Chapter 3

Relativistic dynamics

A particle subject to forces will undergo non-inertial motion. According to Newton, there is a simple relation between force and acceleration,

$$\vec{f} = m\,\vec{a}\,, \tag{3.0.1}$$

and acceleration is the second time derivative of position,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2}\,. \tag{3.0.2}$$

There is just one problem with these relations — they are wrong! Newtonian dynamics is a good approximation when velocities are very small compared to c, but outside this regime the relation (3.0.1) is simply incorrect. In particular, these relations are inconsistent with our relativity postulates. To see this, it is sufficient to note that Newton's equations (3.0.1) and (3.0.2) predict that a particle subject to a constant force (and initially at rest) will acquire a velocity which can become arbitrarily large,

$$\vec{v}(t) = \int_0^t \frac{d\vec{v}}{dt'}\,dt' = \frac{\vec{f}}{m}\,t \longrightarrow \infty \quad \text{as } t \to \infty\,. \tag{3.0.3}$$

This flatly contradicts the prediction of special relativity (and causality) that no signal can propagate faster than c. Our task is to understand how to formulate the dynamics of non-inertial particles in a manner which is consistent with our relativity postulates (and then verify that it matches observation).

3.1 Proper time

The result of solving for the dynamics of some object subject to known forces should be a prediction for its position as a function of time. But whose time? One can adopt a particular reference frame, and then ask to find the spacetime position of the object as a function of coordinate time t in the chosen frame, $x^\mu(t)$, where as always, $x^0 \equiv c\,t$. There is nothing wrong with this, but it is a frame-dependent description of the object's motion.

For many purposes, a more useful description of the object's motion is provided by using a choice of time which is directly associated with the object in a *frame-independent* manner. Simply imagine that the object carries with it its own (good) clock. Time as measured by a clock whose worldline is the same as the worldline of the object of interest is called the *proper time* of the object. To distinguish proper time from coordinate time in some inertial reference frame, proper time is usually denoted as τ (instead of t).

Imagine drawing ticks on the worldline of the object at equal intervals of proper time, as illustrated in Figure 3.1. In the limit of a very fine proper time spacing $\Delta\tau$, the invariant interval between neighboring ticks is constant, $s^2 = -(c\,\Delta\tau)^2$. In the figure, note how the tick spacing, as measured by the coordinate time x^0, varies depending on the instantaneous velocity of the particle. When the particle is nearly at rest (in the chosen reference frame) then the proper time clock runs at nearly the same rate as coordinate time clocks, but when the particle is moving fast then its proper time clock runs more slowly that coordinate time clocks due to time dilation.

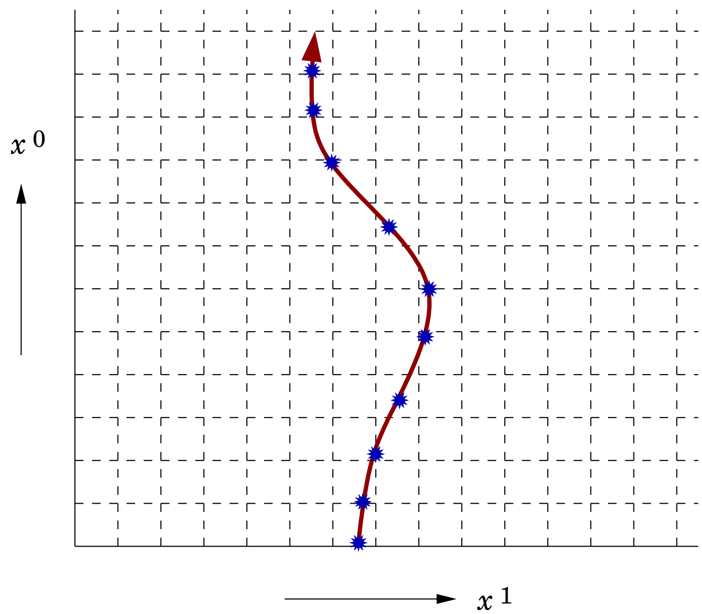

Figure 3.1: The worldline of a non-inertial particle, with tick marks at constant intervals of proper time.

3.2 4-velocity

Using the proper time to label points on the spacetime trajectory of a particle means that its spacetime position is some function of proper time, $x(\tau)$. The time component of x (in a chosen reference frame) gives the relation between coordinate time and proper time of events on the worldline,

$$c\,t = x^0(\tau)\,. \tag{3.2.1}$$

The *four-velocity* of a particle is the derivative of its spacetime position with respect to proper time,

$$u \equiv \frac{dx(\tau)}{d\tau}\,. \tag{3.2.2}$$

Since $x^0 = ct$, the time component of the 4-velocity gives the rate of change of coordinate time with respect to proper time,

$$u^0 = c\,\frac{dt}{d\tau}\,. \tag{3.2.3}$$

The spatial components of the 4-velocity give the rate of change of the spatial position with respect to proper time, $u^k = dx^k/d\tau$. This is *not* the same as the ordinary 3-velocity $\vec{v}$, which is the rate of change of position with respect to coordinate time, $v^k = dx^k/dt$. But we can relate the two using calculus,

$$u^k = \frac{dx^k}{d\tau} = \frac{dt}{d\tau}\,\frac{dx^k}{dt} = \frac{u^0}{c}\,v^k\,. \tag{3.2.4}$$

From our discussion of time dilation, we already know that moving clocks run slower than clocks at rest in the chosen reference frame by a factor of γ. In other words, it must be the case that

$$\frac{u^0}{c} = \frac{dt}{d\tau} = \gamma = \left[1 - \frac{\vec{v}^{\,2}}{c^2}\right]^{-1/2}\,. \tag{3.2.5}$$

Combined with Eq. (3.2.4), this shows that the spatial components of the 4-velocity equal the three-velocity times a factor of γ,

$$u^k = \gamma\, v^k = \frac{v^k}{\sqrt{1 - \vec{v}^{\,2}/c^2}}\,. \tag{3.2.6}$$

We can now use Eqs. (3.2.5) and (3.2.6) to evaluate the square of the 4-velocity,

$$u^2 = -(u^0)^2 + (u^k)^2 = -\gamma^2 \left(c^2 - \vec{v}^{\,2}\right) = -c^2\,. \tag{3.2.7}$$

So a four-velocity vector *always* squares to $-c^2$, regardless of the value of the 3-velocity.

Let's summarize what we've learned a bit more geometrically. The worldline $x(\tau)$ describes some trajectory through spacetime. At every event along this worldline, the four-velocity $u = dx/d\tau$ is a 4-vector which is tangent to the worldline. When one uses proper time to parametrize the worldline, the tangent vector u has a constant square, $u^2 = -c^2$. So you can think of u/c as a tangent vector which has unit "length" everywhere along the worldline. The fact that u^2 is negative shows that the 4-velocity is always a timelike vector.

