Chapter 1

Math Methods: A Quick Review

In your Elementary Mathematical Physics course (227/8) you learned about a variety of functions and techniques that will be useful in Physics 226. We will attempt to review the most relevant of those, especially for the analysis of Special Relativity, in these notes. The reader is also strongly encouraged to review the lecture notes from the last time I taught Phys. 227-228 (2008-2009), which are available here. The content of essentially all of the first ten lectures has application to our studies in Physics 226. Also note that these 227/8 notes include worked examples and samples of how the computer program *Mathematica* can be used to both think about (make plots, *etc.*) and solve relevant exercises.

1.1 Power Series

A extremely powerful tool for both understanding and evaluating functions is the power series expansion, typically the Taylor series expansion,

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \,, \tag{1.1.1}$$

where, for the expansion about the origin, the coefficients are the derivatives at the origin divided by $n!$,

$$c_n = \frac{d^n f(z)}{d^n z}|_{z=0} \frac{1}{n!} \,. \tag{1.1.2}$$

The analytic properties of the function $f(z)$ in the complex z-plane are then characterized by the convergence propertied of this series. Note that this power series, where it converges, serves to define the function whether z is a real number, a complex number or matrix valued. We will use this last point later in this Chapter.

A particularly useful approximate result arises when we have a small parameter, say $|\delta| \ll 1$, so that, for example,

$$(1+\delta)^{\alpha} \approx 1 + \alpha\delta + (\alpha(\alpha-1)/2!)\,\delta^2 + \mathcal{O}(\delta^3) \,. \tag{1.1.3}$$

This expansion is valid independent of the signs of α and δ, but actually requires that the product $|\alpha\delta| \ll 1$ to be useful. However, the exponent α is typically of order unity.

1.2 The Exponential Function

One of the most useful functions of mathematical physics is the exponential function. It is effectively defined as the solution of the following (trivial) first order differential equation plus boundary/initial condition,

$$\frac{df(z)}{dz} = f(z); \quad f(0) = 1 \Rightarrow f(z) = \exp(z) = e^z \,. \tag{1.2.1}$$

So the exponential function is the eigenfunction of the derivative operator with a specific boundary condition, *i.e.*, specific normalization. Iterating the form of Eq. (1.2.1) leads to the conclusion that all derivatives of the exponential function at the origin have unit value,

$$\frac{d^n f(z)}{d^n z}|_{z=0} = 1 \,. \tag{1.2.2}$$

Thus the Taylor series expansion of this function about the origin is given by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \,. \tag{1.2.3}$$

Since only the $n = 0$ term contributes at the origin, this sum clearly satisfies the boundary condition,

$$e^0 = 1 \,, \tag{1.2.4}$$

as long as you know that, by convention, $0! = 1$ (but note that $(-1)! = \infty$). Likewise taking a derivative yields

$$\frac{de^z}{dz} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z \,. \tag{1.2.5}$$

A little 227 style analysis confirms that this sum is convergent in the entire z plane (*i.e.*, it is singular only at ∞), and serves to define the exponential function as an analytic function everywhere in the finite (complex) z plane. To explore this function we first focus on the behavior of the exponential function separately along both the real and imaginary axes.

1.3 Along the Real Axis: The Hyperbolic Functions

Focusing on the case $z = x + iy \to x$ with x (and y) purely real, we can easily confirm that the series in Eq. (1.2.3) increases quickly with increasing positive x and diverges to ∞ as $x \to +\infty$. On the hand, for negative x, there is substantial cancellation between the terms, which alternate in sign, and the exponential function decreases rapidly as x becomes more negative. In the limit $x \to -\infty$ the exponential function vanishes ($e^{-\infty} \equiv 0$). This behavior is illustrated in Fig. 1.1. Since the logarithm function is the inverse of the exponential function, a semi-log plot, as on the right in Fig. 1.1, yields "linear" behavior in the plot, *i.e.*, $\ln e^x = x$.

