Chapter 5

Quarks and hadrons

Every atom has its ground state — the lowest energy state of its electrons in the presence of the atomic nucleus — as well as many excited states, which can decay to the ground state via emission of photons. Nuclei composed of multiple protons and neutrons also have their ground state plus various excited nuclear energy levels, which typically also decay via emission of photons (plus α and β radiation). But what about individual protons or neutrons?

It was asserted earlier that individual nucleons are also composite objects, and may be viewed as bound states of quarks. And just as atoms and nuclei have excited states, so do individual nucleons.

The force which binds quarks together into bound states is known as the *strong interaction*, and the theory which describes strong interactions (on distance scales small compared to a fermi) is called *quantum chromodynamics*, often abbreviated as QCD. The quarks carry a corresponding charge, the analogue of electric charge, which is labeled "color" charge, leading to the "chromo" in the name. Unlike electric charge, for which there is only a single variety - either plus or minus (with the underlying symmetry group $U(1)$), there are three possible "colors" for the color charge of a quark, along with the corresponding "anti-colors". The group describing the underlying symmetry is $SU(3)$. [1] We will have more to say about QCD as we progress. But the justification for the validity of the following qualitative description of quarks and their bound states lies in the success of QCD. Using this theory, one can do detailed quantitative calculations of the masses and other properties of bound states of quarks and compare with experimental results. The theory works. (In fact, the story of how this theory has been verified in experiment, even though the theory has quarks as degrees of freedom, while experiment *never* detects individual quarks, is an interesting one indeed. We will have only a limited opportunity to discuss it this quarter, but I will note that an essential feature of this story is the emergence of "jets" of hadrons in the final states of high energy collisions, and I have spent much of my scientific career clarifying both the theoretical and experimental properties of these jets.)

In this Chapter we will introduce a variety of new concepts, which we will return in more detail in subsequent discussion. In particular, we will be using several techniques from group theory. Now would be a good time to read both Chapters 10 and 11.

[1] In the group theory language of Chapter 10 $U(1)$ is an Abelian group and the particle corresponding to the interaction, the photon, does *not* interact directly with itself, *i.e.*, the photon has zero electric charge. $SU(3)$, on the other hand, is a non-Abelian group and the gluons, the analogs of the photon, come in 8 varieties and *do* interact directly with each other, *i.e.*, have nonzero color charge.

5.1 Quark flavors

Quarks are spin-1/2 particles (fermions), which come in various species, referred to as *flavors*. Different quark flavors have been given somewhat whimsical names, as shown in Table 5.1 (values from the PDG). Note that the table includes values for the quark "masses", but care must be taken when interpreting these values as individual, *isolated* quarks are *never* observed experimentally. On the other hand, it should be clear that the masses for different flavors vary substantially.

flavor	**symbol**	**mass**	**charge**
up	u	$\approx 2.3^{+0.7}_{-0.5}$ MeV/c^2	$\frac{2}{3}\,\lvert e\rvert$
down	d	$\approx 4.8^{+0.7}_{-0.3}$ MeV/c^2	$-\frac{1}{3}\,\lvert e\rvert$
strange	s	$\approx 95 \pm 5$ MeV/c^2	$-\frac{1}{3}\,\lvert e\rvert$
charm	c	1.275 ± 0.025 GeV/c^2	$\frac{2}{3}\,\lvert e\rvert$
bottom	b	4.18 ± 0.03 GeV/c^2	$-\frac{1}{3}\,\lvert e\rvert$
top	t	$173.5 \pm 0.6 \pm 0.8$ GeV/c^2	$\frac{2}{3}\,\lvert e\rvert$

Table 5.1: Known quark flavors

Along with quarks, there are, of course, also *antiquarks*, denoted $\bar{u}$, $\bar{d}$, $\bar{s}$, etc., with the *same* masses but opposite electric charge as their partner. (So, for example, the $\bar{u}$ antiquark has charge $-2/3$ and the $\bar{d}$ has charge $+1/3$ - note the *non*-integer values.) As suggested above, quarks are distinguished from leptons by an additional quantum number that is called *color*, which takes three possible values: say red, blue, or green (and anti-red, anti-blue and anti-green for the antiquarks). These names are simply labels for different quantum states of the quark.[2] Since quarks have spin 1/2, they can also be labeled by their spin projection, ↑ or ↓, along any chosen spin quantization axis. Hence, for each quark flavor, there are really six different types of quark, distinguished by the color (red, blue, green) and spin projection (up, down).

In addition to the curious names, two other things in Table 5.1 should strike you as odd: the enormous disparity of masses of different quarks, spanning five orders of magnitude, and the fact that quarks have fractional charge (in units of $|e|$). Both issues are at the core of ongoing research, including that at the LHC, seeking evidence of the dynamics of mass generation and the connection between quarks and leptons.

5.2 Hadrons

No (reproducible) experiments have detected any evidence for free, *i.e.*, isolated quarks. Moreover, there is no evidence for the existence of any isolated charged particle whose electric charge is not an integer multiple of the electron charge. This is referred to as *charge quantization*. Consistent with these observational facts, the theory of strong interactions predicts that quarks will always be

[2] These names are purely conventional — one could just as well label the different "color" states as 1, 2, and 3. But the historical choice of names explains why the theory of strong interactions is called quantum chromodynamics: a quantum theory of the dynamics of "color" — although this color has nothing to do with human vision!

trapped inside bound states with other quarks and antiquarks, never separated from their brethren by distances larger than about a fermi.[3] The bound states produced by the strong interactions are called *hadrons* (*hadros* is Greek for strong).

Quantum chromodynamics, in fact, predicts that only certain types of bound states of quarks can exist, namely those which are "colorless". (This can be phrased in a mathematically precise fashion in terms of the symmetries of the theory. More on this later. For now consider this situation as being similar to that in atomic physics where the low energy states, the atoms, are electrically neutral.) Recall that to make white light, one mixes together red, blue, and green light. Similarly, to make a colorless bound state of quarks one sticks together three quarks, one red, one blue, and one green. But this is not the only way. Just as antiquarks have electric charges which are opposite to their partner quarks, they also have "opposite" color: anti-red, anti-blue, or anti-green. So another way to make a colorless bound state is to combine three antiquarks, one anti-red, one anti-blue, and one anti-green. A final way to make a colorless bound state is to combine a quark and an antiquark, say red and anti-red, or better the truly colorless combination $r\bar{r} + b\bar{b} + g\bar{g}$. Bound states of three quarks are called *baryons*, bound states of three antiquarks are called *antibaryons*, both of which are fermions, while quark-antiquark bound states are called *mesons* and are bosons.

How these rules emerge from QCD will be described in a bit more detail later. For now, let's just look at some of the consequences. The prescription that hadrons must be colorless bound states says nothing about the flavors of the constituent quarks and antiquarks. In the language of quantum mechanics we say that color dependent operators and flavor dependent operators commute, $[\text{color}, \text{flavor}] = 0$. Since quarks come in the multiple flavors of Table 5.1, we can (and will) enumerate the various possibilities for the hadrons. Similar comments apply to the spatial (orbital angular momentum) and spin parts of the wave function (*i.e.*, the dynamics of these various parts commute to a good approximation), and we can think of the wave functions describing the hadrons, again to a good approximation, as *products* of a color wave function, a flavor wave function, a spatial wave function and a spin wave function. The most important violation of this assumption of the factorization of the wave function arises due to the role of the Pauli Exclusion Principle for baryons, as we will shortly see.

For the lowest mass hadrons we may assume that the quarks are essentially at rest (except for the constraints of quantum mechanics) and that the orbital angular momentum vanishes, $\vec{L} = 0$. Thus the rest energy of such a hadron (like any bound state, although it is admittedly a bit more subtle in this case) may be regarded as the sum of the rest energies of its constituents plus the energy associated with the binding interaction. For the lowest mass hadrons, including nucleons, we will see that most of their total rest energy (mass) comes from the binding energy, *i.e.*, the "gluon cloud", the analog of the electromagnetic potential for an atom. But the masses of quarks also play a part. Looking at the quark masses listed in Table 5.1, it is apparent that u, d and s quarks are quite light compared to the mass (≈ 1 GeV/c^2) of a nucleon, while the other quark flavors are considerably heavier. So it should not be surprising that the lightest hadrons will be those which are bound states of u and d quarks. Further, since a proton is composed of uud while a neutron is udd (*i.e.*,

[3] Except at sufficiently high temperatures. Above a temperature of $T_c \approx 2 \times 10^{12}$ K (or $kT \approx 170$ MeV), hadrons "melt" or "vaporize" and quarks are liberated. This is important in the physics of the early universe, since temperatures are believed to have exceeded this value in the earliest moments after the big bang. Temperatures above T_c can also be produced, briefly, in heavy ion collisions. A nice overview of heavy ion collisions and quark gluon plasma may be found at `www.bnl.gov/rhic/physics.asp`. There is also an ongoing heavy ion program at the LHC, *i.e.*, some running time is dedicated to accelerating and colliding heavy nuclei rather than protons.

the difference is only $1u \to 1d$), we are gratified to see that m_n is only slightly larger than m_p (see Eqs. (4.1.3) and (4.1.4)). Substituting a strange quark for a u or d quark should be expected to raise the mass of the resulting bound by roughly 100 MeV (due to the larger s quark mass). And hadrons containing the other quark flavors (c, b, or t) should be substantially heavier.

As noted above, when enumerating possible combinations of quarks that could form hadrons, we must also consider the role of spin and flavor (electric charge, strangeness, *etc.*). For this purpose it is helpful to recall what you have learned in Quantum Mechanics about how to combine spins (consistently with the principles of QM). In particular, combining the angular momentum of two spin 1/2 particles can yield either spin 1 or 0, and the corresponding states have definite symmetry under the interchange of the two spins (symmetric for the spin 1 state and antisymmetric for the spin 0). Three spin 1/2 particles can combine to form either spin 3/2 or 1/2.

