Chapter 7

Weak interactions

As already discussed, weak interactions are responsible for many processes which involve the transformation of particles from one type to another. Weak interactions cause nuclear beta decay, as well as the decays of muons, charged pions, kaons, and many other hadrons. All processes which involve production or scattering of neutrinos, the conversion of quarks from one flavor to another, or the conversion of leptons from one type to another, involve weak interactions.

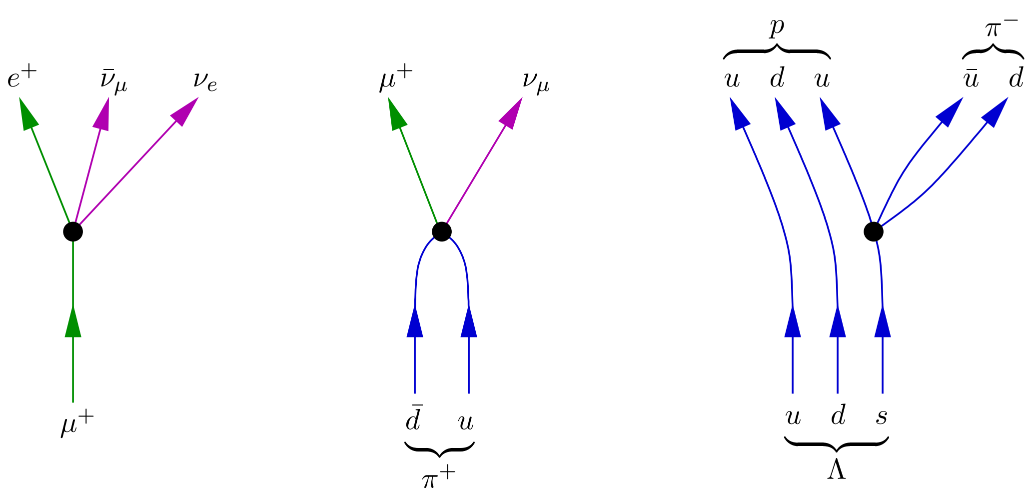

Figure 7.1: Depictions, at the level of quarks and leptons, of the weak decays $\mu^+ \to e^+ + \bar{\nu}_\mu + \nu_e$, $\pi^+ \to \mu^+ + \nu_\mu$, and $\Lambda \to p + \pi^-$.

Figures 7.1 and 7.2 depict, at the level of quarks and leptons, some of these weak interaction processes. As these figures illustrate, every weak interaction involves four fermions, either one fermion turning into three (as in muon decay) or two incoming fermions scattering and producing two outgoing fermions (as in neutrino scattering). As the above Λ baryon decay illustrates, there can also be *spectator* quarks which are constituents of the hadrons involved but not direct participants in the weak interaction process.

The complete Hamiltonian which describes particle interactions can be written as a sum of contributions from strong, electromagnetic, and weak interactions,

$$H = H_{\text{strong}} + H_{\text{EM}} + H_{\text{weak}} \,. \tag{7.0.1}$$

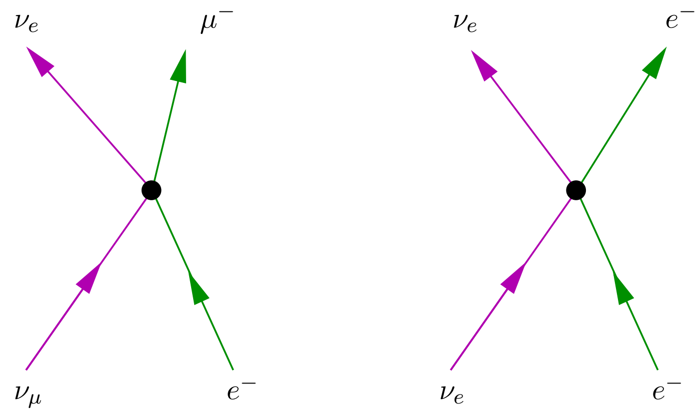

Figure 7.2: Left: inelastic neutrino scattering, $\nu_\mu + e^- \to \nu_e + \mu^-$. Right: elastic neutrino scattering, $\nu_e + e^- \to \nu_e + e^-$.

Because weak interactions are truly weaker than strong or electromagnetic interactions, it is useful to think of $H_{\rm weak}$ as a small perturbation to the dynamics generated by strong and electromagnetic interactions.