It is useful to define even and odd functions in terms of the real exponential (remember that symmetries are important), which yields the so-called hyperbolic functions:

$$\begin{aligned} \cosh(x) &\equiv \frac{e^x + e^{-x}}{2}, \quad \cosh(x) = \cosh(-x) \,, \\ \sinh(x) &\equiv \frac{e^x - e^{-x}}{2}, \quad \sinh(x) = -\sinh(-x) \,. \end{aligned} \tag{1.3.1}$$

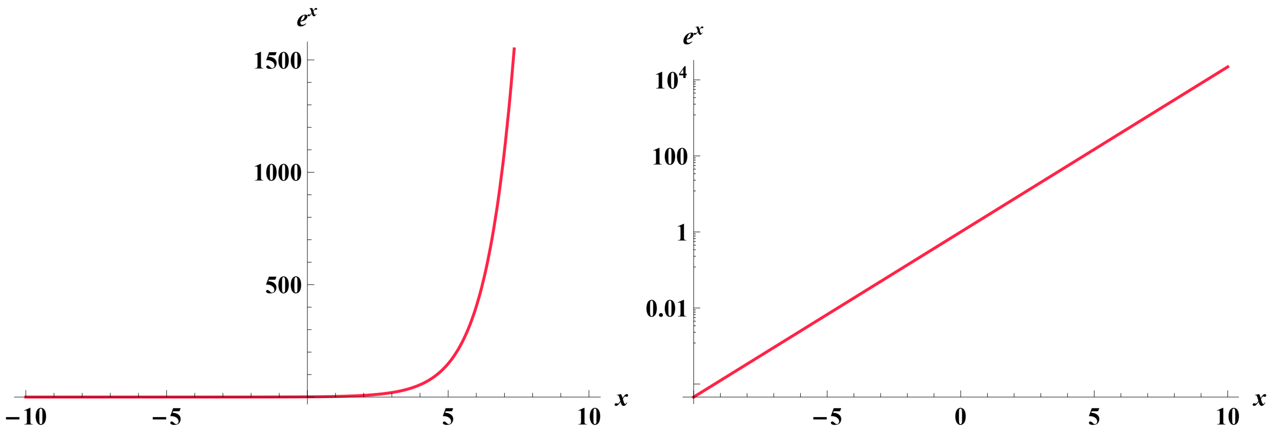

Figure 1.1: The exponential function along the real axis: LEFT- linear scale, RIGHT- logarithmic scale.

Using Eq. (1.2.2) the hyperbolic functions have the expected series expansions in terms of even and odd powers;

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\,, \qquad \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}\,. \tag{1.3.2}$$

Other useful properties follow from these definitions and Eq. (1.2.1). In particular, we have

$$\frac{d\cosh(x)}{dx} = \frac{e^x - e^{-x}}{2} = \sinh(x)\,, \qquad \frac{d\sinh(x)}{dx} = \frac{e^x - e^{-x}}{2} = \cosh(x)\,. \tag{1.3.3}$$

So it follows that the hyperbolic functions are eigenfunctions of the second order derivative (with eigenvalue +1),

$$\frac{d^2\cosh(x)}{d^2x} = \cosh(x)\,, \qquad \frac{d^2\sinh(x)}{d^2x} = \sinh(x)\,. \tag{1.3.4}$$

We also have

$$\cosh^2(x) = \frac{e^{2x} + e^{-2x} + 2}{4}\,, \qquad \sinh^2(x) = \frac{e^{2x} + e^{-2x} - 2}{4}\,, \tag{1.3.5}$$

so that

$$\cosh^2(x) - \sinh^2(x) = 1\,. \tag{1.3.6}$$

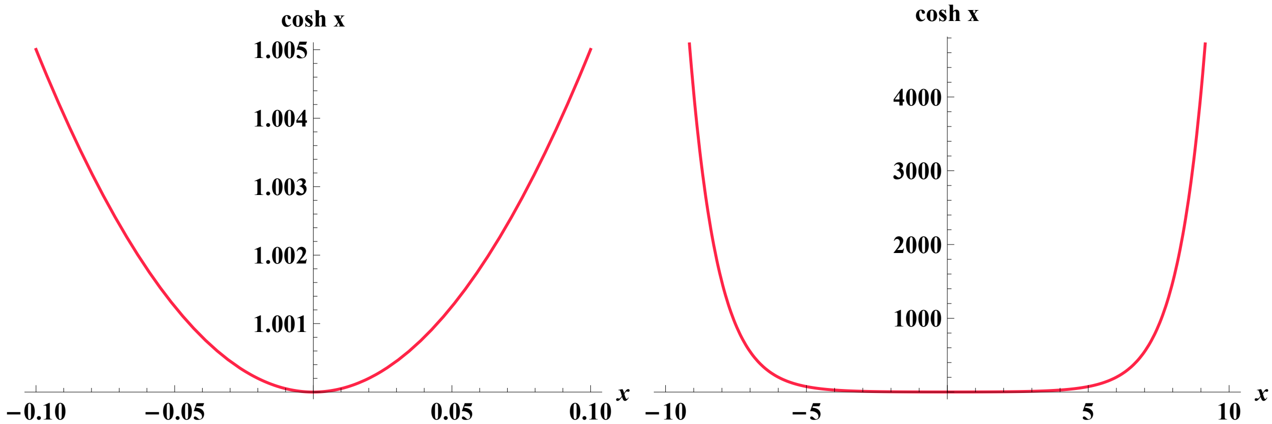

Figure 1.2: The hyperbolic function $\cosh x$: LEFT- small x, RIGHT- large x.