Having picked a specific reference frame in which to evaluate the components of the four-velocity u, Eqs. (3.2.5) and (3.2.6) show that the components of u are completely determined by the ordinary 3-velocity $\vec{v}$, so the information contained in u is precisely the same as the information contained in $\vec{v}$. You might then ask "why bother with 4-velocity?" The answer is that four-velocity u is a more natural quantity to use — it has geometric meaning which is *independent* of any choice of reference frame. Moreover, the components u^μ of four-velocity transform linearly under a Lorentz boost in exactly the same fashion as any other 4-vector. [See Eq. (2.7.5)]. In contrast, under a Lorentz boost the components of 3-velocity transform in a somewhat messy fashion. (Example problem 3.10.1 below works out the precise form for the case of parallel velocities.)

3.3 4-momentum

The *rest mass* of any object, generally denoted m, is the mass of the object as measured in its rest frame. The *four-momentum* of a particle (or any other object) with rest mass m is defined to be m times the object's four-velocity,

$$p = m\, u\,. \tag{3.3.1}$$

For systems of interacting particles, this is the quantity to which conservation of momentum will apply. Spatial momentum components (in a given reference frame) are just the spatial components of the 4-momentum. The definition of momentum which you learned in introductory physics, $\vec{p} = m\,\vec{v}$, is wrong — this is a non-relativistic approximation which is only useful for slowly moving objects. This is important, so let us repeat,

$$\vec{p} \neq m\,\vec{v}\,. \tag{3.3.2}$$

Momentum is *not* mass times 3-velocity. Rather, momentum is mass times 4-velocity.[1]

[1]Many introductory relativity books introduce a velocity-dependent mass $m(v) \equiv m\,\gamma$, in order to write $\vec{p} = m(v)\,\vec{v}$, and thereby avoid introducing four-velocity, or any other 4-vector. This is pedagogically terrible and offers no benefit whatsoever. If you have previously seen this use of a velocity-dependent mass, erase it from your memory!

If the spatial components of the four-momentum are the (properly defined) spatial momentum, what is the time component p^0? There is only one possible answer — it must be related to energy.[2] In fact, the total energy E of an object equals the time component of its four-momentum times c, or

$$p^0 = E/c\,. \tag{3.3.3}$$

Using the relation (3.3.1) between 4-momentum and 4-velocity, plus the result (3.2.5) for u^0, allows one to express the the total energy E of an object in terms of its rest mass and its velocity,

$$E = c\,p^0 = mc\,u^0 = mc^2\gamma = \frac{mc^2}{\sqrt{1-\vec{v}^{\,2}/c^2}} = mc^2\cosh\eta\,, \tag{3.3.4}$$

where the last form uses the relation (2.6.2) between rapidity and γ. In other words, the relativistic gamma factor of any object is equal to the ratio of its total energy to its rest energy,

$$\gamma = \frac{E}{mc^2}\,. \tag{3.3.5}$$

When the object is at rest, its kinetic energy (or energy due to motion) vanishes, but its *rest energy*, given by Einstein's famous expression mc^2, remains. If the object is moving slowly (compared to c), then it is appropriate to expand the relativistic energy (3.3.4) in powers of $\vec{v}^{\,2}/c^2$. This gives

$$E = mc^2 + \tfrac{1}{2}m\vec{v}^{\,2} + \cdots\,, \tag{3.3.6}$$

and shows that for velocities small compared to c, the total energy E equals the rest energy mc^2 plus the usual non-relativistic kinetic energy, $\frac{1}{2}m\vec{v}^{\,2}$, up to higher order corrections which, relative to the kinetic energy, are suppressed by additional powers of $\vec{v}^{\,2}/c^2$. One can define a relativistic *kinetic energy* K, as simply the difference between the total energy and the rest energy, $K = E - mc^2$.

Combining the 4-momentum definition (3.3.1), and the relation (3.2.4) between three- and four-velocity components, yields the relation between the spatial components of the relativistic momentum and the 3-velocity,

$$\vec{p} = m\,\vec{v}\,\gamma = \frac{m\,\vec{v}}{\sqrt{1-\vec{v}^{\,2}/c^2}} = m\hat{v}\,\sinh\eta\,, \tag{3.3.7}$$

where the last form uses rapidity and a unit spatial vector $\hat{v}$ pointing in the direction of the 3-velocity. Expanding in powers of v/c shows that, for low velocities, the relativistic spatial momentum reduces to the non-relativistic form,

$$\vec{p} = m\,\vec{v} + \cdots\,, \tag{3.3.8}$$

up to higher order corrections suppressed by powers of $\vec{v}^{\,2}/c^2$.

We saw above that four-velocities square to $-c^2$. Because four-momentum is just mass times four-velocity, the four-momentum of any object with mass m satisfies

$$p^2 = -m^2c^2\,. \tag{3.3.9}$$

[2]To see why, recall from mechanics (quantum or classical) that translation invariance in space is related to the existence of conserved spatial momentum, and translation invariance in time is related to the existence of a conserved energy. We will discuss this in more detail later. Since Lorentz transformations mix space and time, it should be no surprise that the four-momentum, which transforms linearly under Lorentz transformations, must characterize both the energy *and* the spatial momentum.

Since $p^2 = -(p^0)^2 + (p^k)^2$, and $p^0 = E/c$, this may rewritten (in any chosen inertial reference frame) as

$$E^2 = c^2\,\vec{p}^{\,2} + (mc^2)^2\,, \tag{3.3.10a}$$

or

$$E = \sqrt{c^2\,\vec{p}^{\,2} + (mc^2)^2}\,. \tag{3.3.10b}$$

So if you know the spatial momentum $\vec{p}$ and mass m of some object, you can directly compute its energy E without first having to evaluate the object's velocity.

But what if you want to find the ordinary 3-velocity? Return to the relation $u^k = \gamma\, v^k$ [Eq. (3.2.6)] between 3-velocity and 4-velocity, and multiply both sides by m to rewrite this result in terms of four-momentum. Since spatial momentum $p^k = mu^k$, and total energy $E = mc^2\gamma$, we have $p^k = (E/c^2)\, v^k$ or

$$v^k = \frac{p^k}{E/c^2}\,. \tag{3.3.11}$$

Three-velocity is *not* equal to momentum divided by mass — forget this falsehood! Rather, the ordinary 3-velocity equals the spatial momentum divided by the total energy (over c^2). And its magnitude never exceeds c, no matter how large the momentum (and energy) become.

