Recap:

Shear (strain rate) Deformation:

$$\underline{\underline{\Gamma}} = \left(\dot{\varepsilon}_{ij}\right)_{i,j} \quad , \quad \dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

$$= \frac{1}{2}\left(\vec{\nabla}\vec{v} + \left(\vec{\nabla}v\right)^t\right) \; ; \quad \Gamma_{ij} \equiv \dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ji} = \Gamma_{ji}$$

(symmetric tensor)

If we consider the deformation rate tensor

$$\vec{\nabla}\vec{v} = \left(\frac{\partial v_i}{\partial x_j}\right)_{ij} \quad \text{dyad with 9 elements}$$

and that any tensor can be written as sum of symm. and antisymm. tensor,

$$\vec{\nabla}\vec{v} = \underbrace{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)}_{\substack{\text{symm.} \\ \underline{\underline{\Gamma}}}} + \underbrace{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\substack{\text{antisymm} \\ \underline{\underline{\Omega}} \text{ (vorticity tensor)}}}$$

we have to consider besides shear deformation also vorticity ("rigid" body rotation). It is

$$\underline{\underline{\Omega}} = \frac{1}{2}\left[\vec{\nabla}\vec{v} - \left(\vec{\nabla}\vec{v}\right)^t\right] = \frac{1}{2}\begin{pmatrix} 0 & w_z & -w_y \\ -w_z & 0 & w_x \\ w_y & -w_x & 0 \end{pmatrix}$$

where $\vec{w} = (w_x, w_y, w_z)$ is the vorticity vector $= \vec{\nabla} \times \vec{v}$ (curl)

In cart. coord:

$$w_x = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}$$
$$w_y = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}$$
$$w_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial z}$$

In cyl. coord.:

$$w_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}$$
$$w_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}$$
$$w_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

The vorticity is related to the angular velocity $\vec{a}$, as

$$\vec{w} = 2\vec{a}$$

For irrotational flow: $\vec{w} = 0$ everywhere

Deformations in a sheared fluid

All information on fluid deformation is contained in the shear tensor $\underline{\underline{\Gamma}}$. We have to consider two forms of strain:

(i) Dilatation

$$\underline{\underline{\dot{\varepsilon}_r}} = \frac{1}{3}\underbrace{(\vec{\nabla}\cdot\vec{v})}_{\text{divergence of velocity}}\underline{\underline{\delta}}$$

(in pure dilatation the rate of strain is one third of the rate of dilatation)

(ii) Shape Change (Shear) (subtract dilatation strain rate from shear strain rate)

$$\underline{\underline{\dot{\varepsilon}_\gamma}} = \underline{\underline{\Gamma}} - \frac{1}{3}(\vec{\nabla}\cdot\vec{v})\underline{\underline{\delta}} = \underline{\underline{\Gamma}} - \underline{\underline{\dot{\varepsilon}_r}}$$

Newtonian Fluid / Constitutive Eq

The Constitutive Equation relates the (viscous) stress to the rate of deformation, i.e. $\underline{\underline{\tau}} = f(\underline{\underline{\Gamma}})$

For Newtonian Fluid we assume that the deformation rates $\dot{\varepsilon}_r$ and $\dot{\varepsilon}_\gamma$ are proportional to the stress:

$$\underline{\underline{\tau}} = 2\mu\left[\underline{\underline{\Gamma}} - \frac{1}{3}(\vec{\nabla}\cdot\vec{v})\underline{\underline{\delta}}\right] + 3K\left[\frac{1}{3}(\vec{\nabla}\cdot\vec{v})\underline{\underline{\delta}}\right]$$

μ ... shear viscosity

K ... bulk viscosity

TABLE 6-4

Viscous Stress Components for Newtonian Fluids in Rectangular Coordinates

Newtonian fluid with

$\mu \gg \kappa \Rightarrow \kappa \approx 0$

$\underline{\underline{\tau}} = 2\mu \left[\underline{\underline{\Gamma}} - \frac{1}{3}(\nabla \cdot \vec{v}) \underline{\underline{\delta}} \right]$

$$\tau_{xx} = 2\mu \left[\frac{\partial v_x}{\partial x} - \frac{1}{3} \nabla \cdot \mathbf{v} \right]$$

$$\tau_{yy} = 2\mu \left[\frac{\partial v_y}{\partial y} - \frac{1}{3} \nabla \cdot \mathbf{v} \right]$$

$$\tau_{zz} = 2\mu \left[\frac{\partial v_z}{\partial z} - \frac{1}{3} \nabla \cdot \mathbf{v} \right]$$

$$\tau_{xy} = \tau_{yx} = \mu \left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right]$$

$$\tau_{yz} = \tau_{zy} = \mu \left[\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right]$$

$$\tau_{zx} = \tau_{xz} = \mu \left[\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right]$$

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

see Table 6-5 and 6-6 for cylindrical and spherical coordinates

Incompressible Newtonian fluid

$\kappa \approx 0$ and $\rho = \text{const.}$

$\Rightarrow \vec{\nabla} \cdot \vec{v} = 0 \Rightarrow \underline{\underline{\tau}} = 2\mu \underline{\underline{\Gamma}}$