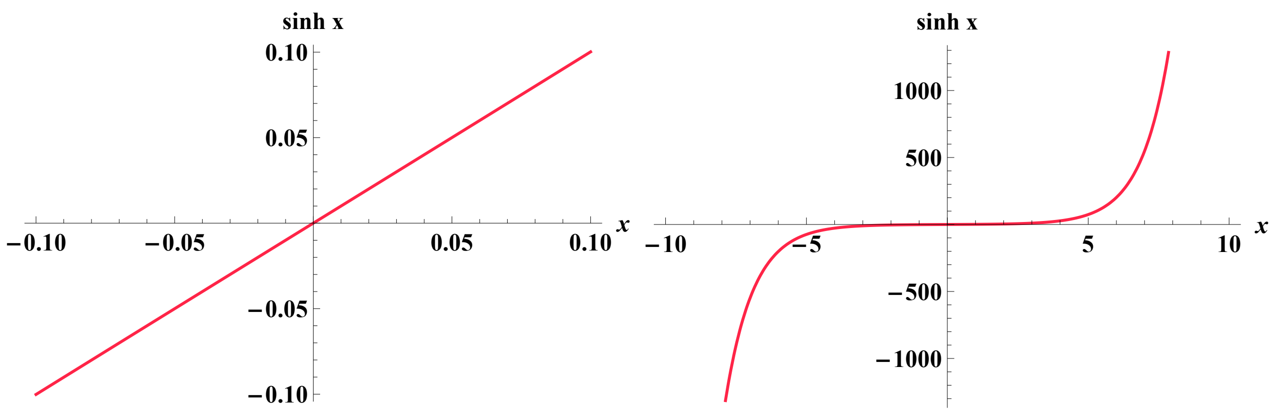

Figure 1.3: The hyperbolic function $\sinh x$: LEFT- small x, RIGHT- large x.

It follows from the behavior of the exponential function that

$$\begin{aligned} \cosh(x \to 0) &\to 1 + \frac{x^2}{2} \to 1\,, \\ \cosh(x \to \pm\infty) &\to +\infty\,, \end{aligned} \tag{1.3.7}$$

while

$$\begin{aligned} \sinh(x \to 0) &\to x \to 0\,, \\ \sinh(x \to \pm\infty) &\to \pm\infty\,. \end{aligned} \tag{1.3.8}$$

The behavior of the hyperbolic function $\cosh x$ is illustrated in Fig. 1.2 for both small (left) and large (right) x. The corresponding plots for $\sinh x$ appear in Fig. 1.3.

A related (and useful) function is the hyperbolic tangent defined by the ratio,

$$\tanh(x) \equiv \frac{\sinh(x)}{\cosh(x)}\,. \tag{1.3.9}$$

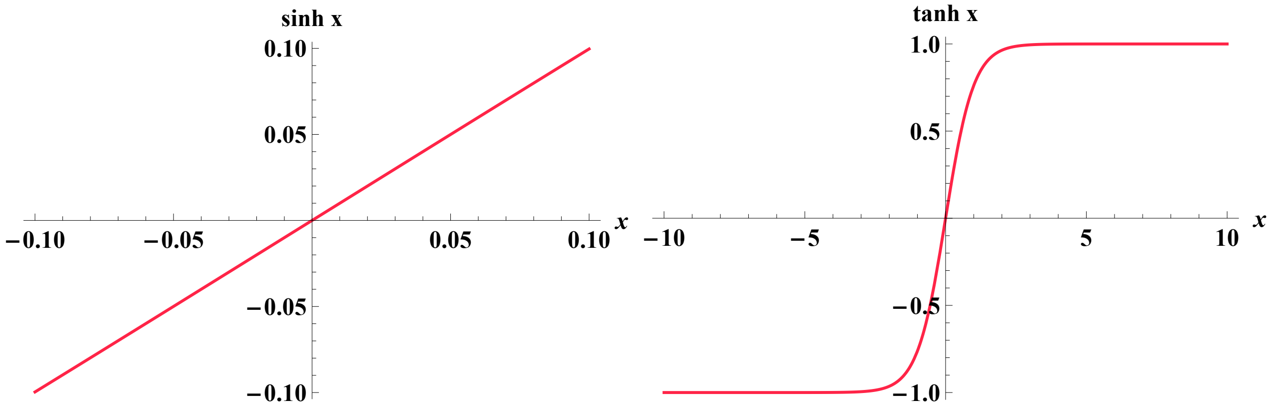

Figure 1.4: The hyperbolic function $\tanh x$: LEFT- small x, RIGHT- large x.