3.4 4-force

In the absence of any forces, the momentum of an object remains constant. In the presence of forces, an object's momentum will change. In fact, force is just the time rate of change of momentum. But what time and what momentum? Newtonian (non-relativistic) dynamics says that $d\vec{p}/dt = \vec{F}$ along with $d\vec{x}/dt = \vec{p}/m$, where $\vec{p}$ is 3-momentum and t is coordinate time. This is wrong — inconsistent with our relativity postulates. A frame-independent formulation of dynamics must involve quantities which have intrinsic frame-independent meaning — such as four-momentum and proper time. The appropriate generalization of Newtonian dynamics which is consistent with our relativity postulates is

$$\frac{dx}{d\tau} = \frac{p}{m}\,, \tag{3.4.1a}$$

$$\frac{dp}{d\tau} = f\,. \tag{3.4.1b}$$

Eq. (3.4.1a) is just the definition (3.2.2) of 4-velocity rewritten in terms of 4-momentum, while Eq. (3.4.1b) is the *definition* of force as a four-vector. The only difference in these equations, relative to Newtonian dynamics, is the replacement of 3-vectors by 4-vectors and coordinate time by proper time.

Equations (3.4.1) are written in a form which emphasizes the role of momentum. If you prefer, you can work with 4-velocity instead of 4-momentum and rewrite these equations as $dx/d\tau = u$ and $du/d\tau = f/m$. Defining the *four-acceleration* $a \equiv du/d\tau = d^2x/d\tau^2$, this last equation is just $f = m\,a$. This is the relativistic generalization of Newton's $\vec{f} = m\,\vec{a}$, with force and acceleration now defined as spacetime vectors.[3]

[3]Eq. (3.4.1b) is equivalent to $f = m\,a$ *provided* the mass m of the object is constant. For problems involving objects whose mass can change, such as a rocket which loses mass as it burns fuel, these two equations are not equivalent and one must use the more fundamental $dp/d\tau = f$.

In non-relativistic dynamics, if you know the initial position and velocity of a particle, and you know the force $\vec{f}(t)$ which subsequently acts on the particle, you can integrate Newton's equations to find the trajectory $\vec{x}(t)$ of the particle. Initial conditions plus a three-vector $\vec{f}(t)$ completely determine the resulting motion. To integrate the relativistic equations (3.4.1), you need initial conditions plus a four-vector force $f(\tau)$. This would appear to be more information (four components instead of three), and yet relativistic dynamics must reduce to non-relativistic dynamics when velocities are small compared to c.

The resolution of this apparent puzzle is that the four-force cannot be a completely arbitrary four-vector. We already know that for any object with mass m, its four-momentum must satisfy $p^2 = -(mc)^2$ [Eq. (3.3.9)]. Take the derivative of both sides with respect to proper time. The right hand side is constant in time (provided that the object in question is some stable entity with a fixed rest mass), so its proper time derivative vanishes. The derivative of the left hand side gives twice the dot product of p with f, and hence the four-force must always be orthogonal to the four-momentum,

$$p \cdot f = 0 \,. \tag{3.4.2}$$

Written out in components, this says that $p^0 f^0 = p^k f^k$, or

$$f^0 = \frac{p^k f^k}{p^0} = \frac{\vec{v}}{c} \cdot \vec{f} \,, \tag{3.4.3}$$

showing that the time component of the 4-force is completely determined by the spatial force components (and the 3-velocity).

3.5 Constant acceleration

Let us put this formalism into action by examining the case of motion under the influence of a constant force. But what is a "constant" force? We have just seen that the four-force must always be orthogonal to the momentum. So it is impossible for the 4-force $f(\tau)$ to be a fixed four-vector, independent of τ. However, it is possible for the force to have components which are constant when viewed in a frame which is *instantaneously* co-moving with the accelerating object.

Suppose a particle begins at the spacetime origin with vanishing 3-velocity (or 3-momentum) at proper time $\tau = 0$, and a force of magnitude F, pointing in the x^1 direction, acts on the particle. So the components of the initial spacetime position, four-velocity, and four-force are $x_0^\mu = (0,0,0,0)$, $u_0^\mu = (c,0,0,0)$, and $f_0^\mu = (0,F,0,0)$, respectively. The four-velocity at later times may be written as some time-dependent Lorentz boost acting on the initial four-velocity,

$$u(\tau) = \Lambda_{\text{boost}}(\tau)\, u_0 \,. \tag{3.5.1}$$

The condition that the force is constant in a co-moving frame amounts to the statement that the *same* Lorentz boost relates the four-force at any time τ to the initial force,

$$f(\tau) = \Lambda_{\text{boost}}(\tau)\, f_0 \,. \tag{3.5.2}$$

At all times, $u^2 = -c^2$ (because u is a four-velocity), and $f^2 = F^2$ because the magnitude of the force is assumed to be constant.

Since the initial force points in the x^1 direction, the particle will acquire some velocity in this direction, but the x^2 and x^3 components of the velocity will always remain zero. Hence the boost

$\Lambda_{\rm boost}(\tau)$ will always be some boost in the x^1 direction, and the force $f(\tau)$ will likewise always have vanishing x^2 and x^3 components. In other words, the 4-velocity and 4-force will have the form

$$u^\mu(\tau) = \big(u^0(\tau), u^1(\tau), 0, 0\big) \,, \qquad f^\mu(\tau) = \big(f^0(\tau), f^1(\tau), 0, 0\big) \,, \tag{3.5.3}$$

with $u^0(0) = c$, $u^1(0) = 0$ and $f^0(0) = 0$, $f^1(0) = F$. The dot product $f \cdot u = -f^0u^0 + f^1u^1$ must vanish, implying that $f^0/f^1 = u^1/u^0$. So the components of the force must be given by

$$f^\mu(\tau) = \frac{F}{c}\, \big(u^1(\tau), u^0(\tau), 0, 0\big) \,. \tag{3.5.4}$$

We want to solve $m\, du/d\tau = f(\tau)$. Writing out the components explicitly (and dividing by m) gives

$$\frac{du^0(\tau)}{d\tau} = \frac{F}{mc}\, u^1(\tau) \,, \qquad \frac{du^1(\tau)}{d\tau} = \frac{F}{mc}\, u^0(\tau) \,. \tag{3.5.5}$$

This is easy to solve if you remember that $\frac{d}{dz} \sinh z = \cosh z$ and $\frac{d}{dz} \cosh z = \sinh z$. To satisfy Eq. (3.5.5), and our initial conditions, we need

$$u^0(\tau) = c \cosh \frac{F\tau}{mc} \,, \qquad u^1(\tau) = c \sinh \frac{F\tau}{mc} \,. \tag{3.5.6}$$

