Chapter 10

Introduction to Group Theory

Since symmetries described by groups play such an important role in modern physics, we will take a little time to introduce the basic structure (as seen by a physicist) of this most interesting field of study. We will first define groups in the abstract and then proceed to think about their representations, typically in the form of matrices. In physics we are typically interested in "real" operators that act on "real" states, which provide concrete representations of the more abstract concept of groups. The underlying arithmetic will be familiar from your previous experience with matrices.

10.1 Definitions

Group (G): A set of (perhaps abstract) elements (*i.e.*, things) - $g_1, ..., g_n$, plus a definition of the multiplication operation, *i.e.*, a definition of the product or combination of two of the elements, such that

1. $g_i \bullet g_j = g_k \in G$ - products of elements are also elements of the group (a property of groups called closure),
2. the multiplication operation is associative - $(g_i \bullet g_j) \bullet g_k = g_i \bullet (g_j \bullet g_k)$,
3. the identity element exists as an element of the group, $\mathbf{1} \in G$, where $\mathbf{1} \bullet g_j = g_j \bullet \mathbf{1} = g_j$ (sometimes the left and right identities are distinct, but not generally in the context of physics),
4. the group includes a *unique* inverse for each element, $g_i \in G \Rightarrow g_i^{-1} \in G$ such that $g_i \bullet g_i^{-1} = g_i^{-1} \bullet g_i = \mathbf{1}$ (as with matrices the inverse is sometimes defined separately for left and right multiplication but this situation will not arise in this discussion).

Note: it is *not* necessary that the multiplication be commutative (and the interesting cases are those that are *not* commutative):

- if $g_i \bullet g_j = g_j \bullet g_i$ (commutative), the group is an Abelian group, Abelian group
- if $g_i \bullet g_j \neq g_j \bullet g_i$ (non-commutative), the group is a non-Abelian group. non-Abelian group

10.2 Finite Groups

If the number of elements n is finite ($n < \infty$), then the group is called a finite or discrete group of order n. Finite groups are commonly used in the study of solid state physics where discrete symmetries arise regularly. There is a trivial group corresponding to $n = 1$ with $g = \mathbf{1}$ only. Clearly the group properties are all satisfied but in a trivial way.

How about $n = 2$? Call the elements of the group $\mathbf{1}$ and $\mathbf{P}$. Evidently $\mathbf{P}^{-1} = \mathbf{P}$, $\mathbf{P} \bullet \mathbf{P} = \mathbf{1}$ in order to satisfy the condition of being a group. There are, in fact, two related and physically interesting realizations of this group. One case is the reflection group where $\mathbf{P}$ is a reflection in a plane (*i.e.*, one of the 3 possible planes in 3-D space). For example, reflection in the xy plane means $\mathbf{P}f(x, y, z) = f(x, y, -z)$ so that $\mathbf{P} \bullet \mathbf{P}f(x, y, z) = \mathbf{P}f(x, y, -z) = f(x, y, z)$ as required. Another $n = 2$ group corresponds to reflection through the origin (in 3-D space), $\mathbf{P}(x, y, z) \rightarrow (-x, -y, -z)$. Again $\mathbf{P} \bullet \mathbf{P} = \mathbf{1}$, $\mathbf{P} \bullet \mathbf{P}(x, y, z) = \mathbf{P}(-x, -y, -z) = (x, y, z)$. This is the parity operation that (as we will see) plays an important role in the context of atomic, nuclear, and particle physics. Another (simpler) representation of the group of order 2 are the numbers $(1, -1)$ coupled with ordinary multiplication. The multiplication table for all of these order 2 groups is given in Table 10.1, where the notation is that a given entry is the result of multiplying the column label on the left by the row label (and the fact that $\mathbf{P}$ acts like (-1) is made explicit).

	$\mathbf{1}$	$\mathbf{P(-1)}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{P(-1)}$
$\mathbf{P(-1)}$	$\mathbf{P(-1)}$	$\mathbf{1}$

Table 10.1: Order 2 Group multiplication table

Since the structure of all of the order 2 groups (the multiplication table) is identical, we say that the groups are isomorphic (the formal mathematical term for identical). Note that the order 2 group is (trivially) Abelian, as must be the case if it can be (faithfully) represented by numbers (without needing matrices).