<u>ASIDE</u> Let us review a bit of what you learned in your Quantum Mechanics course. The arithmetic of the construction of the combined spin states (wave functions) is encoded in the so-called Clebsch-Gordan coefficients ($\langle s_1 m_1, s_2 m_2 | SM\rangle$, where s_1 and s_2 are the two spins, m_1 and m_2 are the spin components along say the $\hat{x}^3$ direction and S and M are the combined (total) spin and $\hat{x}^3$ component). A table of these coefficients is included at the end of this chapter. The action of the raising/lowering or "ladder" operators, $\hat{S}_\pm$, allows us to change the value of m for a fixed value of s. These operators obey the relation (the "ket" $|s,m\rangle$ represents a state of definite s and m)

$$\hat{S}_\pm |s,m\rangle = \sqrt{(s \mp m)(s \pm m + 1)}\hbar |s, m \pm 1\rangle \,. \tag{5.2.1}$$

Thus the (normalized) combined states of two spin 1/2 particles can be represented (in a hopefully familiar and obvious notation) as

$$|S,M\rangle = |1,1\rangle = |\uparrow\uparrow\rangle,\ |1,0\rangle = \left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)/\sqrt{2},\ |1,-1\rangle = |\downarrow\downarrow\rangle, \tag{5.2.2}$$

and

$$|0,0\rangle = \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)/\sqrt{2}\,. \tag{5.2.3}$$

In the language of Group Theory and Representations (see Chapter 10, here we are thinking of representations of either $SO(3)$ or $SU(2)$, which are identical for our current purposes) we call spin 1/2 a doublet representation (two states, spin up and spin down) and label it as $\underset{\sim}{2}$. Thus the combination and subsequent reduction (into irreducible representations) of two spin 1/2 states can be written $\underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\sim}{3} \oplus \underset{\sim}{1}$, where the triplet is the spin 1 state (corresponding to the 3 values of M) and the singlet is the spin 0 state (with only a single M value). Since the singlet representation has only 1 element, it is, as expected, invariant under rotations as a rotation acts to transform the elements of a representation into each other (and no transformation can occur for a single element). On the other hand, the 3 elements of the spin 1 (vector) representation can be transformed into one another by a rotation.

The corresponding expression describing the combination of 3 spin 1/2 states is $\underset{\sim}{2} \otimes \underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\sim}{4} \oplus \underset{\sim}{2} \oplus \underset{\sim}{2}$. Thus we obtain the expected spin 3/2 state (the quartet) and *two* spin 1/2 states corresponding to differing internal symmetry; think of combining two of the spins to yield either spin 1 or spin 0 as above and then combine the third spin to yield spin 3/2 and two forms of spin 1/2 (two doublets - either $\underset{\sim}{3} \otimes \underset{\sim}{2} = \underset{\sim}{4} \oplus \underset{\sim}{2}$ or $\underset{\sim}{1} \otimes \underset{\sim}{2} = \underset{\sim}{2}$). (See the discussion in Chapter 11 for the "technology" of combining multiplets of the $SU(n)$ groups.)

Combining electric charge is easy: the charge of a hadron is just the algebraic sum of the charges of its constituent quarks. Combining the rest of the quarks' flavor is more complicated but we can use the same technology as we used to combine spin above (recall the words of the famous theoretical physicist Richard Feynman, "The same equations have the same solutions."). In particular, we can treat the nearly degenerate (nearly equal mass) u and d quarks as being members of a doublet representation of some underlying approximate $SU(2)$ symmetry and combine as above to find the corresponding flavor states. Since this structure parallels that for ordinary spin, the corresponding quantum number was historically labeled *isospin*, and we will use it to understand the flavor structure of the hadrons.

5.3 Mesons

Let us start with mesons and (for the moment) consider those "constructed" from just the lightest two flavors of quarks and antiquarks, $u, \bar{u}$ and $d, \bar{d}$. We know that the color wave function is the trivial color singlet $r\bar{r} + b\bar{b} + g\bar{g}$ (*i.e.*, the singlet, $\underset{\sim}{1}$, in the *color* $SU(3)$ representation decomposition $\underset{\sim}{3} \otimes \overline{\underset{\sim}{3}} = \underset{\sim}{8} \oplus \underset{\sim}{1}$). For the lowest mass states we expect that there is no orbital angular momentum and the spatial wave function is trivial. From our earlier discussion we know that the spin wave function can be reduced to either a spin singlet (spin 0 or scalar particle) or spin triplet (spin 1 or vector particle). Taking the u and d quarks to form an isospin doublet (as outlined above) we combine to find 4 flavor states as outlined in the first line of Table 5.2, where the lines are labeled by the number of strange plus antistrange quarks. Using the isospin language we can define these

$(\#s) + (\#\bar{s})$	$Q = 1$	$Q = 0$	$Q = -1$
0	$u\bar{d}$	$u\bar{u}$, $d\bar{d}$	$d\bar{u}$
1	$u\bar{s}$	$s\bar{d}$, $d\bar{s}$	$s\bar{u}$
2		$s\bar{s}$	

Table 5.2: Possible light quark-antiquark combinations

states in terms of "total isospin I" and a single component I_3. Historically the phases chosen for the antiquark doublet are $\bar{q}^T = (-\bar{d}, \bar{u})$ (note the minus sign which, although unmotivated here, does lead to some attractive features and you will see it in the literature, although sometimes with the minus sign attached to the $\bar{u}$ instead of the $\bar{d}$). Thus the corresponding mesons (of definite isopsin) are defined as

$$|I, I_3\rangle = |1,1\rangle = -|u\bar{d}\rangle,\ |1,0\rangle = (|u\bar{u}\rangle - |d\bar{d}\rangle)/\sqrt{2},\ |1,-1\rangle = |d\bar{u}\rangle, \tag{5.3.1a}$$

$$|0,0\rangle = (|u\bar{u}\rangle + |d\bar{d}\rangle)/\sqrt{2}\,. \tag{5.3.1b}$$

With little extra effort we can expand this discussion to include the *three* lightest quark flavors, u, d, and s, where we treat the s quark as an isospin singlet but carrying the "new" quantum number strangeness. Since it was mesons with the $\bar{s}$ antiquark that were observed first (and labeled as having strangeness +1), the s quark is actually defined to have strangeness -1! By simple counting we see that we have added 4 extra possibilities to Table 5.2 in the second and third lines. We can interpret the states in Table 5.2 more precisely if we take the 3 lightest quarks to be members of a triplet of an approximate $SU(3)$ flavor symmetry, which is apparently even more badly "broken" than the $SU(2)$ of isospin, *i.e.*, the s quark mass differs by about 100 MeV/c^2. Without further discussion here (see

Chapter 10) we will simply assert that, when we combine a triplet and anti-triplet of $SU(3)$, we obtain the expected 9 favor states in the form $\underset{\sim}{3} \otimes \overline{\underset{\sim}{3}} = \underset{\sim}{8} \oplus \underset{\sim}{1}$. (Note this is the same structure we asserted above for color $SU(3)$ - the same equations have the same solutions, only the labels change). In particular, one of the states is a singlet under this flavor $SU(3)$ and we see a new representation, the octet (8 different individual states characterized by differing values of I, I_3 and strangeness).

<u>ASIDE</u> Thus, as noted earlier, in "color space" $q\overline{q}$ includes both an color octet and a color singlet. The latter is just the singlet color state that we have claimed is the physical meson, while the former is precisely the description of the coupling between quarks and gluons, where gluons are members of the octet representation (the "adjoint" representation) of color $SU(3)$.

As suggested by Table 5.2, we want to be able represent the $\underset{\sim}{8}$ of meson flavor in a 2-D form using I_3 (to the right) and strangeness (up) as the axes. So instead of the 1-D structure for the representations of $SU(2)$ in Eq. (5.2.1), described by the single quantum number m, for $SU(3)$ the representations are 2-D and labeled by the *two* quantum numbers I_3 and S (strangeness). We begin by (graphically) representing the quark and antiquark, $\underset{\sim}{3}$ and $\overline{\underset{\sim}{3}}$, including the minus sign in the antiquark,

$$\begin{array}{cccc} \underset{\sim}{3} & d & & u \\ & & s & \end{array} \quad \text{and} \quad \begin{array}{ccccc} \overline{\underset{\sim}{3}} & & \bar{s} & & \uparrow S \\ & \bar{u} & & -\bar{d} & \rightarrow I_3 \end{array} .$$

Table 5.3: Flavor Triplet quarks and antiquarks

We then can combine the flavor (isopsin and strangeness) of the quark and antiquark in a meson to find the flavor $\underset{\sim}{1}$ in this notation as,

$$\underset{\sim}{1} \;\; (u\bar{u} + d\bar{d} + s\bar{s})/\sqrt{3}\,, \tag{5.3.2}$$

which we recognize as having the basic structure as the $SU(3)$ *color* singlet we mentioned earlier (*i.e.*, again "the same equations have the same solutions"). Finally the flavor $\underset{\sim}{8}$ can be represented as

$$\begin{array}{cccccc} \underset{\sim}{8} & & d\bar{s} & & u\bar{s} & \uparrow S \\ & d\bar{u} & (u\bar{u} - d\bar{d})/\sqrt{2} & & -u\bar{d} & \rightarrow I_3 \\ & & (u\bar{u} + d\bar{d} - 2s\bar{s})/\sqrt{6} & & & \\ & & s\bar{u} & & -s\bar{d} & \end{array} .$$

Table 5.4: Quark - antiquark octet

Note that there are three *orthogonal* strangeness zero, $I_3 = 0$ states. One is the $SU(3)$ singlet (Eq. 5.3.2), one is the isoscalar ($I = 0$) state in the $SU(3)$ octet (the lower line in the middle of Table 5.4), and the last is the $I_3 = 0$, $I = 1$ member of the $SU(3)$ octet (the upper line in the middle of Table 5.4).