The ordinary velocity $v^k = u^k\,(c/u^0)$ [Eq. (3.2.4)], so the speed of this particle subject to a constant force is

$$v(\tau) = c \tanh \frac{F\tau}{mc} \,. \tag{3.5.7}$$

Since $\tanh z \sim z$ for small values of the argument, the speed grows linearly with time at early times, $v(\tau) \sim (F/m)\,\tau$. This is precisely the expected non-relativistic behavior. But this approximation is only valid when $\tau \ll mc/F$ and the speed is small compared to c. The argument of the tanh becomes large compared to unity when $\tau \gg mc/F$, and $\tanh z \to 1$ as $z \to \infty$. So the speed of the accelerating particle asymptotically approaches, but never reaches, the speed of light. From the definition (2.6.1) of rapidity, $v/c = \tanh \eta$, one sees that the result (3.5.7) for the speed just corresponds to rapidity growing linearly with proper time,

$$\eta(\tau) = \frac{F\tau}{mc} \,. \tag{3.5.8}$$

At this point, we have determined how the velocity of the particle grows with time, but we need to integrate $dx/d\tau = u$ to find its spacetime position. The integrals are elementary,

$$x^0(\tau) = \int_0^\tau d\tau'\, u^0(\tau') = c \int_0^\tau d\tau' \, \cosh \frac{F\tau'}{mc} = \frac{mc^2}{F} \, \sinh \frac{F\tau}{mc} \,, \tag{3.5.9a}$$

$$x^1(\tau) = \int_0^\tau d\tau'\, u^1(\tau') = c \int_0^\tau d\tau' \, \sinh \frac{F\tau'}{mc} = \frac{mc^2}{F} \left[\cosh \frac{F\tau}{mc} - 1\right] . \tag{3.5.9b}$$

Hyperbolic sines and cosines grow exponentially for large arguments, $\sinh z \sim \cosh z \sim \frac{1}{2}\, e^z$ when $z \gg 1$. Hence, when $\tau \gg mc/F$ the coordinates $x^0(\tau)$ and $x^1(\tau)$ both grow like $e^{F\tau/mc}$ with increasing proper time. But the accelerating particle becomes ever more time-dilated; the rate of change of proper time with respect to coordinate time, $d\tau/dt = c/u^0 = 1/\cosh \frac{F\tau}{mc}$, behaves as $2\, e^{-F\tau/mc} \sim mc/(Ft)$.

3.6 Plane waves

Next, we want to discuss how waves (of any type) may be described using relativistic notation. Consider some plane wave with spatial wave-vector $\vec{k}$ and (angular) frequency ω, as measured in some inertial frame. The amplitude of the wave may be described by a complex exponential, $\mathcal{A}\, e^{i\vec{k}\cdot\vec{x}-i\omega t}$, with the usual understanding that it is the real part of this function which describes the physical amplitude. Such a wave has a wavelength $\lambda = 2\pi/|\vec{k}|$ and planar wave-fronts orthogonal to the wave-vector which move at speed $v = \omega/|\vec{k}|$ in the direction of $\vec{k}$.

As mentioned earlier (2.7.12), it is natural to combine ω and $\vec{k}$ into a spacetime wave-vector k with components

$$k^\mu = (\omega/c,\, k^1, k^2, k^3)\,, \tag{3.6.1}$$

so that $\omega = c\, k^0$ and the complex exponential $e^{i\vec{k}\cdot\vec{x}-i\omega t} = e^{ik\cdot x}$ only involves a spacetime dot product. The virtue of this formulation is that it is frame-independent. The spacetime position x and wave-vector k are geometric entities which you should think of as existing independent of any particular choice of coordinates. The value of the amplitude, $\mathcal{A}\, e^{ik\cdot x}$, depends on the event x and the wave-vector k, but one may use whatever reference frame is most convenient to evaluate the dot product of these 4-vectors.

Just as surfaces of simultaneity are observer-dependent, so is the frequency of a wave. After all, measuring the frequency of a wave involves counting the number of wave crests which pass some detector (or observer) in a given length of time. The time component of the wave-vector gives (by construction) the frequency of the wave as measured by observers who are at rest in the frame in which the components k^μ are defined. Such observers have 4-velocities whose components are just $(c, 0, 0, 0)$ (in that frame). Consequently, for these observers the frequency of the wave may be written as a dot product of the observer's 4-velocity and the wave-vector,

$$\omega_{\text{obs}} = -u_{\text{obs}} \cdot k\,. \tag{3.6.2}$$

This expression is now written in a completely general fashion which is observer-dependent but frame-independent. That is, the expression (3.6.2) depends explicitly on the observer's 4-velocity u_{obs}, but is independent of the frame used to evaluate the dot product between u_{obs} and k. Therefore, the frequency which is measured by *any* observer will be given by (minus) the dot product of the observer's 4-velocity u and the wave-vector k. Once again, this dot product may be evaluated using whatever reference frame is most convenient.

For light waves (in a vacuum), the wave speed $v = c$ and $\omega = c|\vec{k}|$. The resulting spacetime wavevector (3.6.1) is automatically a lightlike 4-vector which squares to zero,

$$k^\mu_{\text{light}} = \frac{\omega}{c}\,(1, \hat{k})\,, \qquad k^2_{\text{light}} = 0\,. \tag{3.6.3}$$

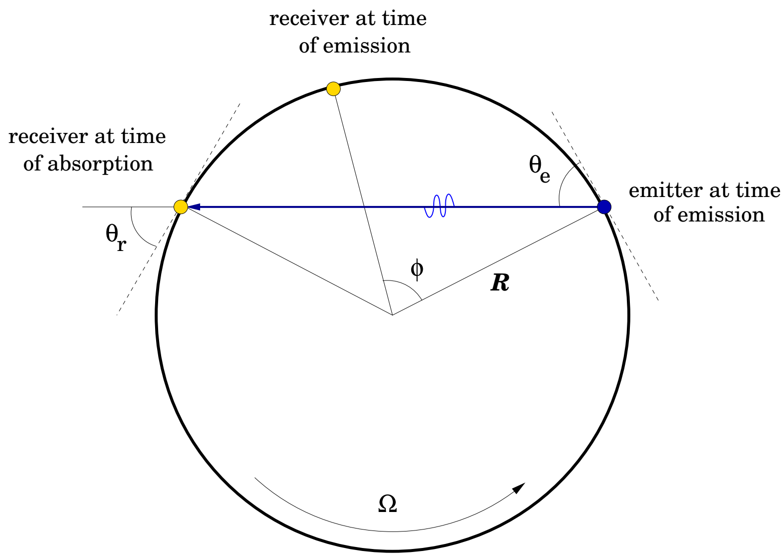

Figure 3.2: Inside a rotating centrifuge, light is emitted at one point and later received at another point. Is there a Doppler shift between the frequencies of emission and reception?

A nice application of Eq. (3.6.2), demonstrating the value of writing physical quantities in frame independent form, is illustrated in Figure 3.2.[4] Mounted on the inner surface of a centrifuge, which is rotating at angular frequency Ω, is an emitter of light at one point, and a receiver at a different point. Let ϕ be the angle between emitter and receiver, relative to the center of the centrifuge, as measured in the inertial lab frame. The (inner) radius of the centrifuge is R. The frequency of the light as measured by an observer who is instantaneously at rest relative to the emitter is $\nu_{\rm e}$. The frequency of the light as measured by an observer who is instantaneously at rest relative to the receiver is $\nu_{\rm r}$. What is the fractional difference $(\nu_{\rm r} - \nu_{\rm e})/\nu_{\rm e}$? How does this frequency shift depend on the angle ϕ and the rotation frequency Ω?