Next consider the representation afforded by the 3 complex numbers $\left(1, e^{2\pi i/3}, e^{4\pi i/3}\right) = \left(1, A, A^2\right)$, which serve to define a order 3 group. Again the multiplication table, as in Table 10.2, is unique and Abelian (the reader is encouraged to prove this).

	$\mathbf{1}$	$\mathbf{A}$	$\mathbf{A}^2$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{A}$	$\mathbf{A}^2$
$\mathbf{A}$	$\mathbf{A}$	$\mathbf{A}^2$	$\mathbf{1}$
$\mathbf{A}^2$	$\mathbf{A}^2$	$\mathbf{1}$	$\mathbf{A}$

Table 10.2: Order 3 Group multiplication table

Groups of order n of the form $\left(1, A, A^2, ..., A^n = 1\right)$ are called cyclic groups. So Table 10.2 indicates that the $n = 3$ group is cyclic. For $n = 4$ the cyclic group has one representation provided by the set of numbers $(1, i, -1, -i)$, where $(i)^{-1} = -i$ and $(-1)^{-1} = -1$. The multiplication table for the

$n = 4$ cyclic group is indicated in Table 10.3, including the just mentioned explicit representation.

	$\mathbf{1}$	$\mathbf{A}(i)$	$\mathbf{A}^2(-1)$	$\mathbf{A}^3(-i)$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{A}(i)$	$\mathbf{A}^2(-1)$	$\mathbf{A}^3(-i)$
$\mathbf{A}(i)$	$\mathbf{A}(i)$	$\mathbf{A}^2(-1)$	$\mathbf{A}^3(-i)$	$\mathbf{1}$
$\mathbf{A}^2(-1)$	$\mathbf{A}^2(-1)$	$\mathbf{A}^3(-i)$	$\mathbf{1}$	$\mathbf{A}(i)$
$\mathbf{A}^3(-i)$	$\mathbf{A}^3(-i)$	$\mathbf{1}$	$\mathbf{A}(i)$	$\mathbf{A}^2(-1)$

Table 10.3: Order 4 Cyclic Group multiplication table

However, for the case of $n = 4$ there is a second possible multiplication table as indicated in Table 10.4. This group, often called the 4's Group, is still Abelian, but is *not* cyclic. While it is still true that we have $\mathbf{AB} = \mathbf{BA} = \mathbf{C}$ (or $\mathbf{AA}^2 = \mathbf{A}^2\mathbf{A} = \mathbf{A}^3$ in the cyclic case), we now have $\mathbf{A}^2 = \mathbf{B}^2 = \mathbf{C}^2 = \mathbf{1}$ (instead of $\mathbf{A}^2 = \mathbf{A}^2$, $(\mathbf{A}^2)^2 = \mathbf{1}$ and $(\mathbf{A}^3)^2 = \mathbf{A}^2$).

	1	**A**	**B**	**C**
1	**1**	**A**	**B**	**C**
A	**A**	**1**	**C**	**B**
B	**B**	**C**	**1**	**A**
C	**C**	**B**	**A**	**1**

Table 10.4: 4's Group multiplication table

Note that the above groups are all Abelian (the elements commute and the multiplication tables are symmetric about the diagonal) and they can be represented by ordinary (complex) numbers. Faithful representations (*i.e.*, representations of the group that are faithful to its properties) of non-Abelian groups will require the use of matrices in order to exhibit nonzero commutators.

In general a group will contain subgroups, *i.e.*, subsets of the elements which themselves form groups. The full (original) group and the unit element are called the trivial subgroups, while other subgroups are called proper subgroups. Clearly the elements $(1, -1) = (1, \mathbf{A}^2)$ constitute a proper subgroup of order 2 of the order 4 cyclic group. In the study of finite groups the concepts of conjugate elements, classes and characters play an important role. Two elements, $\mathbf{A}$ and $\mathbf{B}$, of a group are conjugate if $\mathbf{B} = \mathbf{C}^{-1}\mathbf{AC}$ with $\mathbf{C}$ another element of the group. The set of elements conjugate to $\mathbf{A}$ when the element $\mathbf{C}$ is allowed to vary over all members of the group form a class, *i.e.*, they are all related by a similarity transformation. The elements of a class can be thought as representing the same transformation but for differ choices of the basis vectors (recall that a similarity transformation takes us to a different basis set). Since the trace of a matrix is unchanged by a similarity transformation, the traces of all matrices in a single class must be equal. This common trace is called the character of the class. Thus the characters help us to classify the structure of finite groups.