ASIDE To make the concept of *orthogonal* more explicit think of a 3-D vector space defined by orthogonal "unit vectors" $|u\bar{u}\rangle$, $|d\bar{d}\rangle$ and $|s\bar{s}\rangle$, where

$$\langle u\bar{u}|u\bar{u}\rangle = 1, \, etc., \langle d\bar{d}|u\bar{u}\rangle = 0, \, etc.. \tag{5.3.3}$$

Then it follows that

$$\langle u\bar{u} - d\bar{d}|u\bar{u} + d\bar{d} + s\bar{s}\rangle = 1 - 1 = 0, \, etc.. \tag{5.3.4}$$

Since we expect that both the $SU(3)$ of flavor and the $SU(2)$ of isopsin are "broken" symmetries in nature, we should also expect that the quark content of the *physical* mesons with zero strangeness and zero electric charge may be mixtures of the combinations above.

Summarizing the above discussion there are $(3 \times 3 =)$ 9 different flavor possibilities for *both* spin 0 and spin 1. The actual observed lowest-mass mesons do indeed fall into just this pattern. In the same tabular form the names of the observed scalar mesons are

$$\begin{array}{llllllll}
0^- \underset{\sim}{8} & & K^0 & & & K^+ & & \uparrow S \\
 & \pi^- & & \pi^0\,(I=1) & & & \pi^+ & & \rightarrow I_3 \\
 & & & \eta\,(I=0) & & & & & . \\
 & & K^- & & & \bar{K}^0 & & \\
 & & & & & & & \\
0^- \underset{\sim}{1} & & & \eta'\,(I=0, S=0)\,. & & & &
\end{array}$$

Table 5.5: Scalar mesons

The corresponding "nonet" of vector mesons are labeled

$$\begin{array}{llllllll}
1^- \underset{\sim}{8} & & K^{*0} & & & K^{*+} & & \uparrow S \\
 & \rho^- & & \rho^0\,(I=1) & & & \rho^+ & & \rightarrow I_3 \\
 & & & \omega\,(I=0) & & & & & . \\
 & & K^{*-} & & & \bar{K}^{*0} & & \\
 & & & & & & & \\
1^- \underset{\sim}{1} & & & \phi\,(I=0, S=0)\,. & & & &
\end{array}$$

Table 5.6: Vector mesons

The "small print" associated with these tables includes the following. First, note that we have labeled these multiplets in terms of the spin but with superscript "-", as in 0^- and 1^-. This serves to remind us, as we will discuss in more detail shortly, that these particles have negative intrinsic parity. We will come to understand this as arising from the fact that fermions and antifermions necessarily have *opposite* intrinsic parity. Thus a quark-antiquark pair with only trivial spatial wave function ($L = 0$) must have negative parity. Next, as suggested above, mixing is observed between the strangeless, chargeless states compared to the expectations expressed in Table 5.4. In particular, while the scalar mesons seem to match Table 5.4, the physical ϕ state seems to be essentially pure $s\bar{s}$, while the ω is $(u\bar{u} + d\bar{d})/\sqrt{2}$. As we will see, this last point is suggested by the fact that the ϕ decays dominantly into $K\bar{K}$ states. Finally the structure of decays of the neutral kaons is a special story unto itself and illustrates the "near" conservation of the combined symmetry CP (parity and charge conjugation) by the weak interactions.

These states (particles) are listed again in Table 5.7 for the scalars, along with their dominant decay modes and quark content, while Table 5.8 provides the same information for light spin one mesons.[4] These are the lightest mesons.

meson	**mass**	**lifetime**	**dominant decays**	**quark content**
π^0	135.0 MeV	8.5×10^{-17} s	$\gamma\gamma$	$\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$
π^+	139.6 MeV	2.6×10^{-8} s	$\mu^+\nu_\mu$	$u\bar{d}$
π^-	139.6 MeV	2.6×10^{-8} s	$\mu^-\bar{\nu}_\mu$	$d\bar{u}$
K^+	493.7 MeV	1.2×10^{-8} s	$\mu^+\nu_\mu$, $\pi^+\pi^0$	$u\bar{s}$
K^-	493.7 MeV	1.2×10^{-8} s	$\mu^-\bar{\nu}_\mu$, $\pi^-\pi^0$	$s\bar{u}$
K^0, $\bar{K}^0$	497.6 MeV			$d\bar{s}$, $-s\bar{d}$
$K^0_{\rm S}$	497.6 MeV	8.95×10^{-11} s	$\pi^+\pi^-, \pi^0\pi^0$	$\frac{1}{\sqrt{2}}(d\bar{s} - s\bar{d})$
$K^0_{\rm L}$	497.6 MeV	5.1×10^{-8} s	$3\pi^0, \pi^+\pi^-\pi^0$, $\pi^\pm e^\mp\nu_e$, $\pi^\pm\mu^\mp\nu_\mu$	$\frac{1}{\sqrt{2}}(d\bar{s} + s\bar{d})$
η	547.9 MeV	5.1×10^{-19} s	$\gamma\gamma$, $\pi^+\pi^-\pi^0$, $\pi^0\pi^0\pi^0$	$\approx \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$
η'	957.8 MeV	3.3×10^{-21} s	$\pi^+\pi^-\eta$, $\rho^0\gamma$, $\pi^0\pi^0\eta$	$\approx \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$

Table 5.7: Light spin zero, parity odd mesons.

meson	**mass**	**lifetime**	**dominant decays**	**quark content**
ρ^+, ρ^0, ρ^-	775.5 MeV	4.4×10^{-24} s	$\pi\pi$	$u\bar{d}, \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}), d\bar{u}$
ω	782.7 MeV	7.8×10^{-23} s	$\pi^+\pi^-\pi^0$	$\approx \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$
K^{*+}, K^{*-}	891.7 MeV	1.3×10^{-23} s	$K\pi$	$u\bar{s}$, $s\bar{u}$
$K^{*0}, \bar{K}^{*0}$	895.9 MeV	1.4×10^{-23} s	$K\pi$	$d\bar{s}$, $s\bar{d}$
ϕ	1019.5 MeV	1.5×10^{-22} s	K^+K^-, $K^0_{\rm L}K^0_{\rm S}$	$\approx s\bar{s}$

Table 5.8: Light spin one, parity odd mesons.

[4] As noted earlier Tables 5.7 and 5.8 list parity odd mesons, as this is the parity of quark-antiquark bound states with no orbital excitation. We will discuss parity assignments in the next chapter. Note that the mass values listed in these and subsequent tables should really have units of MeV/c^2 (or the column should be labeled "rest energy" instead of "mass"). We will become increasingly sloppy about this distinction, since one can always insert a factor of c^2, as needed, to convert mass to energy or vice-versa, and in the end we want to set $c = 1$.

In general, and as expected, mesons containing strange quarks are heavier than those with no strange quarks. But among the neutral, $S = 0$ mesons, it is noteworthy (and we have already noted it) that, while the η and η' in Table 5.7 have the naively expected quark content, the ω and ϕ in Table 5.8 do not. This reflects the possibility of quantum mechanical mixing among states with identical quantum numbers. In other words, eigenstates of the Hamiltonian can be linear combinations of basis states which have simple quark content. The form of this mixing will be discussed in more detail later, but the important conclusion here is that the basic description of mesons as bound states of quarks works! Finally It is also worthwhile noting the realization in Table 5.7 of the previous comment about the decays of the neutral kaons ($K^0, \bar{K}^0$). As we will discuss in more detail shortly the weak interactions (and these are weak decays since we must change the flavor of the strange quark) approximately respect CP. Hence the neutral kaons dominantly decay through states of definite CP. The "even" CP state (CP eigenvalue $+1$) is allowed to decay into 2 pions and decays more quickly. Hence the label $K^0_{\mathrm{S(hort)}}$. The CP odd state (CP eigenvalue -1) labeled $K^0_{\mathrm{L(ong)}}$ decays more slowly into 3 pions or the more familiar states with leptons.

5.4 Baryons

One can go through a similar exercise for baryons. The primary differences are the further algebraic complexities of dealing with 3 instead of 2 quarks, and the fact that we are now dealing with 3 *identical* fermions. The exclusion principle now plays a role. In particular, the overall wavefunction describing the 3 quarks must be *anti*-symmetric with respect to the interchange of (all of) the quantum numbers of *any* pair of quarks. So, by way of introduction, we will quickly mention the expected behavior in color, space, spin and flavor and then we will go through the construction of baryons in more detail.

In the same notation we used for mesons (see Chapter 11), we find that combining 3 color triplets yields

$$\underset{\sim}{3} \otimes \underset{\sim}{3} \otimes \underset{\sim}{3} = \underset{\sim}{10} \oplus \underset{\sim}{8} \oplus \underset{\sim}{8} \oplus \underset{\sim}{1} \,. \tag{5.4.1}$$

Thus the expected $3^3 = 27$ color states break-up into a "decuplet" ($\underset{\sim}{10}$), two octets ($\underset{\sim}{8}$), and a singlet ($\underset{\sim}{1}$). The most important feature of these representations is that the decuplet is *symmetric* under the interchange of the color quantum number of *any* pair of quarks, while the singlet is purely *antisymmetric*.[5] The two octets have (different) "mixed" symmetry for different quark pairs (see the discussion below). Most importantly, by our rule that the only viable physical states are color singlets, we choose to put the 3 quarks in the antisymmetric color $\underset{\sim}{1}$. Since Pauli requires the complete wave function to be antisymmetric, the remaining bits of the wavefunction must be overall *symmetric*.

Since we are interested (first) in the least massive baryons, we take the spatial wavefunction to be symmetric (and trivial) with $L = 0$. Thus, due to the statistics of fermions, the combined spin and flavor wavefunctions of the lowest mass baryons must be *symmetric* under the interchange of any pair of quarks.