One approach for solving this problem would involve explicitly constructing the Lorentz transformations which relate the lab frame to the instantaneous rest frames of the emitter and receiver, and then combining these two transformations to determine the net transformation which directly connects emitter and receiver. Given the three-dimensional geometry involved, this is rather involved.

A much better approach is to choose a convenient single frame, namely the lab frame, in which to evaluate the components of the four-vectors appearing in the frame-independent expression (3.6.2) for the frequency. We need to compute

$$\frac{\nu_{\rm r}}{\nu_{\rm e}} = \frac{-u_{\rm r}\cdot k}{-u_{\rm e}\cdot k} = \frac{u_{\rm r}^0\, k^0 - \vec u_{\rm r}\cdot\vec k}{u_{\rm e}^0\, k^0 - \vec u_{\rm e}\cdot\vec k}\,. \tag{3.6.4}$$

Here $u_{\rm e}$ is the four-velocity of the emitter at the moment it emits light, and $u_{\rm r}$ is the four-velocity of the receiver at the moment when it receives the light.

If $\theta_{\rm e}$ denotes the angle between the spatial wavevector and the direction of motion of the emitter (at the time of emission), and $\theta_{\rm r}$ denotes the angle between $\vec k$ and receiver's direction (at the time of reception), then we can express the spatial dot products in terms of cosines of these angles,

$$\frac{\nu_{\rm r}}{\nu_{\rm e}} = \frac{u_{\rm r}^0\, k^0 - |\vec u_{\rm r}||\vec k|\cos\theta_{\rm r}}{u_{\rm e}^0\, k^0 - |\vec u_{\rm e}||\vec k|\cos\theta_{\rm e}}\,. \tag{3.6.5}$$

The speed of the inner surface of the centrifuge is constant, $v = \Omega\, R$, and hence the *speeds* of the emitter and receiver, as measured in the lab frame, are identical — even though their velocity vectors are different. The time component of a 4-velocity, $u^0/c = (1 - v^2/c^2)^{-1/2}$, only depends on

[4]This discussion is an adaptation of an example in *Gravitation* by Misner, Thorne and Wheeler.

the *magnitude* of the velocity $\vec{v}$, and hence $u^0_{\rm r} = u^0_{\rm e}$. The equality of the emitter and receiver speeds also implies that the magnitudes of the spatial parts of the 4-velocities coincide, $|\vec{u}_{\rm r}| = |\vec{u}_{\rm e}|$. So using expression (3.6.5) for the frequency ratio, the only remaining question is how does $\theta_{\rm r}$ compare to $\theta_{\rm e}$?

This just involves ordinary geometry. Looking at the figure, notice that $\theta_{\rm e}$ and $\theta_{\rm r}$ are the angles between the path of the light, which is a chord of the circle, and tangents to the circle at the endpoints of the chord. But the angle a chord makes with these tangents is the same at either end, implying that $\theta_{\rm e} = \theta_{\rm r}$. And this means $\nu_{\rm r} = \nu_{\rm e}$ — there is no Doppler shift no matter how fast the centrifuge rotates!

3.7 Electromagnetism

As already seen in the discussion of lightcones, plane waves, and Doppler shifts, the techniques we are developing are particularly useful for understanding the propagation of light. Unfortunately, we do not have time for extensive explorations of other relativistic aspects of electromagnetism, which will be left for later classes. But one aspect, how to represent the Lorentz force in the framework we have been discussing, is natural to describe here.

As we have seen above, generalizations from non-relativistic to relativistic dynamics are mostly a matter of replacing 3-vectors by 4-vectors (and coordinate time by proper time). But what about electric and magnetic fields? Both are (apparently) 3-vectors, and there is no sensible way to turn them into 4-vectors. It turns out that what is sensible (and natural) is to package the components of $\vec{E}$ and $\vec{B}$, together, into a 4×4 matrix called the *field strength tensor*, whose components are[5]

$$\|F^\mu{}_\nu\| = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix} . \tag{3.7.1}$$

With this repackaging of electric and magnetic fields, the Lorentz force (as a 4-vector) has a remarkably simple form,

$$f^\mu_{\rm Lorentz} = \frac{q}{c}\, F^\mu{}_\nu \, u^\nu \, . \tag{3.7.2}$$

Verifying that this 4-force leads to exactly the same rate of change of energy and momentum as does the traditional form of writing the Lorentz force, $\vec{f} = q\,(\vec{E} + \vec{v} \times \vec{B})$, is an instructive and recommended exercise.

3.8 Scattering

When objects (elementary particles, molecules, automobiles, ...) collide, the results of the collision can differ markedly from the initial objects. Composite objects can fall apart or change form. Interestingly, dramatic changes during collisions can also occur for elementary particles. Studying

[5]The explicit form of the field strength tensor depends on the choice of units one uses for electric and magnetic fields. Expression (3.7.1) is applicable with SI units, where $|\vec{E}|$ is measured in Newtons/Coulomb, and $|\vec{B}|$ in Tesla. One can (and should) check that (3.7.1) is dimensionally consistent — using SI quantities, multiplying by c converts the units for B into the units for E.

the collisions of elementary particles is a primary method used to investigate fundamental interactions (and is the reason for building large high-energy particle colliders such as the LHC).

A complete description of what emerges from a collision (or 'scattering event') depends on microscopic details of the interaction between the incident objects. But certain general principles constrain the possibilities, most importantly, the conservation of energy and momentum. As discussed in section 3.3, the total energy E and spatial momentum $\vec{p}$ of any object may be combined to form the four-momentum $p^\mu = (E/c, \vec{p})$. Consequently, conservation of energy plus conservation of (spatial) momentum may be compactly rephrased as the conservation of four-momentum: in the absence of any external forces, the total four-momentum $p_{\rm tot}$ of any system cannot change,

$$\frac{d}{dt}\, p_{\rm tot}(t) = 0\,. \tag{3.8.1}$$

In a scattering process two or more objects, initially far apart, come together and interact in some manner (which may be very complicated), thereby producing some number of objects that subsequently fly apart. When the incoming objects are far apart and not yet interacting, the total four-momentum is just the sum of the four-momentum of each object,

$$p_{\rm in} = \sum_{a=1}^{N_{\rm in}} p_a\,, \tag{3.8.2}$$

(where $N_{\rm in}$ is the number of incoming objects and the index a labels particles, not spacetime directions). Similarly, when the outgoing objects are arbitrarily well separated they are no longer interacting and the total four-momentum is the sum of the four-momenta of all outgoing objects,

$$p_{\rm out} = \sum_{b=1}^{N_{\rm out}} p_b\,. \tag{3.8.3}$$

Hence, for any scattering processes, conservation of energy and momentum implies that the total incident four-momentum equal the total outgoing four-momentum (regardless of the values of $N_{\rm in}$ and $N_{\rm out}$),

$$p_{\rm in} = p_{\rm out}\,. \tag{3.8.4}$$

As with any four-vector equation, one may choose to write out the components of this equation in whatever reference frame is most convenient. For analyzing scattering processes, sometimes it is natural to work in the rest frame of one of the initial objects (the 'target'); this is commonly called the *lab frame*. Experiments of this variety are known as "fixed target" experiments; the rest-frame of the actual laboratory is the target frame. Alternatively, one may choose to work in the reference frame in which the total spatial momentum vanishes. In this frame, commonly called the *CM frame*,[6] the components of the total four-momentum are

$$p^\mu_{\rm CM} = (E_{\rm CM}/c,\, 0,\, 0,\, 0)\,, \tag{3.8.5}$$

where $E_{\rm CM}$ is the total energy of the system in the CM frame.