It is also useful to ask whether, through a judicious choice of basis (corresponding to a special similarity transformation), we can make all the matrices representing a group block diagonal. These sub-matrices will also constitute representations of the group, and we say that the original representation is reducible. If this diagonalization process is not possible, the original representation is said to be irreducible. Clearly life is easiest if we can work with the lowest dimension (faithful) irreducible representation. This is called the *fundamental* representation. The various fundamental representations play a special role in the process of labeling the multiplets of particles that appear in particle physics.

10.3 Lie Groups

In many (most?) circumstances the groups of interest in physics have an infinite number of elements, but the individual elements are specified by (*i.e.*, are functions of) a finite number (N) of parameters, $g = G(x_1, ..., x_N)$. Of particular interest are those groups where the parameters vary continuously over some range. Thus the number of parameters is finite but the number of group elements is infinite. If the range of all of the parameters is bounded, the group is said to be compact, *e.g.*, the parameter space of the compact group $SO(3)$ (rotations in 3-D as discussed in Chapter 1) is (the interior of) a sphere of radius π. On the other hand, the group of linear translations in 3-D is non-compact, *i.e.*, the magnitude of the translation can be arbitrarily large. Further, the groups we employ in physics often have the added feature that the derivatives ($\partial g/\partial x_i$) with respect to all parameters exist. Groups with this property are called *Lie Groups*. Lie Groups play an essential role in our understanding of particle physics and we will pursue this discussion of Lie Groups a bit further.

First we focus on the behavior near the origin of the parameter space. By definition the group element at the origin in parameter space,

$$g(0, ..., 0) \equiv \mathbf{1}\,, \tag{10.3.1}$$

is the identity element. Near the origin of the parameter space the group elements correspond to infinitesimal transformations (arbitrarily close to the identity operator) and the derivatives are especially important. As a result they have a special name - the *generators* X_k (as already mentioned in Chapters 1 and 5),

$$\frac{\partial g}{\partial x_k}|_{x_j=0,\mathrm{all}\,j} \equiv iX_k\,. \tag{10.3.2}$$

The factor i in the previous equation (and subsequent equations) arises from the conventional choice to deal with Hermitian generators represented by Hermitian matrices. Since the generators are finite in number ($k = 1$ to N), it is easier to discuss the generators than the infinite number of group elements (we are lazy and smart). The generators serve to define a N-dimensional Lie algebra (a vector space) where both addition (of elements of the algebra) and multiplication by constants are defined. The general element of this Lie algebra can be expressed as a linear combination of the generators,

$$\overrightarrow{X} = \sum_{k=1}^{N} c_k X_k\,. \tag{10.3.3}$$

This is analogous to the familiar 3-dimensional vector space except that here the generators are the basis vectors (instead of $\hat{x}, \hat{y}, \hat{z}$). Further, we can think of the generators as allowing a "Taylor series" expansion of the group elements near the origin. General group elements can be obtained from the elements of the algebra via exponentiation.

The algebra also supports the definition of an outer (or vector) product (like the familiar cross product) that produces another element of the algebra, *i.e.*, the algebra is closed under this operation. This "product" is just the familiar commutator

$$[X_k, X_l] \equiv X_k X_l - X_l X_k = iC_{klm} X_m \, . \tag{10.3.4}$$

The tensor C_{klm} is called the structure constant(s) of the algebra. It fully specifies the structure of the algebra and therefore of the structure of the group itself near the origin of the parameter space. For example, the Pauli matrices provide a representation of the unitary group operating on 2-D vectors, or $SU(2)$ (as discussed in Chapter 5). Thus the algebra of $SU(2)$ is given by

$$\left[\frac{\sigma_j}{2}, \frac{\sigma_k}{2}\right] = i\epsilon_{jkl}\frac{\sigma_l}{2} \, , \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{10.3.5}$$