[5]The structure of this antisymmetric 3-quark color wave function is the (hopefully) familiar antisymmetric form in terms of the $SU(3)$ structure constant, $\epsilon_{jkl} q_j q_k q_l$ (ϵ_{jkl} is antisymmetric under the exchange of any pair of indices). We can tell this is a singlet because there are no "left-over" indices and this state cannot be transformed.

baryon	**mass**	**lifetime**	**dominant decays**	**quark content**
p	938.3 MeV	stable	—	uud
n	939.6 MeV	8.8×10^{2} s	$pe^-\bar{\nu}_e$	udd
Λ	1115.7 MeV	2.6×10^{-10} s	$p\pi^-, n\pi^0$	uds
Σ^+	1189.4 MeV	0.80×10^{-10} s	$p\pi^0, n\pi^+$	uus
Σ^0	1192.6 MeV	7.4×10^{-20} s	$\Lambda\gamma$	uds
Σ^-	1197.4 MeV	1.5×10^{-10} s	$n\pi^-$	dds
Ξ^0	1314.9 MeV	2.9×10^{-10} s	$\Lambda\pi^0$	uss
Ξ^-	1321.7 MeV	1.6×10^{-10} s	$\Lambda\pi^-$	dss

Table 5.9: Light spin 1/2, parity even baryons.

Combining three spin 1/2 objects can yield either spin 1/2 or 3/2, or in our new notation

$$\underset{\sim}{2} \otimes \underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\sim}{4} \oplus \underset{\sim}{2} \oplus \underset{\sim}{2}\,, \qquad (5.4.2)$$

where the spin 3/2 is the quartet that is again the symmetric state. The two spin 1/2 doublets, like the color octets above, are of mixed symmetry. To illustrate that point more explicitly we can assemble the 3 quark state by first putting together 2 quarks. Now the spin arithmetic is the same as for the quark-antiquark above, $\underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\sim}{3} \oplus \underset{\sim}{1}$, where the triplet is symmetric under interchange of the 2 quarks and the singlet is antisymmetric. To be explicit, label the first two quarks as 1 and 2, $\underset{\sim}{2}_1 \otimes \underset{\sim}{2}_2 = \underset{\sim}{3}_{S12} \oplus \underset{\sim}{1}_{A12}$ with S for symmetric and A for antisymmetric. Now include quark 3 to yield $\underset{\sim}{3}_{S12} \otimes \underset{\sim}{2}_3 = \underset{\sim}{4}_{S123} \oplus \underset{\sim}{2}_{S12,3}$ and $\underset{\sim}{1}_{A12} \otimes \underset{\sim}{2}_3 = \underset{\sim}{2}_{A12,3}$. The differing *mixed* qualities of the resulting symmetries for the doublets should be clear from the following more explicit version of Eq. (5.4.2)

$$\underset{\sim}{2}_1 \otimes \underset{\sim}{2}_2 \otimes \underset{\sim}{2}_3 = \underset{\sim}{4}_{S123} \oplus \underset{\sim}{2}_{S12,3} \oplus \underset{\sim}{2}_{A12,3}\,. \qquad (5.4.3)$$

Similar mixed underlying symmetry is what distinguishes the $SU(3)$ octets above.

baryon	**mass**	**lifetime**	**dominant decays**	**quark content**
$\Delta^-, \Delta^0, \Delta^+, \Delta^{++}$	1232 MeV	6×10^{-24} s	$p\pi$, $n\pi$	ddd, udd, uud, uuu
$\Sigma^{*-}, \Sigma^{*0}, \Sigma^{*+}$	1385 MeV	2×10^{-23} s	$\Lambda\pi$	dds, uds, uus
Ξ^{*-}, Ξ^{*0}	1530 MeV	7×10^{-23} s	$\Xi\pi$	dss, uss
Ω^-	1672 MeV	0.82×10^{-10} s	ΛK^-, $\Xi\pi$	sss

Table 5.10: Light spin 3/2, parity even baryons.

As expected the lightest observed baryons are, in fact, either spin 1/2 or spin 3/2. Tables 5.9 and 5.10 list the lightest spin 1/2 and spin 3/2 baryons, respectively. The intrinsic parity of these states is the same as the intrinsic parity as a quark, which, by *convention*, is chosen to be positive. Finally, we must consider the $SU(3)$ flavor structure. Again we note that "the same equations have the same solutions" and we use the same representation structure as for the $SU(3)$ of color above in Eq. (5.4.1), $\underset{\sim}{3}\otimes\underset{\sim}{3}\otimes\underset{\sim}{3} = \underset{\sim}{10}\oplus\underset{\sim}{8}\oplus\underset{\sim}{8}\oplus\underset{\sim}{1}$. The only step remaining is to pull the various factors together while ensuring overall antisymmetry under interchange of any 2 quarks.

We now discuss that final step in some detail (reviewing some of our previous discussions). Our goal is to understand the experimental observations summarized in Tables 5.9 and 5.10. In particular, Table 5.10 shows that the ten lightest $J = 3/2$ baryons form a decuplet representation of the $SU(3)$ of flavor. This simply corresponds to matching the *symmetric* spin state (the $\underset{\sim}{4}$) with the *symmetric* flavor state (the $\underset{\sim}{10}$) (with the *antisymmetric* color $\underset{\sim}{1}$ state and the trivial but *symmetric* $L = 0$ spatial state). From the masses listed in Table 5.10 one sees that the Σ^* baryons, which contain one strange quark, are heavier than the Δ baryons, which contain only u and d quarks, by about 150 MeV. The Ξ^* baryons, which contain two strange quarks are heavier than the Σ^* by an additional ≈ 150 MeV, and the Ω^- baryon, containing three strange quarks, is yet heavier by about the same increment. This is consistent with our expectations that substituting heavier quarks for lighter quarks should increase the mass of bound states (by approximately the mass change of the substituted quark), since the binding dynamics due to the color interactions remains the *same* independent of the quark flavors involved.

As Table 5.9 shows, there are only eight light $J = 1/2$ baryons and they form a flavor $SU(3)$, spin 1/2 baryon *octet*. There is only *one* combination of flavor octet and spin doublet with the correct overall symmetric behavior, which we then combine (as above) with the *antisymmetric* color $\underset{\sim}{1}$ state and the trivial but *symmetric* $L = 0$ spatial state. As noted earlier, this difference between $J = 3/2$ and $J = 1/2$ bound states can be understood as a consequence of the Pauli exclusion principle.

5.5 Baryon wavefunctions

To understand how the fermionic nature of quarks produces the observed pattern of spin and flavor for the baryons, we first review in more detail the structure of the color wave function. Our goal is to characterize the form of the colorless bound state. What does this really mean? Just as you can think of a spin 1/2 particle as having a wavefunction which is a two-component complex vector,

$$\begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = \begin{pmatrix} \langle\uparrow|\psi\rangle \\ \langle\downarrow|\psi\rangle \end{pmatrix}, \qquad (5.5.1)$$

the wavefunction of a quark (of definite flavor and spin) is a three-component vector in "color-space",

$$\vec{\Psi} \equiv \begin{pmatrix} \psi_{\rm r} \\ \psi_{\rm g} \\ \psi_{\rm b} \end{pmatrix} = \begin{pmatrix} \langle{\rm red}|\psi\rangle \\ \langle{\rm green}|\psi\rangle \\ \langle{\rm blue}|\psi\rangle \end{pmatrix}. \qquad (5.5.2)$$

The component $\psi_{\rm r}$ gives the amplitude for the quark color to be red, $\psi_{\rm g}$ is the amplitude to be green, *etc.* The assertion that hadrons must be "colorless" really means that the multi-quark wavefunction must not depend on the choice of basis in three-dimensional "color" space, *i.e.*, is unchanged by rotations in color space. Since the quark (color) wavefunction is a three-component vector, to build a colorless state from three quarks, A, B and C, one must combine the three color vectors describing the individual quarks, $\vec{\psi}_A$, $\vec{\psi}_B$, and $\vec{\psi}_C$, in such a way that the result is basis independent. In practice this means that the three three-component color vectors must be combined to yield an expression with *no* left-over color indices, *i.e.*, no index on which a color transformation could act.

This may sound peculiar, but the mathematical problem is the same as asking how to build a rotationally-invariant scalar from three (ordinary) 3-vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$, in such a way that the result is a linear function of each of the vectors. You already know the (essentially unique) answer, namely the scalar triple-product of the three vectors, $\vec{A}\cdot(\vec{B}\times\vec{C})$. This triple product may be expressed in a variety of ways, including as the determinant of the components,

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \det\begin{vmatrix} A_1\, B_1\, C_1 \\ A_2\, B_2\, C_2 \\ A_3\, B_3\, C_3 \end{vmatrix} = \epsilon_{ijk}\, A_i\, B_j\, C_k\,. \tag{5.5.3}$$

In the last expression ϵ_{ijl} is the totally antisymmetric (3x3x3) tensor which equals +1 when (ijk) is any cyclic permutation of (123), -1 when (ijk) is any cyclic permutation of (321), and zero otherwise.[6] Recall that a determinant changes sign if any two columns (or rows) are interchanged. Consequently, the triple product is antisymmetric under any interchange of two of the vectors, which is encoded in the antisymmetry of the ϵ_{ijk} symbol. This is exactly the antisymmetric character that we claimed above for the color singlet state that arises when we combine 3 color triplets, $\underset{\sim}{3}\otimes\underset{\sim}{3}\otimes\underset{\sim}{3} = \underset{\sim}{10}\oplus\underset{\sim}{8}\oplus\underset{\sim}{8}\oplus\underset{\sim}{1}$ (see footnote 5). As the *unique* 3x3x3 antisymmetric tensor it must also be the *structure constant* for both of the groups $SU(2)$ and $SO(3)$, as discussed in Chapter 10.3. Thus the color singlet state of 3 quarks has the form

$$\Psi_{qqq,\text{singlet}} = \epsilon_{ijk}\, \Psi_{q_1,i}\, \Psi_{q_2,j}\, \Psi_{q_3,k}\,, \tag{5.5.4}$$

where the Ψ_q's are the 3-component quark color (triplet) vectors of Eq. (5.5.2).