[6] CM means 'center of mass', but this historical name is really quite inappropriate for relativistic systems, which may include massless particles that carry momentum but have no rest mass. The widely used 'CM' label should always be understood as referring to the zero (spatial) momentum frame.

As an application of these ideas, consider the scattering of protons of energy $E_{\rm in} = 1\,{\rm TeV}$ on protons at rest (in ordinary matter). The proton rest energy $m_p\,c^2$ is a bit less than $1\,{\rm GeV}$. Using Eq. (3.3.5), one sees that a proton with $1\,{\rm TeV}$ energy is ultrarelativistic, $\gamma = E_{\rm in}/(m_p\,c^2) \approx 10^3$. When an ultrarelativistic proton strikes a target proton at rest, both protons can be disrupted and new particles may be created. Schematically,

$$p + p \to X\,,$$

where X stands for one or more outgoing particles. What is the largest mass of a particle which could be produced in such a collision?

The total energy of the incident particles (in the rest frame of the target) is $E_{\rm tot} = E_{\rm in} + m_p c^2 \approx 1.001\,{\rm TeV}$. If all of this energy is converted into the rest energy of one or more outgoing particles, then these collisions could produce particles with mass up to $E_{\rm tot}/c^2 \approx 10^3\,m_p$. This would be consistent with conservation of energy. But this is *wrong*, as it completely ignores conservation of momentum. In the rest frame of the target, the total spatial momentum $\vec p_{\rm tot}$ is non-zero (and equal to the momentum $\vec p_{\rm in}$ of the projectile proton). If there is a single outgoing particle X, it cannot be produced at rest — it must emerge from the collision with a non-zero spatial momentum equal to $\vec p_{\rm tot}$. That means its energy will be greater than its rest energy.

To determine the largest mass of a particle which can be produced in this collision, one must simultaneously take into account conservation of both energy and momentum. That is, one must satisfy the four-vector conservation equation (3.8.4). In the lab frame, if we orient coordinates so that the z-axis is the collision axis, then

$$p_{\rm in} = p_{\rm projectile} + p_{\rm target} = \begin{pmatrix} E_{\rm in}/c \\ 0 \\ 0 \\ p_{\rm in} \end{pmatrix} + \begin{pmatrix} m_p\,c \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.8.6}$$

If a single particle X emerges, then its four-momentum is the total outgoing four-momentum,

$$p_{\rm out} = p_X = \begin{pmatrix} E_X \\ p_X^1 \\ p_X^2 \\ p_X^3 \end{pmatrix}. \tag{3.8.7}$$

Demanding that $p_{\rm in}$ coincide with $p_{\rm out}$ determines $\vec p_X = p_{\rm in}\,\hat e_3$ and $E_X = E_{\rm in} + m_p c^2$. Eq. (3.3.10), applied to the projectile proton (with known mass), may be used to relate the incident spatial momentum and energy, $p_{\rm in}^2 = (E_{\rm in}/c)^2 + (m_p\,c)^2$. The same relation (3.3.10), applied to the outgoing particle X, connects its energy E_X and momentum $\vec p_X$ to the desired maximum mass m_X, $(m_X c^2)^2 = E_X^2 - (c\,\vec p_X)^2$. Inserting numbers and computing E_X, $|\vec p_X| = p_{\rm in}$, and finally m_X is straightforward. But even less work is required if one recalls [from Eq. (3.3.9)] that the square of any four-momentum directly gives the rest mass of the object, $p^2 = -m^2 c^2$. Hence

$$\begin{aligned} -m_X^2\,c^2 = p_X^2 = p_{\rm out}^2 = p_{\rm in}^2 &= (p_{\rm projectile} + p_{\rm target})^2 \\ &= p_{\rm projectile}^2 + p_{\rm target}^2 + 2\,p_{\rm projectile}\cdot p_{\rm target} \\ &= -2\,m_p^2\,c^2 - 2E_{\rm in}\,m_p\,. \end{aligned} \tag{3.8.8}$$

Consequently, $m_X = \sqrt{2m_p(m_p + E_{\rm in}/c^2)} = m_p\sqrt{2 + 2E_{\rm in}/(m_p c^2)} \approx \sqrt{2002}\,m_p \approx 45\,m_p$. Even though the projectile proton has an energy a thousand times greater than its rest energy, the maximum mass particle which can be created in this collision is only 45 times heavier than a proton.

Most of the energy of the projectile is needed to provide the kinetic energy of the outgoing particle X, which is necessarily associated with the conserved spatial momentum. More generally, the maximum mass grows (only) like the square root of the lab frame energy, $m_X^{\rm max} \sim \sqrt{2E_{\rm in} m_p/c^2}$, when $E_{\rm in} \gg m_p c^2$.

This illustrates why "colliders" in which two beams of particles are aimed at each other, so that the lab and CM frames coincide, are more efficient when hunting for new heavy particles. If the colliding particles have equal mass, then they will also have equal energy ($E_{\rm in}$) when their spatial momenta are equal and opposite. In this case, the total spatial momentum vanishes and the maximum mass of a produced particle is limited only by the total energy, $m_X^{\rm max} = 2E_{\rm in}/c^2$, which grows *linearly* with the beam energy $E_{\rm in}$.

3.9 Units and sizes

It may be helpful at this point to say a few words about units and the size of things. For "dimensionfull" quantities (*i.e.*, quantities which are not pure numbers and whose measurement requires some standard for comparison), the value of the quantity depends on one's choice of units. It only makes sense to say that a dimensionfull quantity is "large" or "small" in comparison to some other quantity with the *same* units. For velocities, the universal value of the speed of light makes c the natural standard for comparison; an object is moving slowly (and non-relativistic dynamics can be a good approximation) if its speed is small *compared* to the speed of light, $|v|/c \ll 1$. Similarly, classical mechanics can provide a good approximation when quantum interference effects produced by a wave function such as (2.7.13) vary so rapidly that they become unresolvable. This is the case when $p \cdot x$ is large *compared* to Planck's constant $\hbar$.

In the SI (or MKS) system, there are *three* independent fundamental units, length (m), mass (kg), and time (s). These units are convenient for describing many phenomena which occur on human scales. But they are *not* convenient for describing atomic, nuclear, or particle physics phenomena. For example, the mass of a proton is 1.67×10^{-27} kg, and the spatial size of a proton is conveniently measured in fermi, not meters. A fermi (fm) is shorthand for one femtometer, 1 fm = 1 femtometer = 10^{-15} m. Likewise, the lifetime of a typical particle that decays via the strong interactions (discussed in the next chapter) is of order 10^{-23} s, roughly the time needed for light to travel across a distance of 1 fm.