with the structure constant equal to the (now familiar) fully antisymmetric 3x3x3 tensor ϵ_{jkl} (the Levi-Civita symbol). In fact, this is the *unique* such 3x3x3 antisymmetric tensor and all Lie groups with 3 generators must have the same structure constant and thus the same Lie algebra. In particular, the group of rotations in 3D, $SO(3)$ (as discussed in Chapter 1), has the same algebra,

$$[J_j, J_k] = i\epsilon_{jkl} J_l \, , \tag{10.3.6}$$

with the J_l standing for the generator of rotations. In quantum mechanics this operator will become familiar as the angular momentum operator. An explicit form for these matrices (appropriate for the rotations of ordinary location 3-vectors as in Chapter 1) is given by the following (and the reader is encouraged to check the commutation relations in Eq. 10.3.6),

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \; J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} , \; J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{10.3.7}$$

While the precise form of these matrices may not be intuitively obvious, the general form should be clear from our understanding of how ordinary rotations work. For example, a rotation about the 1-axis is a rotation in the 2-3 plane. It should serve to mix the 2 and 3 components of an ordinary 3-vector. Modulo the issue of phases, this is precisely what the form of the J_1 matrix in Eq. (10.3.7) does. It transforms a 3-component into a 2-component and a 2-component into a 3-component.

Since the algebras of $SU(2)$ and $SO(3)$ are identical, it is amusing to ask how the groups themselves differ. This issue is addressed below.

While it will not be demonstrated here, a very important concept is the connection between the symmetries of a physical system and the conserved quantities which characterize its dynamics. For example, if a system is rotationally symmetric, *i.e.*, invariant under the operation of the group elements of $SO(3)$, its motion will exhibit a constant, *i.e.*, conserved, angular momentum. Translational invariance in space implies conserved linear momentum, while translation invariance in time implies energy conservation. These connections are realizations of Noether's Theorem (*i.e.*, symmetries mean conserved quantities, see `http://courses.washington.edu/partsym/14Spr/Noether.pdf`).

To complete this introduction to group theory we will look in general terms at the two Lie groups that seem to arise most often in physics - the Orthogonal group $SO(n)$ and the Unitary group $SU(n)$. The former appears in the study of real n-D vector spaces, *e.g.*, Euclidean space-time, and are defined by being transformations of the vector space that preserve the length of vectors or, more generally preserving any scalar product (appropriately defined, if there is a nontrivial metric). Thus, if two vectors have zero scalar product in one reference frame, *i.e.*, they are orthogonal, they will remain orthogonal in the rotated frame - hence the name of the group. The Unitary group appears in the study of complex n-D vector spaces, *e.g.*, quantum mechanics, and are defined by again preserving the length of (state) vectors, *i.e.*, probability. [Note that in both cases scalar products are those products that "use" all indices - nothing is left to "operate on". Hence scalar products are left unchanged by the transformations.]

To see what the properties of the groups these statements imply consider first a n-D real vector and its square, r_1 and $r_1 \bullet r_1 = r_1^T r_1$. Now consider the same vector in a transformed reference frame (or after transforming the vector) where the transformation (rotation) is represented by a real matrix Λ (not a boost here), $r_1' = \Lambda r_1$. We demand for the Orthogonal group that $r_2 \bullet r_1$ be preserved by the transformation for any r_1, r_2,

$$r_2'^T r_1' = (\Lambda r_2)^T \lambda r_1 = r_2^T \Lambda^T \Lambda r_1 = r_2^T r_1 \,, \tag{10.3.8}$$

which leads us to

$$\Lambda^T \Lambda = \mathbf{1} \,,\, \Lambda^{-1} = \Lambda^T \,. \tag{10.3.9}$$

