad w = 0$

a) incompressibility : $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ continuity eqn. for $\rho = const$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}(a(x^2-y^2)) = 2ax \\ \frac{\partial v}{\partial y} &= \frac{\partial}{\partial y}(-2axy) = -2ax \end{aligned}\right\} \quad 2ax - 2ax = 0 \ \checkmark$$

b)

$$u = \frac{\partial \psi}{\partial y} = a(x^2-y^2) \Rightarrow \psi = \int u\,dy + f(x) = ax^2y - \frac{ay^3}{3} + f(x)$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left(ax^2y - \frac{ay^3}{3} + f(x)\right) = -2axy + f'(x)$$

From problem statement $v = -2axy \Rightarrow f'(x) = 0 \Rightarrow f(x) = c$

$$\Rightarrow \boxed{\psi = ax^2y - \frac{ay^3}{3} + c}$$

②

$$u = \frac{y^3}{3} + 2x - x^2y \qquad v = xy^2 - 2y - \frac{x^3}{3}$$

incompressible : $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y^3}{3} + 2x - x^2y\right) = 2 - 2xy$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(xy^2 - 2y - \frac{x^3}{3}\right) = 2xy - 2$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2 - 2xy + 2xy - 2 = 0 \ \checkmark$$

irrotational : $\vec{\nabla} \times \vec{w} = 0$ 2D in x-y plane $\Rightarrow \vec{w} = (0, 0, w_z)$

as per lecture $w_z = \frac{\partial}{\partial x} v - \frac{\partial}{\partial y} u \quad \Big| \begin{array}{l} u = v_x \\ v = v_y \end{array}$

$$\left.\begin{aligned} &\Rightarrow \frac{\partial}{\partial x}\left(xy^2 - 2y - \frac{x^3}{3}\right) = y^2 - x^2 \\ &\Rightarrow \frac{\partial}{\partial y}\left(\frac{y^3}{3} + 2x - x^2y\right) = y^2 - x^2 \end{aligned}\right\} \quad y^2 - x^2 - (y^2 - x^2) = 0$$

$\Rightarrow w_z = 0 \Rightarrow$ irrotational