As most of the physics we will discuss in this course is both relativistic and quantum mechanical, it will often be convenient to use units in which the speed of light and Planck's constant $\hbar$ have numerical values close to unity. In fact, one is free to choose "natural" units in which both c and $\hbar$ are *exactly* equal to unity. By declaring that

$$c = 2.99\,792\,458 \times 10^8 \text{ m/s} = 1\,, \tag{3.9.1}$$

one is choosing to regard time and distance as having the *same* units; one second is the *same* as $2.99 \cdots \times 10^8$ meters. As a measure of distance, one second means one "lightsecond," the distance light travels in a second. The speed of light, when expressed in m/s, is just a conversion factor between two different units for distance, meters and seconds, in the same way that $1 = 2.54$ cm/in or $1 = 6$ ft/fathom are conversion factors relating other measures of distance.

Similarly, by declaring that

$$\hbar = 1.05\,457\,148 \cdots \times 10^{-34} \text{ J s} = 1\,, \tag{3.9.2}$$

1 kg	$=$	$5.61 \times 10^{26} \text{ GeV}$	$[\text{GeV}/c^2]$
1 m	$=$	$5.07 \times 10^{15} \text{ GeV}^{-1}$	$[\hbar c/\text{GeV}]$
1 s	$=$	$1.52 \times 10^{24} \text{ GeV}^{-1}$	$[\hbar c/\text{GeV}]$
$1 \text{ fm} \equiv 10^{-15} \text{ m}$	$=$	$5.07 \, \text{GeV}^{-1}$	$[\hbar c/\text{GeV}]$
$(1 \text{ fm})^2 = 10 \, \text{mb} = 10^{-30} \text{ m}^2$	$=$	$25.7 \, \text{GeV}^{-2}$	$[(\hbar c/\text{GeV})^2]$
$\hbar c$	$=$	$197 \, \text{MeV fm}$	
$(\hbar c)^2$	$=$	$0.389 \, \text{GeV}^2 \, \text{mb}$	

Table 3.1: Useful approximate conversion factors. The last column shows the appropriate units with $\hbar$ and c included.

one is choosing to regard energy and frequency (inverse time) as having the same units. Since quantum states with energy E evolve in time with an amplitude $e^{-iEt/\hbar}$, one sees that their frequency of oscillation is always directly related to their energy by a factor of Planck's constant, $\omega = E/\hbar$. This relation applies to photons, electrons, and any other particle. So it is natural to regard Planck's constant $\hbar$, expressed in J/s (or any other traditional units), as just a conversion factor between two different measures for energy (or frequency).

With c set equal to unity, time has the same dimensions as distance. Moreover, mass, momentum and energy all have the same units (since factors of c can convert one to the other). With $\hbar$ also set equal to unity, mass and energy have the same units as 1/distance or 1/time. The net result is that there is only one fundamental independent dimension, say energy, which requires a choice of unit. We could use Joules, ergs, or any other measure of energy, but it will be most convenient to choose a unit which is comparable to energy scales relevant for particle physics — such as the proton's rest energy, $m_p \, c^2$. This is about 1.5×10^{-10} J, showing that Joules are not a very nice choice for our purposes. It is preferable, and conventional, to instead use SI-prefixed (*e.g.*, kilo-, mega-, giga-, ...) electron volts, namely keV $= 10^3$ eV, MeV $= 10^6$ eV, GeV $= 10^9$ eV, TeV $= 10^{12}$ eV, *etc.* As the proton rest energy is very close to one GeV, $m_p \, c^2 = 0.938$ GeV, giga-electron volts (GeV) will be especially convenient.

As noted above, the fermi is a useful measure for lengths in particle physics applications. A convenient conversion factor is $1 = 197$ MeV fm, (or $\hbar c = 197$ MeV fm with $\hbar$ and c retained), so 1 fm $\approx 1/(0.2 \, \text{GeV})$. For measuring areas (*e.g.*, cross sections for scattering), the "barn", defined as 10^{-28}m^2, is commonly used in nuclear physics. For particle physics applications, millibarn (mb $= 10^{-31}$ m^2), microbarn (μb $= 10^{-34} \, \text{m}^2$), or nanobarn (nb $= 10^{-37} \, \text{m}^2$) are generally more convenient. One square fermi is 10 millibarn. Table 3.1 lists a number of conversion factors relating traditional and natural particle physics units. In these notes, we will initially retain explicit factors of c and $\hbar$, but you should gradually become comfortable using natural units with $c = \hbar = 1$.

As a final illustration of the relation between different units, Table 3.2 compares the sizes, in both meters and GeV^{-1}, of a wide variety of objects. Note the huge range of sizes that characterize our universe. The last quantity listed, the Planck length, is the length scale, or inverse mass scale, where quantum fluctuations in the geometry of spacetime (*i.e.*, quantum gravity effects) are believed to become significant.

observable universe	$\sim$	10^{26} m	$\approx$	5×10^{41} GeV^{-1}	($\sim 10^{11}$ galaxies)
galaxy supercluster	$\sim$	10^{24} m	$\approx$	5×10^{39} GeV^{-1}	
galaxy	$\sim$	10^{21} m	$\approx$	5×10^{36} GeV^{-1}	($\sim 10^{11}$ stars)
star	$\sim$	10^{9} m	$\approx$	5×10^{24} GeV^{-1}	
Earth	$\sim$	10^{7} m	$\approx$	5×10^{22} GeV^{-1}	
human	$\sim$	10^{0} m	$\approx$	5×10^{15} GeV^{-1}	
atom	$\sim$	10^{-10} m	$\approx$	5×10^{5} GeV^{-1}	
nucleus	$\sim$	10^{-14} m	$\approx$	5×10^{1} GeV^{-1}	
proton	$\sim$	10^{-15} m	$\approx$	5×10^{0} GeV^{-1}	
present observational limit	$\sim$	10^{-19} m	$\approx$	5×10^{-4} GeV^{-1}	
Planck length	$\sim$	10^{-35} m	$\approx$	5×10^{-20} GeV^{-1}	

Table 3.2: Characteristic sizes of various objects (to within factors of 2–3).

3.10 Example problems

3.10.1 Relativistic velocity addition

Q: Frame S' moves in the x^1 direction with velocity v_0 relative to frame S. A point particle moves with velocity v' in the x^1 direction as seen in frame S'. Find the 3-velocity of the particle in frame S.