The complete wavefunction describing three quarks in a bound state must characterize not only the color of the quarks, but also their flavor, spin, and spatial location. To a good approximation, the wavefunction describing the lowest mass baryons will be a *product* of a color wavefunction (depending only on the color vectors as in Eq. (5.5.4)), a spatial wavefunction (depending only on the quark positions, or orbital angular momentum), and a flavor & spin wavefunction,

$$\Psi = \Psi_{\text{color}} \times \Psi_{\text{space}} \times \Psi_{\text{spin+flavor}}\,. \tag{5.5.5}$$

The essential point of the above discussion about triple products is that the color wavefunction for three quarks is antisymmetric under any interchange of the color vectors of any two quarks. The lightest hadrons which can be built from a given set of quark flavors will have a spatial wavefunction which is symmetric under interchange of quark positions, *i.e.*, it will have no orbital angular momentum. If this is not true (*i.e.*, if the orbital angular momentum is nonzero) then the wavefunction will have spatial nodes across which it changes sign, and this behavior invariably increases the kinetic energy of the state. The Pauli principle (or the fact that quarks are fermions) requires that the *total* wavefunction must be antisymmetric under interchange of any two quarks — which means simultaneous interchange of the positions, spins, flavors, and colors of the two quarks. Since the color part of the wavefunction must be antisymmetric, while the spatial part should be symmetric,

[6] The geometric definition of a cross product only makes sense for real vectors, but the expressions involving components of the vectors are equally well-defined for complex vectors.

this means that the flavor plus spin part must also be *symmetric* under permutations.[7] Thus the symmetry properties of the spin and flavor wave functions must be *matched* to each other in order to provide the overall symmetry. This is why the two quantities are coupled in Eq. (5.5.5) (*i.e.*, we were planning ahead).

For a spin 3/2 baryon, such as the Δ^{++}, the flavor structure of the wavefunction is trivial, and totally symmetric, since all three quarks are the same type, namely uuu. For the $S_z = 3/2$ state, the spin structure is also trivial, and totally symmetric, since all three quarks must individually have $S_z = 1/2$ if the total spin projection is 3/2.[8] Therefore the combined spin+flavor part of the wavefunction,

$$\Psi^{\Delta^{++}}_{\text{spin+flavor}} \sim (uuu) \times (\uparrow\uparrow\uparrow) \; , \tag{5.5.6}$$

satisfies the above condition of symmetry under permutation of quark spins and flavors. Analogous spin+flavor wavefunctions may be constructed for all baryons (with any spin projection) in the spin 3/2 decuplet. Just like the example in Eq. (5.5.6), these wavefunctions are independently symmetric under permutations of quark spins, or permutations of quark flavors. For completeness the quark content (in the same language as for the mesons) of the decuplet is presented in Table 5.11 and the particle names in Table 5.12 (which agree with the results already included in Table 5.10).

$\underset{\sim}{10}$	ddd		ddu		duu		uuu	$\uparrow S$
		dds		dus		uus		$\rightarrow I_3$
			dss		uss			
				sss				

Table 5.11: 3 quark decuplet

$\frac{3}{2}^+\underset{\sim}{10}$	Δ^-		Δ^0		Δ^+		Δ^{++}	$\uparrow S$
		Σ^{*-}		Σ^{*0}		Σ^{*+}		$\rightarrow I_3$
			Ξ^{*-}		Ξ^{*0}			
				Ω^-				

Table 5.12: Baryon decuplet

Note that, since both the $\underset{\sim}{10}$ of flavor $SU(3)$ and the $\underset{\sim}{4}$ of spin are individually symmetric wavefunctions, there is no real coupling of spin and flavor in this case (the overall wave function is just a simple product). But note also that the need to match the symmetry properties of the spin and

[7] This was actually an issue in the early days ($\sim$ 1970) when quarks had been postulated as the underlying degrees of freedom but the color quantum number had not yet appeared. The spatial, flavor and spin wavefunctions that matched the observed states are clearly symmetric, but the quarks are fermions. Where was Pauli? Then color appeared to save statistics *and* provide the needed interactions!

[8] This ignores the possibility of further constituents in the baryon in addition to the three up quarks. Using an improved description of the structure of baryons does not change the essential conclusions of the following discussion.

flavor wavefunctions means that, at least for the lowest mass baryons (with $L = 0$), there can be no spin 1/2 flavor decuplets ($\underset{\sim}{10}$) and no spin 3/2 flavor octets ($\underset{\sim}{8}$).

For $J = 1/2$ baryons, the situation is more complicated. As noted earlier the two possible spin doublets constructed from 3 spin 1/2 quarks have mixed symmetry with respect to interchanging pairs of the quark spins. Luckily the two flavor octets that can be constructed from 3 flavor triplets (quarks) have similar mixed symmetry and there is a combined spin 1/2, flavor octet 3 quark state that is overall symmetric under the simultaneous interchange of both the spin and flavor quantum numbers of any pair of quarks. To construct this state we recognize first that the case where all three quarks have the same flavor is *not* possible in the mixed symmetry octet. A flavor wavefunction such as uuu is totally symmetric. This explains why there are *no* light spin 1/2 baryons composed of three up (or three down, or three strange) quarks, in contrast to the case for spin 3/2 baryons. But if there are at least two distinct quark flavors involved, then it is possible to build a flavor wavefunction with the required symmetry. As an example, let us build a spin+flavor wavefunction for the proton. We need two u quarks and one d quark. A spin wavefunction of the form $(\uparrow\downarrow - \downarrow\uparrow)\uparrow$ describes a state in which the first two quarks have their spins combined to form an (antisymmetric) $S = 0$ state, so that adding the third spin yields a total spin of 1/2, as desired. Since this spin wavefunction is antisymmetric under interchange of the first two spins, we need a flavor wavefunction which is also antisymmetric under interchange of the first two quark flavors, namely $(ud - du)u$. If we multiply these, we have a spin+flavor wavefunction,

$$[\,(ud - du)u\,] \times [\,(\uparrow\downarrow - \downarrow\uparrow)\uparrow\,] = (udu - duu)\,(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)\,, \tag{5.5.7}$$

which is symmetric under combined spin and flavor exchange of the first two quarks. But we need a wavefunction which is symmetric under interchange of *any* pair of quark spins and flavors. This can be accomplished by simply adding terms which are related to the above by cyclic permutations (or in another words by repeating the above construction when it is the second and third, or first and third quarks which are combined to form spin zero). The result, which is unique up to an overall normalization factor, has the form

$$\Psi^{\text{proton}}_{\text{spin+flavor}} = (udu - duu)\,(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) + (uud - udu)\,(\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow) + (uud - duu)\,(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow)\,. \tag{5.5.8}$$

To be explicit we can also write this wavefunction with the terms multiplied out and normalized as

$$\begin{aligned}\Psi^{\text{proton}}_{\text{spin+flavor}} = \frac{1}{\sqrt{18}}\,[&2u\uparrow u\uparrow d\downarrow +2u\uparrow d\downarrow u\uparrow +2d\downarrow u\uparrow u\uparrow \\ &- u\downarrow u\uparrow d\uparrow -u\uparrow u\downarrow d\uparrow -u\downarrow d\uparrow u\uparrow \\ &-u\uparrow d\uparrow u\downarrow -d\uparrow u\uparrow u\downarrow -d\uparrow u\downarrow u\uparrow]\,.\end{aligned} \tag{5.5.9}$$

You can, and should, check that these expressions satisfy the required condition of symmetry under interchange of spins and flavors of *any* pair of quarks. It should be clear that in this case the spin and flavor structures are truly intertwined. The wavefunction in Eq. (5.5.9) represents, for the case of 2 u quarks and 1 d quark, the unique member of an $SU(3)$ flavor octet and spin doublet (with $S_3 = +1/2$) that is symmetric under the interchange of any pair of quarks.

For fun (*e.g.*, in the HW) try to generate the following wavefunction for the neutron.

$$\begin{aligned}\Psi^{\text{neutron}}_{\text{spin+flavor}} = \frac{1}{\sqrt{18}}\,[&2d\uparrow d\uparrow u\downarrow +2d\uparrow u\downarrow d\uparrow +2u\downarrow d\uparrow d\uparrow \\ &- d\downarrow d\uparrow u\uparrow -d\uparrow d\downarrow u\uparrow -d\downarrow u\uparrow d\uparrow \\ &-d\uparrow u\uparrow d\downarrow -u\uparrow d\uparrow d\downarrow -u\uparrow d\downarrow d\uparrow]\,.\end{aligned} \tag{5.5.10}$$

Similar constructions can be performed for all the other members of the spin 1/2 baryon octet. The expected quark content of the baryon octet is given in Table 5.13.

$$
\begin{array}{cccccccc}
\underset{\sim}{8} & & udd & & uud & & \uparrow S & \\
& sdd & & sud\,(I=1) & & suu & & \to I_3 \\
& & & sud\,(I=0) & & & & \\
& & ssd & & ssu & & &
\end{array}\,.
$$

Table 5.13: Baryon octet

The corresponding identification with the lowest mass baryons is provided in Table 5.14.

$$
\begin{array}{cccccccc}
\frac{1}{2}^{+}\,\underset{\sim}{8} & & n & & p & & \uparrow S & \\
& \Sigma^- & & \Sigma^0\,(I=1) & & \Sigma^+ & & \to I_3 \\
& & & \Lambda^0\,(I=0) & & & & \\
& & \Xi^- & & \Xi^0 & & &
\end{array}\,.
$$

Table 5.14: Baryon octet

One notable feature of the set of octet baryons, shown in Tables 5.13 and 5.14 , is the presence of two different baryons whose quark content is *sud*, specifically the Λ and the Σ^0. This is not inconsistent. When three distinct flavors are involved, instead of just two, there are more possibilities for constructing a spin+flavor wavefunction with the required symmetry. A careful examination (left as a problem) shows that there are precisely two independent possibilities, completely consistent with the observed list of spin 1/2 baryons.