It follows from the above properties of the hyperbolic sine and cosine that for small x values ($x \ll 1$), $\tanh(x)$ behaves like $\sinh(x)$, *i.e.*, like x, while its magnitude is bounded by 1 for large x. We have

$$\tanh(x \ll 1) \to x\,,$$
$$\tanh(x \to \pm\infty) \to \pm 1\,, \tag{1.3.10}$$

as illustrated in Fig. 1.4.

As we will see when we discuss Special Relativity in detail, the hyperbolic functions play an essential role in explicitly representing transformations between reference frames which are moving with a fixed velocity with respect to each other. Next we consider the exponential function along the *imaginary* axis.

1.4 Along the Imaginary Axis: The Sinusoidal Functions

Consider the exponential function with a purely imaginary argument, $z = iy$ (with y real). We can write the series form in terms of separate even, real and odd, imaginary parts ($i^2 = -1$, $i^3 = -i$, $i^4 = 1$, *etc.*) ,

$$e^{iy} \equiv \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!}\,. \tag{1.4.1}$$

This immediately suggests the usual series definitions of the sinusoidal functions:

$$\cos(y) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!}\,,$$
$$\sin(x) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\,. \tag{1.4.2}$$

Note that these expressions are very similar to the hyperbolic forms in Eq. (1.3.2) *except* for the alternating signs in the sums. Thus Eq. (1.4.1) can be written as (called Euler's formula)

$$e^{iy} = \cos(y) + i\sin(y),\ \cos(y) = \mathrm{Re}(e^{iy}),\ \sin(y) = \mathrm{Im}(e^{iy})\,, \tag{1.4.3}$$

which leads to the analogs of Eq. (1.3.1) illustrating similar symmetry properties,

$$\begin{aligned} \cos(y) &= \frac{e^{iy} + e^{-iy}}{2}, \quad \cos(y) = \cos(-y)\,, \\ \sin(y) &= \frac{e^{iy} - e^{-iy}}{2i}, \quad \sin(y) = -\sin(-y)\,. \end{aligned} \tag{1.4.4}$$

We can also immediately obtain the analogs of Eq. (1.3.3)

$$\begin{aligned} \frac{d\cos(y)}{dy} &= \frac{ie^{iy} - ie^{-iy}}{2} = -\sin(y)\,, \\ \frac{d\sin(y)}{dy} &= \frac{ie^{iy} + ie^{-iy}}{2i} = \cos(y)\,. \end{aligned} \tag{1.4.5}$$

So it follows that the sinusoidal functions are eigenfunctions of the second derivative operator (with eigenvalue -1),

$$\begin{aligned} \frac{d^2\cos(y)}{d^2y} &= = -\cos(y)\,, \\ \frac{d^2\sin(y)}{d^2y} &= -\sin(y)\,. \end{aligned} \tag{1.4.6}$$

The next step is to determine the analogs of Eqs. (1.3.5) and (1.3.6),

$$\begin{aligned} \cos^2(y) &= \frac{e^{2iy} + e^{-2iy} + 2}{4}\,, \\ \sin^2(x) &= \frac{e^{2iy} + e^{-2iy} - 2}{-4}\,, \end{aligned} \tag{1.4.7}$$

so that

$$\cos^2(y) + \sin^2(y) = 1\,. \tag{1.4.8}$$

Note, in particular, the similarities and *differences* from the hyperbolic case.

Now consider the values of these functions. From the series expansions, it follows that at the origin, like the hyperbolic functions, we have

$$\begin{aligned} \cos(0) &= 1\,, \\ \sin(0) &= 0\,. \end{aligned} \tag{1.4.9}$$

However, away from the origin (along the imaginary axis for e^z), things are more interesting. The alternating signs in the series expansions suggests that any growth will be smaller than that of the hyperbolic functions. In fact, since the expression e^{iy} is a pure phase with modulus 1, $|e^{iy}| = 1$, it is not surprising that the sinusoidal functions have magnitudes bounded above by 1, as is already suggested by Eq. (1.4.8). The most straightforward approach is to simply evaluate the series in Eq. (1.4.2). It is perhaps surprising, based on simply looking at the series expression, that the numerical results lead to the conclusion that both of these functions are periodic with period 2π,

$$\begin{aligned} \cos(y + 2\pi) &= \cos(y)\,, \\ \sin(y + 2\pi) &= \sin(y)\,, \\ \sin(y) &= \cos\left(\frac{\pi}{2} - y\right)\,. \end{aligned} \tag{1.4.10}$$