7.1 Muon decay

Consider (anti)muon decay, $\mu^+ \to e^+ + \bar\nu_\mu + \nu_e$. Let the ket $|\mu(\vec p{=}0)\rangle$ denote an initial state containing a single μ^+ at rest. Let the bra $\langle e(\vec p_e)\, \bar\nu_\mu(\vec p_{\bar\nu})\, \nu_e(\vec p_\nu)|$ denote a final state describing a positron with spatial momentum $\vec p_e$, a muon antineutrino with momentum $\vec p_{\bar\nu}$, and an electron neutrino with momentum $\vec p_\nu$. The existence of muon decay means that the time evolution of the initial state $|\mu(\vec p{=}0)\rangle$ will have a non-zero projection onto the final state $\langle e(\vec p_e)\, \bar\nu_\mu(\vec p_{\bar\nu})\, \nu_e(\vec p_\nu)|$. This can only happen if the Hamiltonian, which generates time evolution, has a non-zero matrix element connecting these states. And this can only be due to the weak interaction part of the Hamiltonian. In other words, the existence of muon decay implies that the amplitude

$$M \equiv \langle e(\vec p_e)\, \bar\nu_\mu(\vec p_{\bar\nu})\, \nu_e(\vec p_\nu)|H_{\rm weak}|\mu(\vec p{=}0)\rangle \ , \tag{7.1.1}$$

is non-zero. The rate of decay must be proportional to the square of this amplitude. Because there are many different final states corresponding to different values of the final momenta p_e, $p_{\bar\nu}$ and p_ν, the complete decay rate Γ will involve a sum over all possible final states. Schematically,

$$\Gamma \quad \sim \sum_{\text{final states}} |M|^2 \,. \tag{7.1.2}$$

The amplitude M must vanish, due to momentum conservation, if $\vec p_e + \vec p_{\bar\nu} + \vec p_\nu \neq 0$. When momentum is conserved, $p_{\bar\nu}$ will equal $-(p_\nu + p_e)$, so M may be regarded as function of two independent momenta, p_e and p_ν. This amplitude can, in principle, depend in some complicated fashion on these two final momenta. But the simplest possibility is for the amplitude to have negligible dependence on the outgoing momenta. Physically, this corresponds to a point-like interaction, for which the spatial variation of wavefunctions (due to their momentum) plays no role.

This guess turns out to work remarkably well. If the amplitude M is momentum independent then, with just a little calculation, one can perform the sum over final states in Eq. (7.1.2) and predict the muon decay spectrum as a function of positron energy. (That is, the fraction of decays in which the positron has energy between E and $E+dE$.) Figure 7.3 shows the comparison between experimental data for the decay spectrum and the result of this calculation. The agreement is excellent.

