Chapter 11

Multiplets and Young Diagrams

11.1 Basic Definitions

For the question of decomposing products of $SU(N)$ representations into irreducible representations(*e.g.*, $N = 2$ for spin or $N = 3$ for color or flavor), the most efficient notation is that of Young diagrams. These are just left justified arrays of boxes with a specific set of (seemingly ad hoc) rules for their manipulation and interpretation. Without derivation the rules include the following.

1. Each horizontal row of boxes is at least as long as the horizontal row below it.

2. We can think of the horizontal direction as symmetrization (with respect to some internal index) and the vertical direction as anti-symmetrization. There are at most N rows for the case of $SU(N)$.

3. For the $SU(3)$ representation (p,q) the first row has p more boxes than the second row and the second row has q more boxes than the third row. Thus we have

$$(1,0) = \underset{\sim}{3} = \begin{array}{|c|}\hline \ \\ \hline\end{array},\ (0,1) = \underset{\sim}{\bar{3}} = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array},\ (0,0) = \underset{\sim}{1} = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array},$$

$$(1,1) = \underset{\sim}{8} = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array},\ (2,0) = \underset{\sim}{6} = \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}, (3,0) = \underset{\sim}{10} = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline\end{array}. \tag{11.1.1}$$

4. The counting of states within a given representation (Young diagram) involves three steps. First you "fill in" the boxes starting with the upper left hand corner based on the symmetry group. For $SU(N)$ you put N in the upper left corner box and then increase the number when moving to the right and decrease the number when moving down (see the examples below). For example, for $SU(3)$ the diagram $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array}$ becomes

$$\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array} \Rightarrow \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}. \tag{11.1.2}$$

The next step involves again putting numbers in the boxes but this time the number correspond to the length the "hook" that has that box as the "elbow" of the "hook" (this is the most confusing part of this game). For each individual box in the Young diagram we count the number of boxes to the right of the starting box in the *same* row and the number of boxes below the given box in the *same* column. The sum of these two integers, plus 1 for the original box, is the length of the (right) "hook" with the original box at the "elbow". We put this length (*i.e.*, this integer) in the box and proceed until all boxes contain the lengths of the associated hooks.[1] As an example, consider again the diagram $\ydiagram{2,1}$. Applying the hook rules just defined populates the Young diagram in the following way,

$$\ydiagram{2,1} \Rightarrow \begin{ytableau} 3 & 1 \\ 1 \end{ytableau} . \tag{11.1.3}$$

Without proof, we state that the number of states in the representation corresponding to a Young diagram is given by the product of all of the numbers in the boxes based on the N of the symmetry divided by the product of all the numbers in the boxes corresponding to the lengths of the hooks. For the example above for $SU(3)$, we have

$$N\left(\ydiagram{2,1}\right) \equiv \begin{ytableau} 3 & 4 \\ 2 \end{ytableau} \Big/ \begin{ytableau} 3 & 1 \\ 1 \end{ytableau} \equiv \frac{3 \cdot 4 \cdot 2}{3 \cdot 1 \cdot 1} = 8\,, \tag{11.1.4}$$

as expected. Two other examples to test your understanding are

$$\begin{aligned} N\left(\ydiagram{1,1,1}\right) &= \begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau} \Big/ \begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau} = \frac{3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 1\,, \\ N\left(\ydiagram{3}\right) &= \begin{ytableau} 3 & 4 & 5 \end{ytableau} \Big/ \begin{ytableau} 3 & 2 & 1 \end{ytableau} = \frac{3 \cdot 4 \cdot 5}{3 \cdot 2 \cdot 1} = 10\,. \end{aligned} \tag{11.1.5}$$

11.2 Combine Multiplets

To actually combine multiplets, *i.e.*, define a product of representations, we need to carefully label things. Here we use the notation of the PDG.

(See `http://pdg.lbl.gov/2013/reviews/rpp2013-rev-young-diagrams.pdf`.)

Consider the product of 2 octets defined with the following notation

$$\underset{\sim}{8} \otimes \underset{\sim}{8} = \ydiagram{2,1} \otimes \begin{ytableau} a & a \\ b \end{ytableau} , \tag{11.2.1}$$

where we use boxes to represent the first octet and lettered boxes for the second (with"a" for the first row, "b" for the second row, etc.). Now we proceed to "add the boxes" with the following rules.

1. Start with the left-hand Young diagram (the empty boxes) $\ydiagram{2,1}$.

[1] We thank auditor Hanan Bell, Spring 2014, for suggesting this description of the hooks. Several other descriptions are possible.