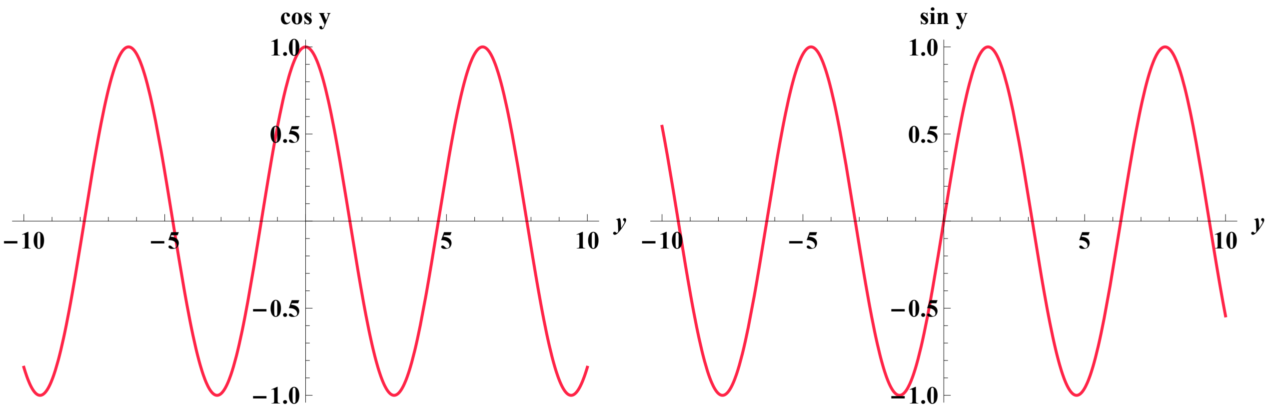

Figure 1.5: The sinusoidal functions: LEFT- $\cos y$, RIGHT- $\sin y$.

In fact, the transcendental number π can be determined numerically by solving for the smallest positive real number for which the series expression for the sine function vanishes. In this path to π no reference is made to circles or trigonometry, which is how this quantity is usually first introduced. We are also led directly to the special cases,

$$\cos(\pi) = -1 = e^{i\pi},\ \cos(2\pi) = e^{i2\pi} = 1,\ \cos\left(\frac{\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) = 0\,,$$
$$\sin(\pi) = \sin(2\pi) = 0,\ \sin\left(\frac{\pi}{2}\right) = 1,\ \sin\left(\frac{3\pi}{2}\right) = -1\,. \tag{1.4.11}$$

This behavior is illustrated in Fig. 1.5. The analogue to Eq. (1.3.9) is the sinusoidal tangent function defined by

$$\tan y \equiv \frac{\sin y}{\cos y}\,, \tag{1.4.12}$$

and illustrated in Fig. 1.6. Note that, since the numerator of this expression is maximum when the denominator vanishes, this function has regularly spaced singularities along the real axis (separated by zeros).

Another way to approach the discussion of periodicity is to recall that, as we already noted, the series expansion for the exponential function in Eq. (1.2.3) defines an *entire* function with no singularities in the finite complex z plane. We also need to recall (from Phys 227) that a complex number can be written in terms of it real and imaginary parts, $z = x + iy$, or it terms of its magnitude and phase, $z = |z|e^{i\phi}$ with $\phi = \tan^{-1}(y/x)$. Thus, if we evaluate the exponential function just *above* the real axis, $z_+ = 1e^{i\delta}$ $\delta \ll 1$, we must get the *same* value if we go around the unit circle and approach the real axis from below, $z_+ = 1e^{(i2\pi - i\delta)}$. It is consistent for the exponential function to be branch cut *free* if and only if the sinusoidal functions are periodic, $e^{iy} = e^{iy+i2\pi}$.