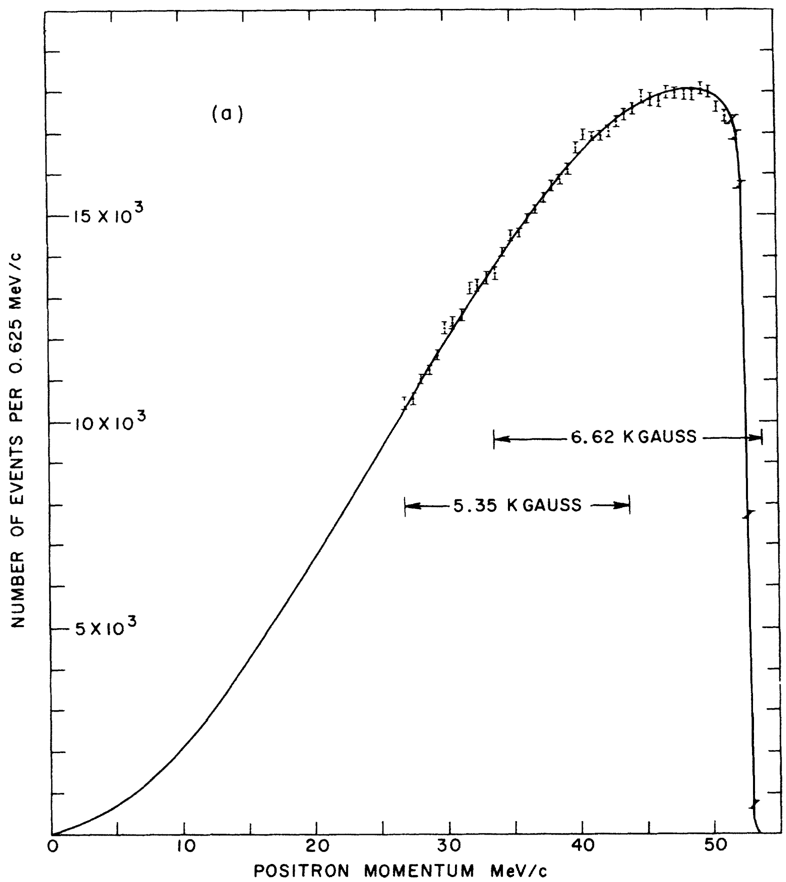

Figure 7.3: Energy spectrum of positrons emitted from decays of positively charged muons. The solid curve is the theoretical prediction; data points are shown with error bars. [From M. Bardon *et al.*, Phys. Rev. Lett. 14, 449 (1965)].

To characterize the value of the amplitude M, it will be useful to begin with some dimensional analysis. To make this as easy as possible, it will be convenient to use "natural units" in which $\hbar = c = 1$. Since c has ordinary dimensions of [length/time], setting $c = 1$ means that we are regarding length and time as having the same dimensions. Since $\hbar$ has dimensions of [energy $\times$ time], setting $\hbar = 1$ means that we are regarding energy and frequency (or inverse time) as having the same dimensions. Setting both $\hbar$ and c to unity means that we are treating length and inverse energy as dimensionally equivalent. After using natural units in any calculation, one can always reinsert factors of $\hbar$ and c as needed to restore conventional dimensions. In particular, the value $\hbar c \simeq 197$ MeV fm may be regarded as a conversion factor which allows one to convert lengths measured in femtometers into lengths measured in MeV^{-1}, 1 fm $= \frac{1}{197}$ MeV^{-1}.

The Hamiltonian is the operator which measures energy. Its eigenvalues are the energies of stationary states. Therefore, the Hamiltonian must have dimensions of energy. If $|\Psi\rangle$ is any physical, normalized state, then the matrix element $\langle\Psi|H|\Psi\rangle$ is the expectation value of the energy in state $|\Psi\rangle$. Hence, matrix elements of the Hamiltonian, such as the muon decay amplitude M, also have dimensions of energy, provided the states appearing in the matrix element are normalized.

The wavefunction describing a particle with definite momentum $\vec{p}$ is proportional to the plane wave $e^{i\vec{p}\cdot\vec{x}/\hbar}$. To normalize such a state, it is convenient to imagine that space is not infinite, but rather is limited to some finite, but arbitrarily large region $\mathcal{V}$. The condition that a state is normalized then becomes $1 = \int_{\mathcal{V}} d^3x\ |\Psi(\vec{x})|^2$, where the integral only includes the interior of the region $\mathcal{V}$. For simplicity, suppose that this region is a cube of size L (and hence volume L^3). A normalized state describing a particle with momentum $\vec{p}$ will thus have a wavefunction $\Psi(\vec{x}) = e^{i\vec{p}\cdot\vec{x}/\hbar}/L^{3/2}$. The absolute square of this wavefunction gives a constant probability density of $1/L^3$ whose volume integral over the region $\mathcal{V}$ equals one, as desired.

Now consider the muon decay amplitude M. The initial muon, with zero spatial momentum, will have a constant wavefunction, $\psi_\mu(\vec{x}) = 1/L^{3/2}$. The final positron, with momentum $\vec{p}_e$, will have a plane-wave wavefunction $\psi_e(\vec{x}) = e^{i\vec{p}_e\cdot\vec{x}/\hbar}/L^{3/2}$, and similarly the final neutrino and antineutrino will have wavefunctions $\psi_{\nu_e}(\vec{x}) = e^{i\vec{p}_\nu\cdot\vec{x}/\hbar}/L^{3/2}$ and $\psi_{\bar{\nu}_\mu}(\vec{x}) = e^{i\vec{p}_{\bar{\nu}}\cdot\vec{x}/\hbar}/L^{3/2}$, respectively.