2. Now add the “a”’s in all ways that produce a valid Young diagram, but with no more than a single “a” in each column (initially symmetric labels cannot be antisymmetrized)

$$\begin{ytableau} \; & \; \\ \; \end{ytableau} \Rightarrow \begin{ytableau} \; & \; & a & a \\ \; \end{ytableau} \oplus \begin{ytableau} \; & \; & a \\ \; & a \end{ytableau} \oplus \begin{ytableau} \; & \; & a \\ \; \\ a \end{ytableau} \oplus \begin{ytableau} \; & \; \\ \; & a \\ a \end{ytableau}. \tag{11.2.2}$$

3. Starting in the *second* row (where the “b”’s were initially) add the “b”’s subject to the constraint that, reading from right to left starting at the right end of the first row and then moving on to the second row, the number of “a”’s must be $\geq$ the number of “b”’s ($\geq$ the number of “c”’s, *etc.*) at each point in the reading process. Thus, when reading through the boxes in the prescribed fashion, we can come to the first “b” only after we have passed *at least* one “a”. We come to the second “b” only after passing the second “a”, *etc.* The allowed Young diagrams for our current example are

$$\begin{aligned} &\begin{ytableau} \; & \; & a & a \\ \; & b \end{ytableau} \oplus \begin{ytableau} \; & \; & a & a \\ \; \\ b \end{ytableau} \oplus \begin{ytableau} \; & \; & a \\ \; & a & b \end{ytableau} \oplus \begin{ytableau} \; & \; & a \\ \; & a \\ b \end{ytableau} \oplus \begin{ytableau} \; & \; & a \\ \; & b \\ a \end{ytableau} \oplus \begin{ytableau} \; & \; \\ \; & a \\ a & b \end{ytableau} \\ = &\begin{ytableau} \; & \; & \; & \; \\ \; & \; \end{ytableau} \oplus \begin{ytableau} \; & \; & \; & \; \\ \; \\ \; \end{ytableau} \oplus \begin{ytableau} \; & \; & \; \\ \; & \; & \; \end{ytableau} \oplus \begin{ytableau} \; & \; & \; \\ \; & \; \\ \; \end{ytableau} \oplus \begin{ytableau} \; & \; & \; \\ \; & \; \\ \; \end{ytableau} \oplus \begin{ytableau} \; & \; \\ \; & \; \\ \; & \; \end{ytableau}. \end{aligned} \tag{11.2.3}$$

4. Using the rules noted earlier we can work out the multiplicity of each of these irreducible representations (using the notation introduced above).

$$\begin{aligned} &\begin{ytableau} \; & \; & \; & \; \\ \; & \; \end{ytableau} \oplus \begin{ytableau} \; & \; & \; & \; \\ \; \\ \; \end{ytableau} \oplus \begin{ytableau} \; & \; & \; \\ \; & \; & \; \end{ytableau} \oplus \begin{ytableau} \; & \; & \; \\ \; & \; \\ \; \end{ytableau} \oplus \begin{ytableau} \; & \; & \; \\ \; & \; \\ \; \end{ytableau} \oplus \begin{ytableau} \; & \; \\ \; & \; \\ \; & \; \end{ytableau} \\ = &\begin{ytableau} 3 & 4 & 5 & 6 \\ 2 & 3 \end{ytableau} \Big/ \begin{ytableau} 5 & 4 & 2 & 1 \\ 2 & 1 \end{ytableau} \oplus \begin{ytableau} 3 & 4 & 5 & 6 \\ 2 \\ 1 \end{ytableau} \Big/ \begin{ytableau} 6 & 3 & 2 & 1 \\ 2 \\ 1 \end{ytableau} \oplus \begin{ytableau} 3 & 4 & 5 \\ 2 & 3 & 4 \end{ytableau} \Big/ \begin{ytableau} 4 & 3 & 2 \\ 3 & 2 & 1 \end{ytableau} \\ &\oplus \begin{ytableau} 3 & 4 & 5 \\ 2 & 3 \\ 1 \end{ytableau} \Big/ \begin{ytableau} 5 & 3 & 1 \\ 3 & 1 \\ 1 \end{ytableau} \oplus \begin{ytableau} 3 & 4 & 5 \\ 2 & 3 \\ 1 \end{ytableau} \Big/ \begin{ytableau} 5 & 3 & 1 \\ 3 & 1 \\ 1 \end{ytableau} \oplus \begin{ytableau} 3 & 4 \\ 2 & 3 \\ 1 & 2 \end{ytableau} \Big/ \begin{ytableau} 4 & 3 \\ 3 & 2 \\ 2 & 1 \end{ytableau} \\ = &\frac{3\cdot 4\cdot 5\cdot 6\cdot 2\cdot 3}{5\cdot 4\cdot 2\cdot 2\cdot 1\cdot 1} \oplus \frac{3\cdot 4\cdot 5\cdot 6\cdot 2\cdot 1}{6\cdot 3\cdot 2\cdot 2\cdot 1\cdot 1} \oplus \frac{3\cdot 4\cdot 5\cdot 2\cdot 3\cdot 4}{4\cdot 3\cdot 3\cdot 2\cdot 2\cdot 1} \\ &\oplus \frac{3\cdot 4\cdot 5\cdot 2\cdot 3\cdot 1}{5\cdot 3\cdot 3\cdot 1\cdot 1\cdot 1} \oplus \frac{3\cdot 4\cdot 5\cdot 2\cdot 3\cdot 1}{5\cdot 3\cdot 3\cdot 1\cdot 1\cdot 1} \oplus \frac{3\cdot 4\cdot 2\cdot 3\cdot 1\cdot 2}{4\cdot 3\cdot 3\cdot 2\cdot 2\cdot 1} \\ = &\utilde{27} \oplus \utilde{10} \oplus \utilde{\overline{10}} \oplus \utilde{8} \oplus \utilde{8} \oplus \utilde{1} \\ = &(2,2) \oplus (3,0) \oplus (0,3) \oplus (1,1) \oplus (1,1) \oplus (0,0). \end{aligned} \tag{11.2.4}$$