We close this discussion by exploring the situation in the complex plane more generally. With the definitions above we have

$$\cos(z) = \cos(x + iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)\,,$$
$$\sin(z) = \sin(x + iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)\,. \tag{1.4.13}$$

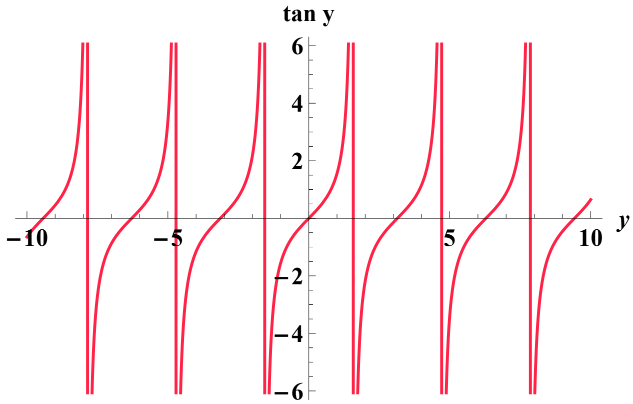

Figure 1.6: The sinusoidal tangent function.

This should look familiar to the addition of angles formulas you learned in high school,

$$\begin{aligned} \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\,, \\ \sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)\,, \end{aligned} \tag{1.4.14}$$

plus

$$\cos(iy) = \cosh(y),\ \sin(iy) = i\sinh(y)\,, \tag{1.4.15}$$

which follow directly from the series expressions above.

1.5 Rotations in 3-D

As a final mathematical physics subject to review we turn to rotations in 3 dimensions (3-D), which will allow us to use the sinusoidal functions from above and prepare for the idea of Lorentz transformations (in 4-D). For a discussion of rotations as a specific set of transformations described by the mathematics of Group Theory see Chapter 10. For the current discussion consider vectors in 3-D, for example, the position vector $\vec{r}$ (or the velocity $\vec{v} = \dot{\vec{r}}$, where the dot signifies the derivative with respect to time), measured with respect to a chosen origin. To use specific vector and matrix notation we introduce 3 orthonormal unit basis vectors, $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ to obtain a completely defined reference frame. (Note that you may be more familiar with the $\hat{x}$, $\hat{y}$ and $\hat{z}$ notation, but the 1-2-3 notation is more common in the 4-D world to which we are headed.) These are represented by

$$\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ \hat{e}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1.5.1}$$

A general 3-D position vector is then represented by

$$\vec{r} = x^1\hat{e}_1 + x^2\hat{e}_2 + x^3\hat{e}_3 = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}. \tag{1.5.2}$$

With the usual definition of spherical coordinates with polar angle θ (measured from the $\hat{e}_3$ direction) and azimuthal angle ϕ (measured from the $\hat{e}_1$ direction in the $\hat{e}_1$-$\hat{e}_2$ plane), along with the usual trigonometric definitions of the sinusoidal functions, we have the familiar vector components

$$x^1 = |\vec{r}| \sin\theta \cos\phi,\ x^2 = |\vec{r}| \sin\theta \sin\phi,\ x^3 = |\vec{r}| \cos\theta\,. \tag{1.5.3}$$

Now we want to consider performing a rotation. There are actually two types of rotations possible. We could choose to rotate the location vector $\vec{r}$ with the basis vectors *fixed*, called an "active" rotation, or we could instead rotate the basis vectors with the location vector fixed, *i.e.*, rotate the reference frame, called a "passive" rotation. The mathematics is the same if the two angles of rotation differ by a sign. In this class we will be concentrating on transformations between different reference frames and thus on passive rotations.

We can think of generating a rotation through a specified angle α as being represented by the exponentiation of the appropriate "generator" of an *infinitesimal* rotation (see Chapter 10 for more details). As you may have learned in your Quantum Mechanics (or Classical Mechanics, or 227/8) class and we will discuss more later, the generators of an infinitesimal rotation about any of the 3 axes (*i.e.*, a rotations in the plane orthogonal to that axis, are given by the 3 components of the angular momentum operator modulo a factor of $\hbar$ that carries the appropriate units). These 3 operators obey the commutation relation (here we include the explicit factor of $\hbar$),

$$[J_k, J_l] \equiv J_k J_l - J_l J_k = i\hbar\epsilon_{klm} J_m \quad [k,l,m = 1,2,3]\,, \tag{1.5.4}$$

which is the *algebra* of the Rotation Group, $SO(3)$.[1]

A useful 3-D representation of these operators is given by

$$J_1 = \hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},\ J_2 = \hbar \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix},\ J_3 = \hbar \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.5.5}$$