Since the point-like weak interaction event can occur at any point in space, the complete amplitude will involve an integral over space, with an integrand which is the product of the amplitude $\psi_\mu(\vec{x})$ to find the muon at some point $\vec{x}$, times the product of conjugate wavefunctions $\psi_e(\vec{x})^*\,\psi_{\nu_\mu}(\vec{x})^*\,\psi_{\bar{\nu}_e}(\vec{x})^*$, giving the amplitudes for the created positron, neutrino, and antineutrino all to be at point $\vec{x}$, all times some overall constant which will control the rate of this process,

$$M = \left[\int_{\mathcal{V}} d^3x\ \psi_e(\vec{x})^*\,\psi_{\nu_\mu}(\vec{x})^*\,\psi_{\bar{\nu}_e}(\vec{x})^*\,\psi_\mu(\vec{x})\right] \times (\text{const.}). \tag{7.1.3}$$

The overall constant is known as the *Fermi constant*, G_F, divided by $\sqrt{2}$. (Including this factor of $\sqrt{2}$ is merely a convention, but is required so that G_F matches its historical definition.) The integrand appearing in this matrix element is just a constant,

$$\psi_e(\vec{x})^*\,\psi_{\nu_\mu}(\vec{x})^*\,\psi_{\bar{\nu}_e}(\vec{x})^*\,\psi_\mu(\vec{x}) = \frac{e^{-i(\vec{p}_e+\vec{p}_\nu+\vec{p}_{\bar{\nu}})\cdot\vec{x}/\hbar}}{(L^{3/2})^4} = L^{-6}\,, \tag{7.1.4}$$

provided the momenta satisfy conservation of momentum, $\vec{p}_e + \vec{p}_\nu + \vec{p}_{\bar{\nu}} = 0$. Integrating over the region $\mathcal{V}$ thus simply yields a factor of the volume, L^3, of this region. Hence, we find

$$M = \frac{G_F/\sqrt{2}}{L^3}\,. \tag{7.1.5}$$

We noted above that the decay amplitude M must have dimensions of energy. Since $1/L^3$ has dimensions of energy cubed (having set $\hbar = c = 1$), we learn that the Fermi constant G_F must have dimensions of $1/(\text{energy})^2$.

The value of the Fermi constant G_F may be fixed by demanding that the muon decay rate Γ calculated from Eq. (7.1.2) agree with the experimentally determined value. The decay rate is just the inverse of the lifetime, so $\Gamma = 1/\tau_\mu = 1/(2\ \mu\text{s})$. Performing the sum over final states in Eq. (7.1.2) involves integrating over the final momenta subject to the constraints of energy and momentum conservation. Details of this calculation, which is straightforward, will be omitted. One finds that $\Gamma = G_F^2\,m_\mu^5/(192\pi^3)$. Equating this with the inverse of the observed decay rate and solving for G_F yields

$$G_F = 1.2 \times 10^{-5}\ \text{GeV}^{-2} = 12\ \text{TeV}^{-2}\,. \tag{7.1.6}$$

7.2 Neutrino scattering

The significance of this determination of the Fermi constant comes from the fact that a factor of G_F will appear in *every* weak interaction amplitude. Consider, for example, the inelastic neutrino scattering process

$$\nu_\mu + e^- \to \nu_e + \mu^- \,, \tag{7.2.1}$$

depicted in Fig. 7.2 . With sufficient experimental skill and resources, this is a measurable process. The cross section for this scattering process equals the rate of scattering events divided by the incident flux of neutrinos and the density of target electrons. For a neutrino beam with constant flux, the scattering rate is just the probability of scattering in time Δt, divided by Δt. And the probability, as always in quantum mechanics, is the absolute square of a probability amplitude which involves a matrix element of the weak interaction Hamiltonian between the relevant incoming and outgoing states, $M = \langle \text{out}|H_{\text{weak}}|\text{in}\rangle$. This weak interaction amplitude must also be proportional to G_F, so that[1]