The mass values in Table 5.9 show that for spin 1/2 baryons, just as for spin 3/2 baryons, baryons with strange quarks are heavier than those with only up and down quarks; each substitution of a strange quark for an up or down raises the energy of the baryon by roughly 130–250 MeV.

5.6 Baryon number

Baryon number, denoted B, is defined as the total number of baryons minus the number of antibaryons, similarly to how we defined lepton number L in Eq. (4.6.1). Since baryons are bound states of three quarks, and antibaryons are bound states of three antiquarks, baryon number is the same as the number of quarks minus antiquarks, up to a factor of three,

$$B = (\#\text{ baryons}) - (\#\text{ antibaryons}) = \frac{1}{3}\,\big[(\#\text{ quarks}) - (\#\text{ antiquarks})\big]\,. \tag{5.6.1}$$

All known interactions conserve baryon number.[9] High energy scattering processes can change the

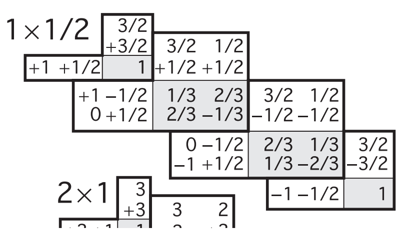
[9] This is not quite true. As with lepton number, the current theory of weak interactions predicts that there are

number of baryons, and the number of antibaryons, but not the net baryon number. For example, in proton-proton scattering, the reaction $p+p \to p+p+n+\bar{n}$ can occur, but not $p+p \to p+p+n+n$.

5.7 Hadronic decays

Turning to the decays of the various hadrons listed in Tables 5.7–5.10 , it is remarkable how much can be explained using a basic understanding of the quark content of the different hadrons together with considerations of energy and momentum conservation. This is essentially the statement that understanding the basic symmetry properties, *i.e.*, the conserved quantum numbers, will get you a long way in the world of particle physics. This discussion will also help to illustrate the basic structure of the Standard Model (SM).

As an example, consider the baryons in the spin 3/2 decuplet. The rest energy of the Δ baryons is larger than that of a nucleon by nearly 300 MeV. This is more than the $\approx$ 140 MeV rest energy of a pion, which is the lightest meson. Consequently, a Δ baryon can decay to a nucleon plus a pion via strong interactions, which is the dominant way to produce pions as long as the process does not change the number of quarks minus antiquarks of each quark flavor (*i.e.*, the strong interactions preserve the net flavor). Specifically, a Δ^{++} can decay to $p\pi^+$, a Δ^+ can decay to either $p\pi^0$ or $n\pi^+$, a Δ^0 can decay to $p\pi^-$ or $n\pi^0$, and a Δ^- can decay to $n\pi^-$. These are the (overwhelmingly) dominant decay modes observed. The short lifetime of Δ baryons, $\tau \simeq 6 \times 10^{-24}$ s or $c\tau \simeq 1.8$ fm, is also indicative of a decay via strong interactions. (After all, "strong" should mean rapid interactions!) To set the scale, note that the time for light to travel a fermi is of order 3×10^{-24} s. The Δ^- barely has time to "figure out" that it exists before it decays.



ASIDE: Let us take this opportunity to both study the isopsin structure of the Δ decays in detail and utilize the Table of Clebsch-Gordan coefficients appended at the end of this Chapter. The appropriate (upper left) section of that Table is shown to the left. The label on this section, "1 x 1/2", tells us this section describes in detail how to combine isospin (or ordinary spin) 1 (a triplet representation) with isospin 1/2 (a doublet representation), $\underset{\sim}{3} \otimes \underset{\sim}{2}$. We know that the resulting combined representations are an isospin 3/2 quartet and an isospin 1/2 doublet, $\underset{\sim}{3} \otimes \underset{\sim}{2} = \underset{\sim}{4} \oplus \underset{\sim}{2}$. This is precisely what we see along the uppermost line in each (funny shaped) box, 3/2 or 1/2. The next row of labels down tells us the specific value of I_3 (or J_3) for the joint representation. The labels to the left in the (funny shaped) boxes are the I_3 (or J_3) values labeling the specific states we are combining. Finally the numbers in the central, shaded regions of each of the (funny shaped) boxes are the actual Clebsch-Gordan coefficients. Strictly speaking, as explained near the top of the Table, the actual coefficient is the *square root* of the number in the Table, not including the sign. More specifically we interpret the upper left (funny shaped) box to mean $|I, I_3\rangle = |\frac{3}{2}, \frac{3}{2}\rangle = |1, 1\rangle|\frac{1}{2}, \frac{1}{2}\rangle$ or, in our particular application, $|\Delta^{++}\rangle = |\pi^+\rangle|p\rangle$. There is only *one* possible final state that conserves isospin. Now consider the more interesting case in the second (funny shaped) box down. We read the Table to mean $|\frac{3}{2}, \frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|1, 1\rangle|\frac{1}{2}, -\frac{1}{2}\rangle + \sqrt{\frac{2}{3}}|1, 0\rangle|\frac{1}{2}, \frac{1}{2}\rangle$ or,

processes which can change baryon number (while conserving $B{-}L$). The rate of these processes is so small that baryon number violation is (so far) completely unobservable.

in our case,

$$|\Delta^+\rangle = \sqrt{\frac{1}{3}}|\pi^+\rangle|n\rangle + \sqrt{\frac{2}{3}}|\pi^0\rangle|p\rangle\,. \tag{5.7.1}$$

Thus, when a Δ^+ decays (strongly), 1/3 of the time it decays to π^+n, while 2/3 of the time it decays to $\pi^0 p$. Note in particular that, in going from the *amplitude* with the Clebsch-Gordan coefficient (with the square root) to the *probability*, which is the amplitude *squared*, we *square* the Clebsch-Gordan coefficient. The second column in the second (funny shaped) box describes the case with total isospin 1/2, which plays no role here as we are considering the strong decay of an isospin 3/2 particle and isospin is conserved by the strong interactions. Next we move down to the third (funny shaped) box to learn that $|\frac{3}{2},-\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|1,0\rangle|\frac{1}{2},-\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}|1,-1\rangle|\frac{1}{2},\frac{1}{2}\rangle$ or, in our case,

$$|\Delta^0\rangle = \sqrt{\frac{2}{3}}|\pi^0\rangle|n\rangle + \sqrt{\frac{1}{3}}|\pi^-\rangle|p\rangle\,. \tag{5.7.2}$$

So a Δ^0 decays (strongly) 2/3 of the time to $\pi^0 n$, while 1/3 of the time it decays to $\pi^- p$. Finally the last (funny shaped) box tells us that the Δ^- decays *uniquely* to $\pi^- n$, as we have noted above. Being able to read the Table of Clebsch-Gordan coefficients is an extremely useful skill and we should all practice it.

Returning to the general discussion, we note that the lifetime of the Δ corresponds to a decay (or resonance) width $\Gamma_\Delta = \hbar/\tau \approx 120$ MeV, which is 10% of the rest energy of this baryon. (We can think of the $\pi - p$ scattering amplitude as having a pole in the complex energy plane at the mass of the Δ, where the pole is off the real axis by the amount Γ_Δ.) While there is a potential issue concerning how long a particle must live in order to be called a particle, we should rather focus on the point that almost all the particles (states) we know about decay eventually (*i.e.*, nearly all of the poles are somewhat off the real axis). The real issue is the practical, experimental one. Does the state live long enough to be detected in the detector before it decays. Or is the lifetime so short that we see evidence of the state's existence *only* from a bump in an interaction rate (*i.e.*, we literally detect the pole in the complex energy plane). This is essentially the distinction in the tables provided by the PDG. In the former case, lifetimes are reported, while in the latter case the width (of the peak) is reported. Thus the PDG reports what is actually measured.

The Σ^* and Ξ^* baryons also have very short lifetimes, on the order of a few times 10^{-23} s. The Σ^* contains one strange quark. The Σ^*'s mass of 1285 MeV/c^2 is larger than the 1116 MeV/c^2 mass of the Λ, the lightest baryon containing a strange quark, by more than the mass of a pion. So strong interactions can cause a Σ^* to decay to a Λ plus a pion, which is the dominant observed decay. Similarly, Ξ^* baryons, containing two strange quarks, can decay via strong interactions to a Ξ (the lightest doubly strange baryon) plus a pion.

Looking ahead to Chapter 6.9 on parity, it is worthwhile noting a characteristic feature of these strong decays of members of the flavor decuplet (the flavor $\underset{\sim}{10}$) into a member of the nucleon octet (the flavor $\underset{\sim}{8}$) plus a pion. In J^P notation (spin and intrinsic parity) we have a $\frac{3}{2}^+$ going into a $\frac{1}{2}^+$ plus a 0^-. Thus naively neither the spin nor parity match, $\frac{3}{2} \neq \frac{1}{2} + 0$ and $+ \neq + \times -$. But total angular momentum is conserved by all interactions and so we conclude that there must be one unit of *orbital* angular momentum in the final state, $L = 1\hbar$ (the proton and pion orbit about each other). This now allows the final state to have total angular momentum $\frac{3}{2}(= \frac{1}{2} + 1)$ and adds an extra parity factor of $(-1)^L$ = -1. Thus everything works for a strong decay.