The reader is encouraged to verify that the matrices in Eq. (1.5.5) satisfy Eq. (1.5.4). Note further that these matrices are traceless and Hermitian ($J_k^\dagger = J_k$) (see also the discussion in Chapter 10). While the precise form of these matrices may not be intuitively obvious, the general form should be clear from our understanding of how ordinary rotations work. For example, a rotation about the 1-axis is a rotation in the 2-3 plane. It should serve to mix the 2 and 3 components of an ordinary 3-vector. Modulo the issue of phases, this is precisely what the form of the J_1 matrix in Eq. (1.5.5) does. It transforms a 3-component into a 2-component and a 2-component into a 3-component (*i.e.*, the only non-zero elements of the matrix perform this transformation).[2]

As an explicit example we consider a rotation by an angle α about the 3-axis. This is obtained by exponentiating the corresponding generator times the angle of rotation. How do we evaluate the exponentiation of a matrix? We simply recall that the exponential is *defined* by the power series of Eq. (1.2.3) and proceed. In particular, we want to evaluate the expression

$$R_3(\alpha) \equiv e^{i\alpha J_3/\hbar} = \sum_{n=0}^{\infty} \frac{(i\alpha J_3/\hbar)^n}{n!} = \mathbf{1} + \sum_{n=1}^{\infty} \frac{(i\alpha J_3/\hbar)^n}{n!}\,, \tag{1.5.6}$$

[1] The symbol ϵ_{klm} (called the Levi-Civita symbol) represents the unique completely *antisymmetric* $3\times3\times3$ matrix, which is normalized so that $\epsilon_{123} = 1$. Cyclical permutations of the indices 123 also yield unity, *e.g.*, $\epsilon_{231} = 1$, while non-cyclical permutations give -1, *e.g.*, $\epsilon_{213} = -1$. Repeated indices yield zero, *e.g.*, $\epsilon_{11k} = 0$.

[2] Being able to think of rotations as *either* occurring in a 2-D plane or about the direction orthogonal to that plane is an accident of (apparently) living where there are precisely 3 spatial dimensions. The "rotation in a plane" interpretation is the one that generalizes to a larger number of spatial dimensions.

with **1** the unit matrix,

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{1.5.7}$$

Note that, since the quantity in the exponent must be dimension-LESS, we have divided out the factor of $\hbar$. To evaluate the rest of the sum we note that each term is a 3×3 matrix determined by the following properties of J_3,

$$(J_3/\hbar)^{2n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \ (J_3/\hbar)^{2n+1} = J_3/\hbar \, . \tag{1.5.8}$$

So, by "pulling apart" the unit matrix and using Eq. (1.5.8), we can rewrite Eq. (1.5.6) as

$$\begin{aligned} R_3(\alpha) = e^{i\alpha J_3/\hbar} &= \mathbf{1} + \sum_{n=1}^{\infty} \frac{(i\alpha J_3/\hbar)^n}{n!} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sum_{n=0}^{\infty} \frac{(i\alpha)^{2n}}{(2n)!} + (J_3/\hbar) \sum_{n=0}^{\infty} \frac{(i\alpha)^{2n+1}}{(2n+1)!} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha)^{2n}}{(2n)!} + i(J_3/\hbar) \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha)^{2n+1}}{(2n+1)!} \, . \end{aligned} \tag{1.5.9}$$

We recognize the series from Eq. (1.4.2) and can write

$$\begin{aligned} R_3(\alpha) = e^{i\alpha J_3/\hbar} &= \mathbf{1} + \sum_{n=1}^{\infty} \frac{(i\alpha J_3/\hbar)^n}{n!} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos\alpha + i(J_3/\hbar)\sin\alpha \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos\alpha + \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sin\alpha \\ &= \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} . \end{aligned} \tag{1.5.10}$$

As expected for a rotation about the 3-axis, the 3-component is unchanged (the 1 in the lower right corner of the rotation matrix), while the 1 and 2-components are mixed via the sinusoidal functions. Note that the full rotation matrix is an Orthogonal matrix ($R_3(\alpha)^{-1} = R_3(\alpha)^T$, the transpose is the inverse) as we would expect for the representation of an element of the 3-D Orthogonal group, $SO(3)$, and has determinant 1 ($1 \times (\cos^2\alpha - (-\sin^2\alpha)) = 1$).