So the characteristic feature of the Orthogonal Group is that it is represented by real orthogonal matrices, *i.e.*, matrices whose inverses are their transposes. If the scalar product is defined with a non-trivial metric g, as with the Lorentz transformations of the group SO(3,1) (where the argument (3,1) reminds us of the plus/minus signs in the metric) we have instead (where $\det[g]^2 = 1$)

$$r_2'^T g r_1' = (\Lambda r_2)^T g \lambda r_1 = r_2^T \Lambda^T g \Lambda r_1 = r_2^T g r_1, \tag{10.3.10}$$

or

$$\Lambda^T g \Lambda = g \,,\, \Lambda^{-1} = g \Lambda^T g \,. \tag{10.3.11}$$

Note that it follows from these equations and the properties of determinants that

$$\det\left[\Lambda^T \Lambda\right] = \det\left[\Lambda^T\right] \det\left[\Lambda\right] = \det\left[\Lambda\right]^2 = 1, \det\left[\Lambda\right] = \pm 1, \tag{10.3.12}$$

or

$$\det\left[g \Lambda^T g \Lambda\right] = \det\left[g\right]^2 \det\left[\Lambda^T\right] \det\left[\Lambda\right] = \det\left[\Lambda\right]^2 = 1 \,. \tag{10.3.13}$$

Recall that typically we want only the "Special" (hence the "S" in the label of the group) or unimodular group (no reflections) and we require that the determinant of Λ be +1 (*i.e.*, a -1 means that a reflection is present in the transformation).

Using (complex) exponentiation (recall Eq. (10.3.2)) to go from the algebra to the group, we write $\Lambda = e^{i\alpha S}$, where α is a *real* parameter and, in order for Λ to be real, S is a purely imaginary nxn matrix (recall Eq. (10.3.7)). The orthogonal form means

$$\left(e^{i\alpha S}\right)^T = e^{i\alpha S^T} = \left(e^{i\alpha S}\right)^{-1} = e^{-i\alpha S} \Longrightarrow S^T = -S = S^* \Longrightarrow S^\dagger = (S^*)^T = S \,. \tag{10.3.14}$$

Thus (with our choice of i factors) the generator of a real, orthogonal transformation is represented by a Hermitian matrix (again recall the matrices in Eq. (10.3.7)). The constraint we imposed on the determinant of Λ translates into a constraint on the trace of S (the reader should convince herself of this result)

$$\det\left[e^{i\alpha S}\right] = +1 \Longrightarrow \mathrm{Tr}\,[S] = 0 \,, \tag{10.3.15}$$

which is trivially satisfied by an anti-symmetric matrix (*e.g.*, any purely imaginary, Hermitian matrix; recall this property for the J_k matrices above). For the more general case of a scalar product defined with a metric g, S is still traceless and satisfies $gS^Tg = S^*$, *i.e.*, S displays mixed symmetry as defined by g.

As an explicit example for $SO(3)$ (recall Chapter 1) consider a rotation by an angle θ about the 3 (or z-axis). It follows from the properties of the matrices in Eq. (10.3.7) that powers of these matrices (like the Pauli matrices) are simple,

$$J_3^{2n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; J_3^{2n+1} = J_3 \,. \tag{10.3.16}$$

Thus a rotation matrix defined by the exponential form $\Lambda = e^{i\alpha S}$ is given by its power series expansion as

$$\begin{aligned} g(\theta) = e^{i\theta J_3} &= \mathbf{1} + \sum_{n=1}^{\infty} \frac{(i\theta J_3)^n}{n!} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!} + J_3 \sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos\theta + iJ_3 \sin\theta \\ &= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{10.3.17}$$

We recognize that, as expected, this transformation does not change the 3 (or z) component of a vector, while the 1-2 ($x - y$) components are mixed in the just the way we expect for a rotation of the basis vectors in the 1-2 plane. (Note that our choice of signs yields a "passive" rotation, we are rotating the basis vectors not the physics vectors.)

Next we can determine the number of independent components of S, *i.e.*, the dimensionality of the corresponding algebra (also called the order of the Lie group). A purely imaginary Hermitian matrix (i times a real, anti-symmetric matrix) has zeroes on the diagonal and all components below the diagonal are determined by (-1 times) those above. Thus we want 1/2 the number of off-diagonal elements in an nxn matrix. Hence the algebra of $SO(n)$ has dimension

$$\begin{aligned} N\left[SO(n)\right] &= \frac{n^2 - n}{2} = \frac{n(n-1)}{2}, \\ N \begin{bmatrix} n = 2 \\ n = 3 \\ n = 4 \end{bmatrix} &= \begin{matrix} 1 \\ 3 \\ 6 \end{matrix}\,. \end{aligned} \tag{10.3.18}$$