The final member of the $J = 3/2$ decuplet, the Ω^- baryon, cannot decay via strong interactions to a lighter baryon plus a pion, because there are no lighter baryons containing three strange quarks (and strong interactions preserve the net number of strange quarks). It could, in principle, decay via strong interactions to a Ξ baryon (containing two strange quarks) and a K meson (containing one strange quark) — but it doesn't have enough energy. Its mass of 1672 MeV/c^2 is less than the sum of Ξ plus K masses. In fact, the Ω^- baryon cannot decay via *any* strong interaction process. Nor can it decay via electromagnetic processes, which also preserve the net flavor. But weak interactions are distinguished by the fact that they can *change* quarks of one flavor into a different flavor. Consequently, the Ω^- baryon can decay via weak interactions to a lighter baryon plus a meson with only 2 strange quarks remaining. The dominant decays involve the conversion of one strange quark into an up quark, leading to final states consisting of a Λ baryon plus a K^- meson, a Ξ^0 baryon plus a π^-, or a Ξ^- plus a π^0. The Ω^- was, in some sense, easy to detect due to its characteristic "cascade" decay (first seen in a bubble chamber photograph)

$$\begin{aligned} \Omega^- \to & \Xi^0 \pi^- \\ & \searrow \Lambda^0 \pi^0 \\ & \qquad \searrow p\pi^- \, . \end{aligned} \tag{5.7.3}$$

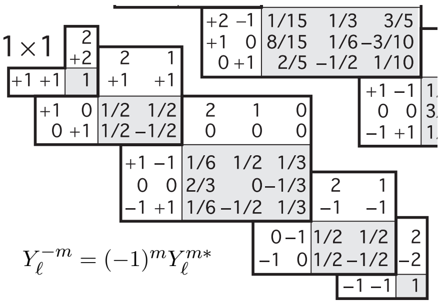
So overall the process in Eq.(5.7.3) is $\Omega^- \to p\,\pi^-\pi^-\pi^0$. Note that these final states conserve baryon number and electric charge, and are allowed by energy conservation. The 10^{-10} s lifetime of the Ω^- is much *longer* than a typical strong interaction decay time, and is indicative of a weak interaction process.

Similar reasoning can be applied to the $J = 1/2$ baryons. The proton is (apparently) stable, while all the other members of the octet decay via weak interactions — except for the Σ^0 which can decay to a Λ plus a photon via electromagnetic interactions. Note that the 7×10^{-20} s lifetime of the Σ^0 is much shorter than a weak interaction lifetime, but is longer than typical strong interaction lifetimes. (In a very real sense the electromagnetic interactions are stronger than the weak interactions but weaker the the strong interactions.) The lifetimes of the Λ, Ξ, and $\Sigma^\pm$ baryons are all around 10^{-10} s, typical of weak interaction decays. The 900 second lifetime of the neutron is vastly longer than a normal weak interaction lifetime. This reflects the fact that neutron decay is just barely allowed by energy conservation. The mass of the final proton plus electron (and antineutrino) is so close to the mass of the neutron that only about 8 MeV, or less than 0.1% of the rest energy of neutron, is available to be converted into the kinetic energy of the decay products.

Before ending this discussion, we can test our newly acquired understanding of conserved quantum numbers and decays by applying it to the case of mesons. Just like the spin 3/2 baryons tend to have strong interaction (*i.e.*, fast) decays into the spin 1/2 barons plus a pion, the vector mesons of Table 5.6 have strong decays into the corresponding scalar mesons (*i.e.*, the states with the same number of strange quarks) plus a pion. Note, in particular, that all of the lifetimes are of order 10^{-22} to 10^{-24} seconds, typical strong (short) lifetimes. Again we should consider the correlation of spin and parity for the decay of a 1^- meson into two 0^- mesons. As with the strong baryon decays above we need 1 unit of orbital angular momentum to allow the conservation of total angular momentum, $1 = 0 + 0 + 1$, which is again just what we need to conserve parity, $-1 = -1 \times -1 \times -1$.

Also worthy of note is that the zero-strangeness, isoscalar state ω decays into 3 pions, while the similar zero-strangeness neutral member of the isovectors, the ρ^0, decays into 2 pions. There is a combined transformation of charge conjugation (C) plus an isospin rotation (historically called "G-parity" and described in more detail in the next Chapter) under which pions and the ω are odd

(-1), while the ρ's are even. Since the strong interactions respect this symmetry, the decay of the ρ^0 must be into an even number of pions, while the ω is into an odd number of pions. More generally a $q\bar{q}$ state made of u and d quarks has the following eigenvalue under G, $(-1)^{L+S+I}$, which yields, as noted above, (-1) for pions ($L = S = 0, I = 1$), (+1) for the ρ ($L = 0, S = I = 1$), and (-1) for the ω ($L = I = 0, S = 1$).



A further detail (not shown explicitly in the table) is that the two pion decay of the neutral ρ is $\rho^0 \to \pi^+ + \pi^-$ and *not* $\rho^0 \to \pi^0 + \pi^0$. This is an interesting realization of the how two isovectors (the pions) combine to form another isovector (the ρ). It supplies another opportunity to look in detail at the table of Clebsch-Gordan coefficients that appears at the end of this Chapter. The appropriate subsection (middle left) of the Table is shown to the left here. For the case of two vectors combining to form another vector ($1 \times 1 \to 1$) where we want to consider the neutral state of the final vector ($M = 0$), we focus on the middle column of the middle (funny shaped) box. We see from the 0 at the center of this box that there is no coupling to the neutral states of the original vectors ($m_1 = m_2 = 0$), *i.e.*, $\rho^0 \nrightarrow \pi^0 + \pi^0$. Instead there are equal couplings, up to the sign, to the 2 charged states,

$$|\rho^0\rangle = \sqrt{\frac{1}{2}}|\pi^+\rangle|\pi^-\rangle - \sqrt{\frac{1}{2}}|\pi^-\rangle|\pi^+\rangle \,. \tag{5.7.4}$$

ASIDE: Another way to look at the absence of this decay, $\rho^0 \nrightarrow \pi^0 + \pi^0$, which illustrates again the "beautiful" self-consistency of physics, is to consider the symmetry required for identical bosons (see problem 11.1 in Das and Ferbel). We already noted that the two pion final state in ρ decay must correspond to $L = 1$ in order to conserve both total angular momentum and parity. Further, a two particle state with relative orbital angular momentum 1 is *antisymmetric* under the interchange of the two particles. This is why it adds a factor of (-1) to the overall parity of the state. On the other hand a two π^0 state must be *symmetric* under the interchange of the 2 π^0's, since they are *identical* bosons. Hence there can be no $L = 1$ two π^0 state and the decay cannot occur. For the state composed of two oppositely charged pions *both* the isospin wave function of Eq. (5.7.4) and the $L = 1$ spatial wave function are antisymmetric under the interchange of the pions yielding the required overall symmetric state.

For the scalar mesons of Table 5.5 energy conservation rules out any strong decays. The (neutral) π^0, η and η' all exhibit electromagnetic decays with photons in the final state and lifetimes of order 10^{-20} seconds (like the Σ^0). The kaons need to convert a strange quark into an up or down quark and so decay weakly with a 10^{-8} second lifetime (except the K_S, which is still a weak decay but slightly faster). Finally the charged pions decay weakly into leptons *only* with a similar 10^{-8} second lifetime.

To complete this discussion of decays we summarize in Table 5.15 the various additive and multiplicative quantum numbers and which interactions conserve them. We will discuss the related symmetries in more detail in the next chapter.

You are encouraged to look at the much more extensive listing of information about known mesons and baryons at the Particle Data Group website . Pick a few particles which have not been discussed above, and see if you can predict the dominant decay modes using the ideas we have discussed in this chapter.

Conserved quantity	**Strong**	**EM**	**Weak**
Additive			
Energy-momentum	Yes	Yes	Yes
Angular Momentum	Yes	Yes	Yes
Electric Charge	Yes	Yes	Yes
Baryon Number	Yes	Yes	Yes
Lepton Number	Yes	Yes	Yes
Quark Flavor	Yes	Yes	No
Isospin	Yes	No	No
Multiplicative			
Partity - P	Yes	Yes	No
Charge Conjugation - C	Yes	Yes	No
Time Reversal - T (or CP)	Yes	Yes	$\sim 10^{-3}$ viol
CPT	Yes	Yes	Yes
G - parity	Yes	No	No

Table 5.15: Conserved quantum numbers.

It is worthwhile emphasizing again what "conserving" means in this context. Consider the decay of one particle into 2 particles, $A \to B + C$. For the scalar quantities like electric charge, baryon number and lepton number (or 4-vectors like energy-momentum) conserving means that the value of this quantity for particle A is equal to the simple sum of these quantities for particles B and C. For more complex quantities like angular momentum, isospin and quark flavor (with nontrivial group theory structure), we need to think a bit harder. Now conserving means that the "total" quantum number like J^2 (*i.e.*, defining the representation) and the simple additive quantities like J_3 (*i.e.*, defining the individual element of the representation) must match before and after the decay. Since the total angular momentum before the decay is just the spin of A, this representation must match one of the possible angular momentum representations that arise when we combine the spin of B with the spin of C and any possible *orbital* angular momentum of the BC pair. We have learned about Young diagrams (Chapter 11 and the sample problems in this chapter) preciously so that we can calculate which angular momentum representations can be present in the final state when we combine the spins of B and C and the orbital angular momentum ($\underset{\sim}{S}^B \otimes \underset{\sim}{S}^C \otimes \underset{\sim}{L}^{BC}$). Likewise, if we specify the spin polarization (S_3^A) in the initial state, it must be matched by that in the final state. Here the arithmetic is simple addition, $S_3^A = S_3^B + S_3^B + L_3^{BC}$. For isospin the calculation is very similar to angular momentum, since it again involves only $SU(2)$ representations (*i.e.*, the same equations have the same solutions). In fact, it is even simpler because there is *no* analogue of *orbital* angular momentum in isospin space! The specific element of the isospin representation is specified by I_3, which satisfies $I_3^A = I_3^B + I_3^C$. For the $SU(3)$ of quark flavor (for the u, d and s quarks) we must match representations of $SU(3)$ before and after the decay. The additive conservation is now of 2 quantum numbers, typically I_3 and strangeness (the number of strange quarks).