A: In frame S', the components of the 4-velocity of the particle are $(u')^\mu = (\gamma_{v'}\, c,\, \gamma_{v'}\, v',\, 0,\, 0)$, with $\gamma_{v'} = (1 - {v'}^2/c^2)^{-1/2}$. Since the S' frame is moving, relative to frame S, by velocity v_0 in the x^1 direction, 4-vector components in frame S are related to those in frame S' by the Lorentz transformation matrix

$$\Lambda(v_0) = \begin{pmatrix} \gamma_0 & \gamma_0\,(v_0/c) & 0 & 0 \\ \gamma_0\,(v_0/c) & \gamma_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with $\gamma_0 = (1 - v_0^2/c^2)^{-1/2}$. Applying this matrix to the components $(u')^\mu$ yields the components u^μ of the particle's 4-velocity in frame S,

$$u = \Lambda(v_0) \begin{pmatrix} \gamma_{v'}\, c \\ \gamma_{v'}\, v' \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c\,\gamma_0\,\gamma_{v'}\,(1 + v_0\, v'/c^2) \\ \gamma_0\,\gamma_{v'}\,(v_0 + v') \\ 0 \\ 0 \end{pmatrix}.$$

The ordinary 3-velocity is related to the 4-velocity via the relation (3.2.4), or $v^k = u^k/(u^0/c)$. Inserting the S frame components u^μ yields a 3-velocity (pointing in the 1 direction),

$$v = v^1 = \frac{v_0 + v'}{1 + v_0\, v'/c^2}\,, \qquad [v^2 = v^3 = 0]\,.$$

The numerator of this answer is the familiar Galilean result, but the denominator reflects relativistic corrections. If either v_0 or v' are small compared to c, then the denominator is close to the 1 and the Galilean result is approximately correct. But if either (or both) of the initial three-velocities approach c, then the final velocity v also approaches, but never exceeds, c.

The reader is encouraged to: (a) verify that the result for u satisfies $u^2 = -c^2$, (b) show that the results for u^0 and v satisfy the usual relation $u^0 = \gamma_v\, c$ with $\gamma_v = (1 - v^2/c^2)^{-1/2}$, and (c) show that if velocities are described by equivalent rapidities, $v_0 = c\,\tanh\eta_0$, $v' = c\,\tanh\eta'$, and $v = c\,\tanh\eta$, then $\eta = \eta_0 + \eta'$. In other words, rapidities (of collinear boosts) add linearly, but 3-velocities do not.

3.10.2 Doppler shift

Q: Using Eqs. (3.6.2) and (3.6.3), derive the relativistic Doppler shift of light — find the frequency seen by an observer moving *away* from a source of light with (ordinary) frequency ν_0, as measured in the rest frame of the source.

A: Take the observer to be moving in the x^1 direction with velocity v. The most convenient reference frame is the rest frame of the source (since this is the frame in which we have information about both the light and the observer). In the source frame, the observer's 4-velocity has components $u^\mu_{\rm obs} = c\,\gamma\,(1,\, v/c,\, 0,\, 0)$. The angular frequency of the light is $\omega_0 = 2\pi\nu_0$, and the spacetime wavevector (for the light moving in the x^1 direction which reaches the observer) has components $k^\mu = (\omega_0/c)\,(1,\, 1,\, 0,\, 0)$. Using (3.6.2), we have

$$\frac{\nu_{\rm obs}}{\nu_0} = \frac{\omega_{\rm obs}}{\omega_0} = -\frac{u_{\rm obs}\cdot k}{\omega_0} = \gamma\,(1 - v/c) = \sqrt{\frac{1 - v/c}{1 + v/c}}\,.$$

For $v > 0$, corresponding to the source and observer receding from each other, we have $\nu_{\rm obs}/\nu_0 < 1$, so the light appears to be red-shifted. For an observer approaching the source, simply change the sign of v; in this case $\nu_{\rm obs}/\nu_0 > 1$ and the light appears blue-shifted to a higher frequency.

3.10.3 Kinetic energy, speed, and momentum[7]

Q: A relativistic particle has kinetic energy equal to twice its rest energy. Find the speed of the particle (relative to c) and its spatial momentum.

A: Total energy is kinetic energy plus rest energy, $E = K + mc^2 = 3\,mc^2$. Total energy is also γ times rest energy, so $\gamma \equiv (1 - v^2/c^2)^{-1/2} = E/(mc^2) = 3$. Solving for v/c gives $v/c = \sqrt{1 - \frac{1}{9}} = 0.943$. The (magnitude of the) particle's spatial momentum is $p = \gamma\, mv = 3mc\,(v/c) = 2.83\,mc$. This could also be evaluated directly using Eq. (3.3.10), which may be rearranged as $p = \sqrt{(E/c)^2 - (mc)^2} = \sqrt{8}\,mc$.

3.10.4 Light propulsion[8]

Q: The most fuel-efficient rocket exhaust is photons (*i.e.*, light), as this has the fastest exit velocity for any given energy. Suppose a rocket, emitting only light (in the backward direction), has initial mass M_i and final mass M_f. Find its final velocity (in the frame in which it starts from rest).

A: The hard way to do this problem is to integrate the relativistic version of Newton's equations (3.4.1) with a time-dependent mass. It is much easier to just use conservation of total energy and momentum. Working in the initial rest frame of the rocket, the total initial energy and spatial momentum are $E_{\rm tot} = M_i\,c^2$ and $\vec p_{\rm tot} = 0$, respectively, since the rocket is at rest. At the final time the rocket, now with mass M_f, is moving in some direction (call it $+\hat x$) with velocity $\vec v$, and all the emitted photons are moving in the opposite $(-\hat x)$ direction. Hence, the total energy at the final

[7] Adapted from Kogut problem 6-11.
[8] Adapted from Kogut problem 6-16.

time is $E_{\text{tot}} = \gamma\, M_f\, c^2 + E_{\text{photons}}$, and the total final spatial momentum $\vec{p}_{\text{tot}} = \gamma\, M_f\, \vec{v} + \vec{p}_{\text{photons}} = (\gamma\, M_f\, v - E_{\text{photons}}/c)\,\hat{x}$. (Note that $|\vec{p}_{\text{photon}}| = E_{\text{photon}}/c$, since the 4-momentum of a photon is a light-like vector.)

Requiring that the final total spatial momentum agree with the initial value of 0 implies that $E_{\text{photons}}/c = \gamma\, M_f\, v$, while demanding that the total initial and final energies agree implies that $E_{\text{photons}}/c = M_i\, c - \gamma\, M_f\, c$. Equating these two results for E_{photons}/c gives

$$M_i = \gamma\, M_f\, \frac{c+v}{c} = M_f \sqrt{\frac{1+v/c}{1-v/c}}\,.$$

Solving for v/c yields

$$\frac{v}{c} = \frac{(M_i/M_f)^2 - 1}{(M_i/M_f)^2 + 1} = \frac{M_i^2 - M_f^2}{M_i^2 + M_f^2}\,.$$

So reaching a relativistic velocity, $v \approx c$, requires that the final mass (including the payload) be very much smaller than the initial mass ($M_f \ll M_i$), even with the most efficient idealized propulsion imaginable.