The corresponding discussion for the Unitary group now involves complex numbers and complex conjugation in the scalar product. Thus, if the unitary transformation is described by a matrix U, we have

$$r_2'^{\dagger} r_1' = (Ur_2)^{\dagger}\, Ur_1 = r_2^{\dagger} U^{\dagger} U r_1 = r_2^{\dagger} r_1, \tag{10.3.19}$$

i.e., U is a unitary matrix. In (complex) exponential notation (where T is not necessarily purely imaginary as U is not real, but β is real)

$$\begin{aligned} U &= e^{i\beta T}, \quad U^\dagger = e^{-i\beta T^\dagger} = U^{-1} = e^{-i\beta T} \\ &\Rightarrow T^\dagger = T \,. \end{aligned} \tag{10.3.20}$$

The generator is again represented by a Hermitian matrix, as we expect from our earlier discussion. We also have

$$\det\left[U^\dagger U\right] = \det\left[U\right]^* \det\left[U\right] = 1, \, \det\left[U\right] = \pm 1 \tag{10.3.21}$$

and we focus on the Special version of the group, $SU(n)$,

$$\det\left[U\right] \equiv 1 \Rightarrow Tr\left[T\right] = 0 \,. \tag{10.3.22}$$

As an explicit example consider the analogue of the rotation in Eq. (10.3.17), but now replacing the generator J_3 with the corresponding Pauli matrix $\sigma_3/2$ (recall Eq. (10.3.5)). These definitions yield the following expressions

$$\begin{aligned} e^{i\sigma_3\theta/2} &= \mathbf{1}\sum_{n=0}^{\infty}\frac{(i\theta/2)^{2n}}{(2n)!} + \sigma_3\sum_{n=0}^{\infty}\frac{(i\theta/2)^{2n+1}}{(2n+1)!} \\ &= \mathbf{1}\cos\theta/2 + i\sigma_3\sin\theta/2 \\ &= \begin{pmatrix} \cos\theta/2 + i\sin\theta/2 & 0 \\ 0 & \cos\theta/2 - i\sin\theta/2 \end{pmatrix} \\ &= \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} . \end{aligned} \tag{10.3.23}$$

So the 2-components of the 2-spinor (for a spinor quantized in the z-direction, recall Chapter 5) are just multiplied by a phase (proportional to the component of the spin, $\pm 1/2$). Note that the transformation is just a factor of -1 for $\theta = 2\pi$. Unlike the rotation of "real" vectors, where a rotation angle of 2π always brings you back to where you started, for spin 1/2 we end up at -1 times where we started! We need to rotate through 4π to get back to where we started (see below).

So we conclude that the algebra of $SU(n)$ is defined by traceless, Hermitian (but not necessarily imaginary) matrices in the appropriate number of dimensions. In n-D a nxn complex matrix has 2 times n^2 components. Being Hermitian reduces this by a factor of 2 and the constraint of zero trace removes another degree of freedom. Thus the order of the special unitary algebra in n-D is

$$\begin{aligned} N\left[SU(n)\right] &= \frac{2n^2}{2} - 1 = n^2 - 1, \\ N\begin{bmatrix} n = 1(\text{really } U(1)) \\ n = 2 \\ n = 3 \end{bmatrix} &= \begin{matrix} 1 \\ 3 \\ 8 \end{matrix} . \end{aligned} \tag{10.3.24}$$

Note that the algebras of $U(1)$ and $SO(2)$ have the same (trivial) dimension (1). You might expect that they are related and they are! They are identical or isomorphic as groups, $U(1) \cong SU(2)$. This becomes obvious when we recall that rotations in a (single) plane, $SO(2)$, can be performed in any order, *i.e.*, $SO(2)$ is an Abelian group like $U(1)$. You might have thought that $SO(2)$ had

2-D representations, unlike $U(1)$, but, in fact, these representations can, by an appropriate choice of basis vectors, be *reduced* to the canonical 1-D representations $e^{in\theta}$, which are the irreducible representations of $U(1)$. Another way to see this is to note that 2-D problems, *i.e.*, $SO(2)$, can always be mapped onto the complex plane, *i.e.*, $U(1)$.