$$\sigma \propto |M|^2 \propto G_F^2 \,. \tag{7.2.2}$$

Now do some more dimensional analysis. A cross section is an area, with dimensions of length squared or (in natural units) [energy]$^{-2}$. The Fermi constant G_F also has dimensions of [energy]$^{-2}$, but G_F appears squared in the cross section. Therefore the cross section must equal G_F^2 times something else with dimensions of [energy]^2. What can this something else depend on? One possibility, which is surely relevant, is the neutrino energy. But the energy of a particle is frame-dependent. One must be able to express the cross section in terms of Lorentz invariant quantities. A Lorentz invariant measure of the scattering energy is $s \equiv -(p_{\nu_\mu} + p_e)^2 = E_{\text{c.m.}}^2$. At low energies, the value of the cross section will also depend on the electron and muon masses. After all, if $E_{\text{c.m.}} < m_\mu c^2$, then the reaction $\nu_\mu + e^- \to \nu_e + \mu^-$ cannot possibly occur. It must be possible to express the cross section in the (dimensionally consistent!) form

$$\sigma = G_F^2 \; s \times f\Big(\frac{m_e}{\sqrt{s}}, \frac{m_\mu}{\sqrt{s}}\Big) \,, \tag{7.2.3}$$

where f is some function of the dimensionless ratios $m_e/E_{\text{c.m.}}$ and $m_\mu/E_{\text{c.m.}}$. (This function will be non-vanishing only when both arguments are less than one.)

The simplest regime to consider is high energy relative to the muon mass, $E_{\text{c.m.}} \gg m_\mu c^2$. In this domain, the ratios $m_e/E_{\text{c.m.}}$ and $m_\mu/E_{\text{c.m.}}$ are both tiny. Since the cross section can be expressed in the form (7.2.3), understanding the behavior of the cross section when the energy is large is the same problem as understanding the behavior of the cross section in a hypothetical world where the value of the electron and muon masses are arbitrarily small.

A crucial observation is that there is no reason to expect anything dramatic, or singular, to happen in the limit of vanishingly small electron and muon mass (at fixed energy $E_{\text{c.m.}}$). In the relativistic relation between energy and momentum, the zero mass limit is perfectly smooth, and just leads to the energy-momentum relation of a massless particle,[2]

$$E(\vec p) = \sqrt{\vec p^{\,2} + m^2} = |\vec p| + \tfrac{1}{2}\frac{m^2}{|\vec p|} + \cdots \underset{m\to 0}{\longrightarrow} |\vec p| \,. \tag{7.2.4}$$

[1]In fact, analytic continuation in the four-momenta relates the amplitude for inelastic neutrino scattering, $\nu_\mu + e^- \to \nu_e + \mu^-$, to the amplitude for μ^+ decay. This relation, which involves replacing particles in the initial state by their antiparticles in the final state (or vice-versa) is known as *crossing symmetry*.

[2]In contrast, the non-relativistic energy $E_{\text{NR}}(\vec p) = \vec p^{\,2}/(2m)$ is *not* well-behaved if $m \to 0$ for fixed momentum $\vec p$.

Similarly, the massless limit of the function $f(\frac{m_e}{\sqrt{s}}, \frac{m_\mu}{\sqrt{s}})$ appearing in the cross section (7.2.3) should be expected to be finite and non-zero, so that $A \equiv f(0,0)$ is just some pure number like 2 or π. A detailed calculation shows that, for the process (7.2.1), the number A is $1/\pi$. Therefore, the inelastic neutrino cross section is given by

$$\sigma_{\nu_\mu e^- \to \nu_e \mu^-} = \frac{G_F^2 \, E_{\text{c.m.}}^2}{\pi} \,, \tag{7.2.5}$$

when $E_{\text{c.m.}} \gg m_\mu c^2$. This quadratic rise of the cross section with center-of-mass energy (for energies above the relevant particle masses) also applies to other weak interaction scattering processes, including neutrino scattering with nucleons and elastic neutrino-electron scattering. In the latter example, the cross section is

$$\sigma_{\nu_e e^- \to \nu_e e^-} = 0.175 \; G_F^2 \, E_{\text{c.m.}}^2 \,. \tag{7.2.6}$$

These predictions of rising neutrino cross sections with increasing energy have been confirmed experimentally for energies in the multi-MeV to multi-GeV range.[3] But the prediction of quadratically rising cross sections raises an immediate puzzle: can cross sections really grow with increasing energy *forever*? Or is there some point at which the behavior must change?

In fact, cross sections cannot become arbitrarily large. The number of scattering events in any scattering experiment is proportional to the cross section. But ultimately, the number of scatterings cannot be larger than the total number of projectiles! A quantum mechanical analysis shows that for point-like (or so-called s-wave) scattering, the cross-section must satisfy the bound

$$\sigma < \frac{\lambda^2}{4\pi} = \frac{\pi}{\vec{p}^{\,2}} \,, \tag{7.2.7}$$

where $\lambda = 2\pi\hbar/|\vec{p}|$ is the de Broglie wavelength of the projectile in the center-of-mass frame. This is referred to as a *unitarity* bound.