Note that, as we have seen before, the “a-b” notation makes clear that the internal symmetry structure of the 2 octets is different. So finally we have the result that

$$\utilde{8} \otimes \utilde{8} = \utilde{27} \oplus \utilde{10} \oplus \utilde{\overline{10}} \oplus \utilde{8} \oplus \utilde{8} \oplus \utilde{1} \tag{11.2.5}$$

Looking ahead to the application to the $SU(3)$ of color we can reproduce some other results that we have already used. Consider the product of a quark and antiquark,

$$\begin{aligned} \underset{\sim}{3} \otimes \underset{\sim}{\bar{3}} &= \ydiagram{1} \otimes \ytableaushort{a,b} = \ytableaushort{{}a,b} \oplus \ytableaushort{{},a,b} \\ &= \underset{\sim}{8} \oplus \underset{\sim}{1}. \end{aligned} \tag{11.2.6}$$

Next consider the product of 3 quarks, but begin by first looking at 2 quarks,

$$\underset{\sim}{3} \otimes \underset{\sim}{3} = \ydiagram{1} \otimes \ydiagram{1} = \ydiagram{2} \oplus \ydiagram{1,1} = \underset{\sim}{6} \oplus \underset{\sim}{\bar{3}}. \tag{11.2.7}$$

With the third quark we have

$$\begin{aligned} \underset{\sim}{3} \otimes \underset{\sim}{3} \otimes \underset{\sim}{3} &= \left(\underset{\sim}{6} \oplus \underset{\sim}{\bar{3}}\right) \otimes \underset{\sim}{3} = \left(\ydiagram{2} \oplus \ydiagram{1,1}\right) \otimes \ydiagram{1} \\ &= \ydiagram{3} \oplus \ydiagram{2,1} \oplus \ydiagram{2,1} \oplus \ydiagram{1,1,1} \\ &= \underset{\sim}{10} \oplus \underset{\sim}{8} \oplus \underset{\sim}{8} \oplus \underset{\sim}{1}. \end{aligned} \tag{11.2.8}$$

In the context of color we are interested only in the color singlets for the mesons and baryons respectively. Note, as already discussed, that the singlet is the completely antisymmetric state. Applied to the $SU(3)$ of flavor, we see again that the mesons should appear in octets and singlets while the baryons should form decuplets, octets (of differing internal permutation symmetry) and singlets of flavor. However, not all of these states can be combined (with space, color and spin wave functions) to yield states with the required overall asymmetry under permutations. For example, the antisymmetric color wave function requires net symmetry in the other quantum numbers. For the ground state we expect the space wave function to be symmetric. The spin wave function is either symmetric ($S = 3/2$) or mixed ($S = 1/2$). Thus only the flavor symmetric $\underset{\sim}{10}$, with spin $3/2$, and the appropriately mixed symmetry $\underset{\sim}{8}$, with spin $1/2$, can appear in the baryon ground state.

For further discussion (including how to connect the integers (p,q) to the "shapes" of the multiplets in isospin-strangeness plane) see the PDG report at

`http://pdg.lbl.gov/2013/reviews/rpp2013-rev-young-diagrams.pdf`

The brief summary is that p counts the number of "spaces" between occupied states at the top (largest strangeness) of the multiplet and q counts the number of spaces between occupied states at the bottom (most negative strangeness).