The groups $SO(3)$ and $SU(2)$ are also related. As mentioned earlier the algebras are identical since there is a there is a unique choice for the anti-symmetric tensor C_{jkl}. It must be equal to ϵ_{jkl} since this is the *only* 3x3x3 fully antisymmetric tensor - an application of the "what else can it be?" theorem! This implies that these groups must be identical near the origin of the 3-D parameter space. On the other hand, the groups are not isomorphic (identical) when we consider the full parameter space. Instead $SU(2)$ is, in some sense, a larger group. For every element of $SO(3)$ there are two elements in $SU(2)$. This relationship is called a homomorphism with a 2 to 1, $SU(2)$ to $SO(3)$, mapping. Another way to think about this is that the parameter space of $SO(3)$ is like the interior of a sphere of radius π (think about this!). Next consider how the parameter space maps onto the group space. Each point inside the sphere specifies a direction from the origin, which is the axis of the rotation, and a distance from the origin, which is the magnitude of the rotation. When we get to the surface at π, we must identify antipodal points, a rotation through π in one direction is equivalent to a rotation of π about the exactly opposite direction. This means we can define a path in both the parameter space and the group space by starting at the origin, going out to π in one direction, hopping to (exactly) the other side of the sphere, and coming back to the origin. This is a *closed* path in the group space that *cannot* be shrunk to zero in the parameter space! Thus the space is not simply connected. On the other hand for $SU(2)$ we define a similar picture but the sphere extends to 2π and now, no matter what direction we left the origin along, we reach the transformation -1 at 2π (recall that, when we rotate a spin 1/2 state by 2π, we don't get back to the original state but to minus the original state). Thus the entire surface at 2π is identified as a single point in group space (the group element -1 with no issues about only antipodal points in this case). All closed paths can be shrunk to zero and the group space is *simply* connected. $SU(2)$ is called the *covering* group for $SO(3)$.

Before we finish this discussion, let's think just a bit more about the representations of groups. Just as in our initial discussion of finite groups, we need the concept of reducible and irreducible. If a representation (*i.e.*, the matrices) of the group elements can be reduced to block diagonal form by some choice of basis vectors,

$$\begin{bmatrix} [\ ] & 0 & 0 \\ 0 & \begin{bmatrix} \ & \ \\ \ & \ \end{bmatrix} & 0 \\ 0 & 0 & \begin{bmatrix} \ & \ & \ \\ \ & \ & \ \\ \ & \ & \ \end{bmatrix} \end{bmatrix}, \tag{10.3.25}$$

then that representation is reducible. If it cannot written in this form, it is irreducible. In fact, the irreducible representations of Abelian groups are all 1-D. The smallest dimension representation that faithfully represents the group, *i.e.*, displays all of its structure (the commutation relations), is called the defining or fundamental representation. All groups have 1-D (scalar) representations but they are not faithful for non-Abelian groups. [Recall that 1-D representations are just numbers, which must commute unlike matrices.] For $SO(3)$ the fundamental representation is the vector representation, **3**. For $SU(2)$ the fundamental representation is the spinor representation, **2**. The half-integer spin representations ($J = 1/2, 3/2,...$), are often referred to as the spinor representations of $SO(3)$ but they are strictly the representations of $SU(2)$. The fundamental representation of the algebra of

$SU(2)$ and $SO(3)$ are provided by the matrices in Eqs. (10.3.5) and (10.3.7), respectively.

Interestingly we can also interpret the algebra itself as providing a representation of the group, called the *adjoint* representation. The generators themselves are the basis vectors. The transformation of a basis vector by a generator is defined as the commutator of the generator with the "basis vector" (*i.e.*, the other generator). Hence the structure constants, the C_{jkl}, define a matrix representation of the algebra and the group (by exponentiation). For $SO(3)$ the adjoint representation and the fundamental representation are identical, *i.e.*, the matrices in Eq. (10.3.7) can be written in the form $[J_j]_{kl} = i\epsilon_{jlk}$, where it is important to note the order of the last two indices. Thus for $SO(3)$ the adjoint representation provided by the structure constants (properly defined) is identical to the fundamental representation. Similarly the generators of $SU(2)$ provide not only the adjoint representation of $SU(2)$ but also form a fundamental (and adjoint) representation of $SO(3)$, *i.e.*, the Pauli matrices transform like a 3-vector under rotation.