For an ultra-relativistic scattering, viewed in the center-of-mass frame, the energy of each particle is almost the same as the magnitude of its momentum (times c), and hence $E_{\text{c.m.}} \simeq 2|\vec{p}\,|$. Equating expression (7.2.5) for the neutrino cross section with the unitarity bound (7.2.7), one finds that the cross section (7.2.5) violates the unitarity bound when the center-of-mass energy exceeds

$$E^* \equiv \sqrt{\frac{2\pi}{G_F}} \approx 700 \text{ GeV} \,. \tag{7.2.8}$$

Therefore, at some energy below 700 GeV, something must dramatically change the behavior of weak interaction cross sections to stop their quadratic rise with increasing energy.

7.3 Weak gauge bosons

In fact, at energies somewhat below E^*, weak interaction cross sections become comparable to electromagnetic cross sections. At this point, one might anticipate significant changes in the behavior

[3] See, for example, the plots of the (anti)neutrino-nucleon total cross section at the particle data group website. Note that for neutrino scattering on a nucleus, the lab frame energy is proportional to the square of the center-of-mass energy, $E_{\text{lab}} \propto E_{\text{c.m.}}^2$, when E_{lab} is large compared to the target mass. So the quadratic rise of the cross section with $E_{\text{c.m.}}$ is equivalent to linear growth as a function of E_{lab}.