5.8 Sample calculations

At several points in this Chapter we used results for the combinations of multiplets of both $SU(2)$ and $SU(3)$. The "slickest" technique for calculating these results is the method of Young diagrams described in Chapter 11. Here we present a brief summary of those results and the reader is strongly encouraged to read Chapter 11 now! This will allow you to understand the (somewhat peculiar) notation used below.

Consider first combining 2 *fundamental* multiplets (representations) as in Eq. (11.2.7). In the Young diagram language we have

$$\square \otimes \square = \square\square \oplus \begin{array}{c}\square\\[-4pt]\square\end{array} . \tag{5.8.1}$$

To proceed we put the appropriate integers in the boxes, evaluate and divide by the "hooks", and calculate. For the $SU(2)$ case of adding 2 doublets (*e.g.*, 2 spin 1/2 fermions) we have

$$\underset{\sim}{2} \otimes \underset{\sim}{2} = \boxed{2} \otimes \boxed{2} = \begin{array}{|c|c|}\hline 2 & 3\\\hline\end{array} \oplus \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} = \frac{2\cdot 3}{2\cdot 1} \oplus \frac{2\cdot 1}{2\cdot 1} = \underset{\sim}{3} \oplus \underset{\sim}{1} . \tag{5.8.2}$$

For combining 2 triplets of $SU(3)$ (of color or flavor) we have

$$\underset{\sim}{3} \otimes \underset{\sim}{3} = \boxed{3} \otimes \boxed{3} = \begin{array}{|c|c|}\hline 3 & 4\\\hline\end{array} \oplus \begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array} = \frac{3\cdot 4}{2\cdot 1} \oplus \frac{3\cdot 2}{2\cdot 1} = \underset{\sim}{6} \oplus \bar{\underset{\sim}{3}} . \tag{5.8.3}$$

Note the (symmetry) distinction here between $\bar{\underset{\sim}{3}}$ and $\underset{\sim}{3}$ ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$ versus $\square$ and the corresponding distinction illustrated in Table 5.3, ∇ versus $\triangle$) that does not exist for the $SU(2)$ $\bar{\underset{\sim}{2}}$ and $\underset{\sim}{2}$, which are identical ($\square$ versus $\square$). This means that in the $SU(3)$ case we can obtain something new and different from combining a $\underset{\sim}{3}$ and $\bar{\underset{\sim}{3}}$ (*i.e.*, as distinct from combining two $\underset{\sim}{3}$'s). We have

$$\underset{\sim}{3} \otimes \bar{\underset{\sim}{3}} = \square \otimes \begin{array}{c}\square\\[-4pt]\square\end{array} = \begin{array}{l}\square\square\\[-4pt]\square\end{array} \oplus \begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\end{array} = \begin{array}{|c|c|}\hline 3 & 4\\\hline 2 & \multicolumn{1}{|c}{}\\\cline{1-1}\end{array} \oplus \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} = \frac{3\cdot 4\cdot 2}{3\cdot 1\cdot 1} \oplus \frac{3\cdot 2\cdot 1}{3\cdot 2\cdot 1} = \underset{\sim}{8} \oplus \underset{\sim}{1} . \tag{5.8.4}$$

Next consider combining three fundamental representations, where we must immediately distinguish $SU(2)$ and $SU(3)$ (we cannot antisymmetrize 3 objects if there are only 2 different kinds). For $SU(2)$ (3 fermions) we have, as in Eq. (5.4.2),

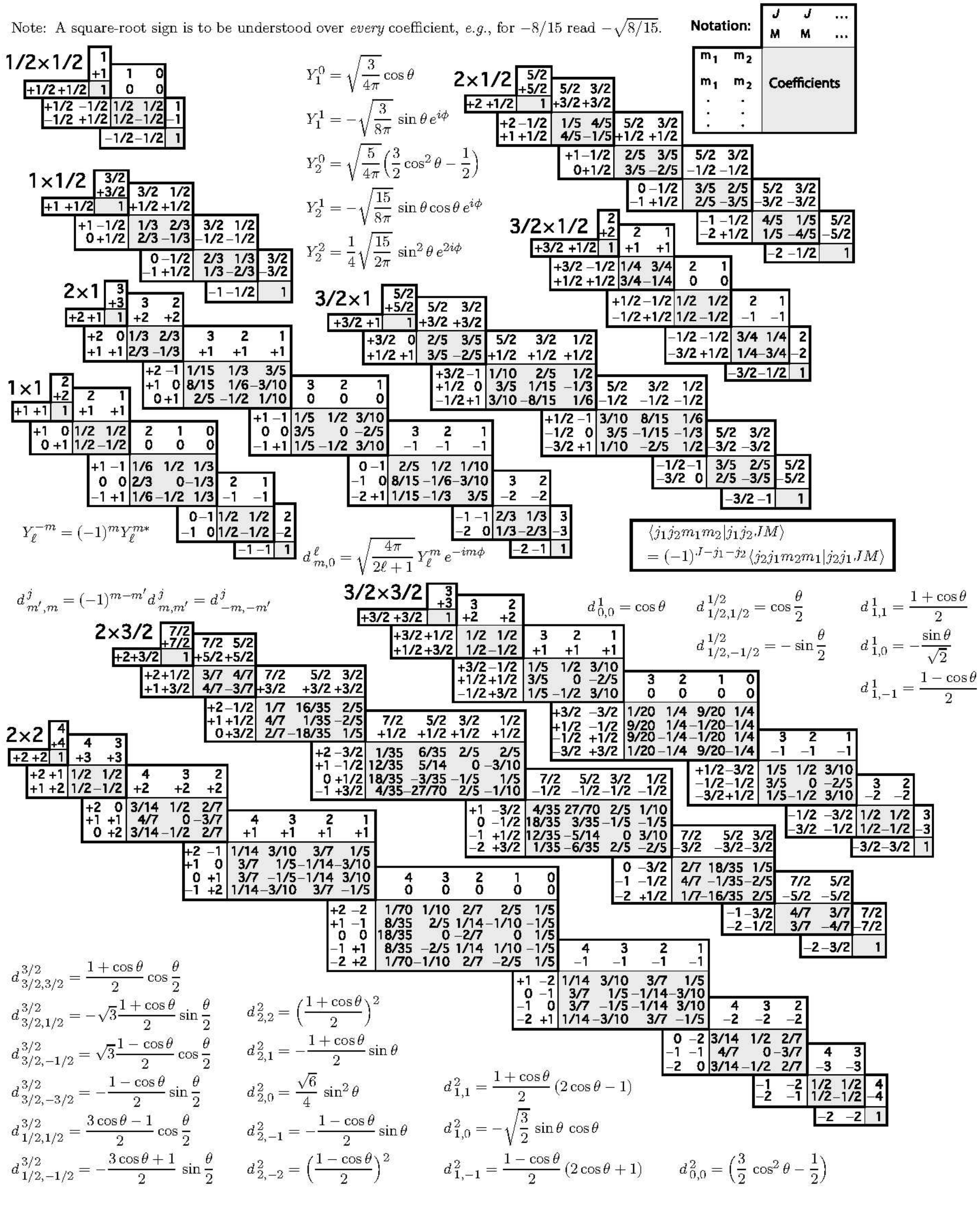
$$\begin{aligned}\boxed{2} \otimes \boxed{2} \otimes \boxed{2} &= \left(\square\square \oplus \begin{array}{c}\square\\[-4pt]\square\end{array}\right) \otimes \square = \square\square\square \oplus \begin{array}{l}\square\square\\[-4pt]\square\end{array} \oplus \begin{array}{l}\square\square\\[-4pt]\square\end{array}\\ &= \begin{array}{|c|c|c|}\hline 2 & 3 & 4\\\hline\end{array} \oplus \begin{array}{|c|c|}\hline 2 & 3\\\hline 1 & \multicolumn{1}{|c}{}\\\cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline 2 & 3\\\hline 1 & \multicolumn{1}{|c}{}\\\cline{1-1}\end{array} = \frac{2\cdot 3\cdot 4}{3\cdot 2\cdot 1} \oplus \frac{2\cdot 3\cdot 1}{3\cdot 1\cdot 1} \oplus \frac{2\cdot 3\cdot 1}{3\cdot 1\cdot 1}\\ &= \underset{\sim}{4} \oplus \underset{\sim}{2} \oplus \underset{\sim}{2} .\end{aligned} \tag{5.8.5}$$

For the distinct case of three $SU(3)$ triplets we have instead, as in Eq. (5.4.1), (note the differences from the $SU(2)$ case above)

$$
\begin{aligned}
\boxed{3}\otimes\boxed{3}\otimes\boxed{3} &= \left(\ydiagram{2}\oplus\ydiagram{1,1}\right)\otimes\ydiagram{1} = \ydiagram{3}\oplus\ydiagram{2,1}\oplus\ydiagram{2,1}\oplus\ydiagram{1,1,1} \\
&= \young(345)\oplus\young(34,2)\oplus\young(34,2)\oplus\young(3,2,1) \\
&= \frac{3\cdot 4\cdot 5}{3\cdot 2\cdot 1}\oplus\frac{3\cdot 4\cdot 2}{3\cdot 1\cdot 1}\oplus\frac{3\cdot 4\cdot 2}{3\cdot 1\cdot 1}\oplus\frac{3\cdot 2\cdot 1}{3\cdot 2\cdot 1} \\
&= \underset{\sim}{10}\oplus\underset{\sim}{8}\oplus\underset{\sim}{8}\oplus\underset{\sim}{1}\,. \qquad (5.8.6)
\end{aligned}
$$

40. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.



$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 40.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).