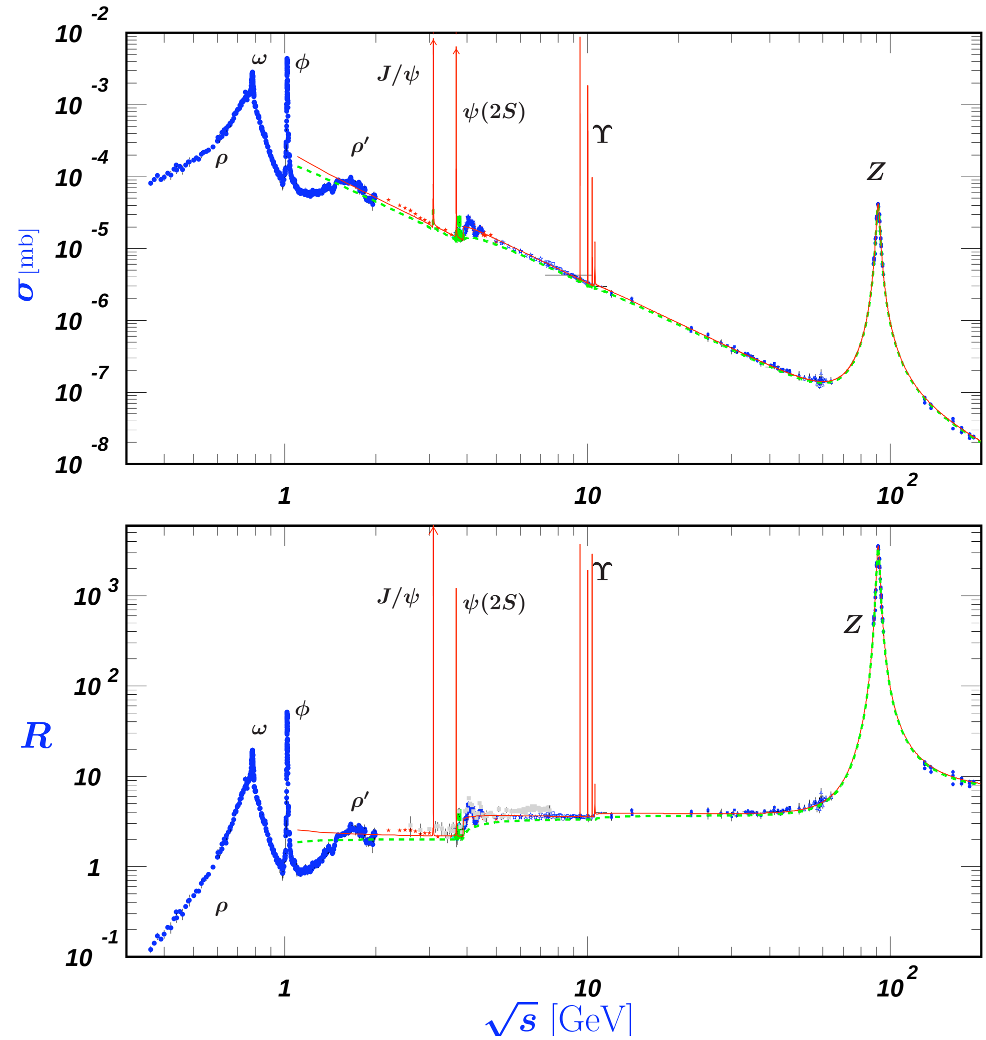

Figure 7.4: Top: Cross section for e^+e^- annihilation to hadrons as a function of $\sqrt{s} = E_{\text{c.m.}}$. Bottom: Ratio of cross sections for e^+e^- annihilation to hadrons versus annihilation to muon pairs, $R = \sigma_{e^+e^- \to \text{hadrons}}/\sigma_{e^+e^- \to \mu^+\mu^-}$. [From http://pdg.lbl.gov/2007/reviews/hadronicrpp.pdf .]

of both electromagnetic and weak interactions. This turns out to be true. Figure 7.4 shows the cross section for electron-positron annihilation into hadrons as a function of $\sqrt{s} = E_{\text{c.m.}}$. At energies below about 50 GeV, one sees that the cross section generally decreases with increasing energy (note the logarithmic scale), but is punctuated by various spin one, parity odd hadronic resonances — the broad ρ and ρ', the narrower ω and ϕ, and the very narrow "spikes" associated with $c\bar{c}$ and $b\bar{b}$ heavy quark states. The J/ψ and $\psi(2s)$ are $c\bar{c}$ bound states with energies close to twice the charm quark mass, while the upsilon (Υ) states near $2m_b$ are $b\bar{b}$ states. But then, at a much higher energy near 90 GeV, there is a very big resonance which is something new. This is not a quark-antiquark bound state, but rather a new type of particle which is called the Z boson. The same resonance appears in neutrino scattering. There is also a closely related pair of charged particles known as the W^+ and W^-. These are not seen in Figure 7.4 because a single W^+ or W^- cannot result from e^+e^- annihilation — this would violate charge conservation!

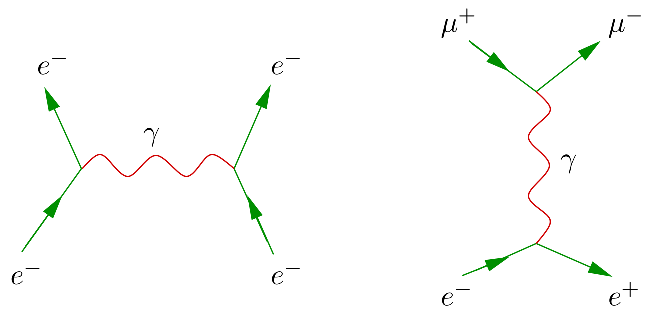

Figure 7.5: Feynman diagrams for Coulomb scattering: $e^-e^- \to e^-e^-$ (left), and electron-positron annihilation to muons: $e^+e^- \to \mu^+\mu^-$ (right).

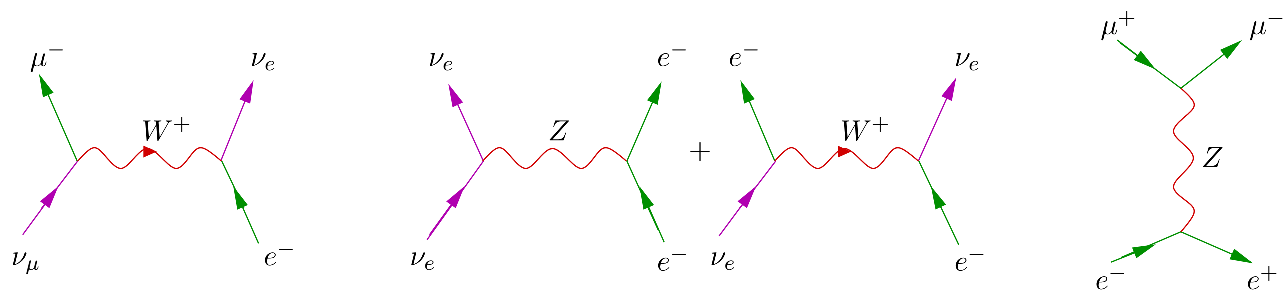

Figure 7.6: Feynman diagrams for inelastic neutrino scattering: $\nu_\mu + e^- \to \nu_e + \mu^-$ (left), elastic neutrino scattering: $\nu_e + e^- \to \nu_e + e^-$ (middle), and the weak interaction contribution to $e^+e^- \to \mu^+\mu^-$ (right).

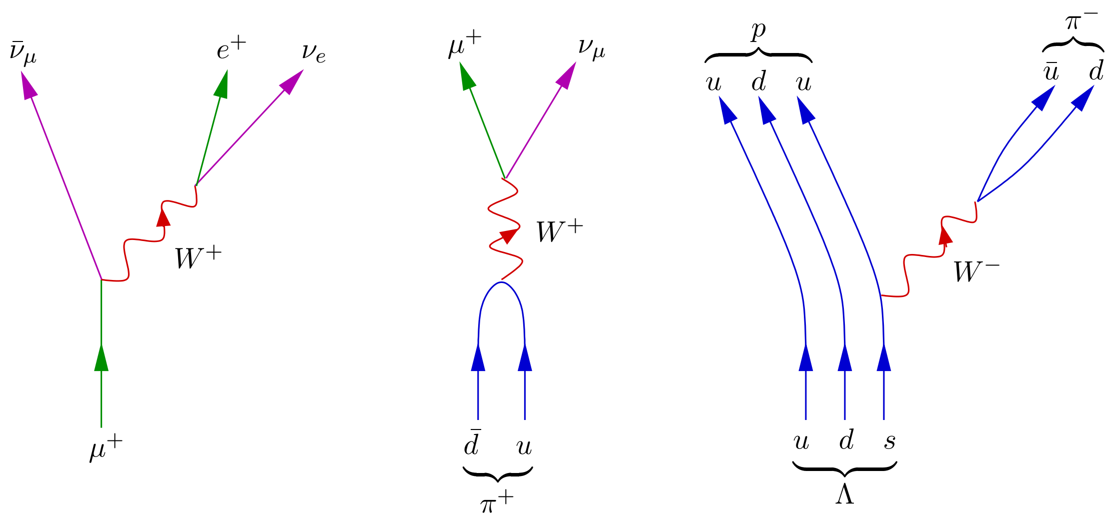

Figure 7.7: Depictions of the weak decays $\mu^+ \to e^+ + \bar{\nu}_\mu + \nu_e$ (left), $\pi^+ \to \mu^+ + \nu_\mu$ (middle), and $\Lambda \to p + \pi^-$ (right), showing the exchange of weak gauge bosons.

Together, the $W^{\pm}$ and Z are known *weak gauge bosons*. They are spin one particles with masses

$$m_W = 80.4 \text{ GeV}, \qquad m_Z = 91.2 \text{ GeV}. \tag{7.3.1}$$

These particles mediate the weak interactions, in the same sense that the photon is responsible for mediating electromagnetic interactions. Coulomb interactions may be viewed as resulting from the exchange of photons between charged particles, and a process like $e^+e^- \to \mu^+\mu^-$ may be regarded as occurring via the annihilation of the electron and positron into a photon, which lives only a very short time before converting into the final μ^+ and μ^-. The diagrams of Figure 7.5 depict these electromagnetic processes.

In the same fashion, weak interactions may be regarded as arising from the exchange of W and Z bosons. Figure 7.6 depicts the same weak interaction scattering processes illustrated in Figure 7.2, plus the weak interaction contribution to $e^+e^- \to \mu^+\mu^-$, showing the exchange of weak gauge bosons. Figure 7.7 does the same for the weak decays of Figure 7.1 . The diagrams of Figures 7.5–7.6 are examples of *Feynman diagrams*. They actually do more than merely depict some process — these diagrams encode precise rules for how to calculate the quantum mechanical amplitude associated with each process. But developing this in detail will have to be left for a later class.

With this brief sketch of the current understanding of weak interactions, we must conclude our introduction to particles and symmetries. I hope it has whetted your appetite to learn more about this subject.