If we explicitly apply this rotation to a general location vector, we obtain (recall Eq. (1.5.2), recall also that we are performing a "passive" rotation where the location we are describing remains *fixed* while the axes of the reference frame are rotated)

$$\vec{r}' = R_3(\alpha)\vec{r} = \begin{pmatrix} \cos\alpha x^1 + \sin\alpha x^2 \\ -\sin\alpha x^1 + \cos\alpha x^2 \\ x^3 \end{pmatrix} . \quad (1.5.11)$$

Returning to the spherical coordinate notation of Eq. (1.5.3), we have

$$\begin{aligned} x'^1 &= |\vec{r}| \sin\theta(\cos\phi\cos\alpha + \sin\phi\sin\alpha) = |\vec{r}|\sin\theta\cos(\phi-\alpha), \\ x'^2 &= |\vec{r}| \sin\theta(\sin\phi\cos\alpha - \cos\phi\sin\alpha) = |\vec{r}|\sin\theta\sin(\phi-\alpha), \\ x'^3 &= x^3 = |r|\cos\theta\,, \end{aligned} \quad (1.5.12)$$

where the last steps in the first two lines use the expressions in Eq. (1.4.14). This result should be intuitively reasonable. First, as already noted, a rotation about the 3-axis does *not* change the 3-component, *i.e.*, the polar angle θ is unchanged. Such a rotation does, however, mix the 1 and 2-components. Since the rotation of the axes by angle α is in the same sense as the definition of the azimuthal angle ϕ, the azimuthal angle of the (unrotated) location vector as measured in the rotated reference frame is *reduced* by α, $\phi' = \phi - \alpha$. This explicit example also serves to illustrate the again intuitive result that such a rotation does *not* change the length of the rotation vector, which is given by the "scalar" product of the vector with itself denoted by $|\vec{r}|^2 = \vec{r}\cdot\vec{r}$. This is called a scalar product precisely because it is *not* changed by rotations. (The label scalar is to be contrasted with objects labeled as vectors that are changed by rotations.) In detail we have

$$\begin{aligned} \vec{r}\cdot\vec{r} &= (x^1)^2 + (x^2)^2 + (x^3)^2 = |\vec{r}|^2\left(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\right) \\ &= |\vec{r}|^2\left(\sin^2\theta + \cos^2\theta\right) = |\vec{r}|^2\,, \end{aligned} \quad (1.5.13)$$

and

$$\vec{r}'\cdot\vec{r}' = (x'^1)^2 + (x'^2)^2 + (x'^3)^2 = |\vec{r}|^2\left(\sin^2\theta\cos^2(\phi-\alpha) + \sin^2\theta\sin^2(\phi-\alpha) + \cos^2\theta\right) = |\vec{r}|^2\,. \quad (1.5.14)$$

The scalar product of two *different* 3-vectors is also unchanged by rotations, since it depends only on the polar angle between the directions of the two vectors. Again in detail we have

$$\begin{aligned} \vec{r}_1\cdot\vec{r}_2 &= (x_1^1)(x_2^1) + (x_1^2)(x_2^2) + (x_1^3)(x_2^3) \\ &= |\vec{r}_1||\vec{r}_2|\left(\sin\theta_1\sin\theta_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2) + \cos\theta_1\cos\theta_2\right) \\ &= |\vec{r}_1||\vec{r}_2|\left(\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2) + \cos\theta_1\cos\theta_2\right) \\ &= |\vec{r}_1||\vec{r}_2|\cos\Delta\theta_{12} = \vec{r}'_1\cdot\vec{r}'_2\,, \end{aligned} \quad (1.5.15)$$

where the last step arises from recognizing the expression for the cosine of the angle between two directions, $\cos\Delta\theta_{12}$, expressed in spherical coordinates $((\theta_1,\phi_1),(\theta_2,\phi_2))$ (which you learned about in Phys 227/8). Since this expression involves only *differences* between the spherical coordinate angles, it will be unchanged when these angles are changed in identical ways by a rotation. The corresponding group of transformations (rotations) is labeled the Orthogonal Group since orthogonal vectors remain orthogonal after the transformation.

The reader is encouraged to evaluate more general 3-D rotations and prepare to consider 4-D transformations - the Lorentz transformations. These will be expressed in terms of 4×4 matrices similar, but not identical, to the rotation matrices above. 4-D Lorentz *scalar* products of a pair of 4-vectors (invariant under Lorentz transformations) will play a central role in our discussion.

1.6 Examples

To complete this discussion we present the results of rotation about the 1 and 2 axes, results the reader should check. It follows from the explicit expressions in Eq. (1.5.5) that we have the analogues of Eq. (1.5.8) as

$$(J_1/\hbar)^{2n} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ (J_1/\hbar)^{2n+1} = J_1/\hbar\,, \tag{1.6.1}$$

and

$$(J_2/\hbar)^{2n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ (J_2/\hbar)^{2n+1} = J_2/\hbar\,. \tag{1.6.2}$$

Thus we can nearly guess the corresponding rotations by an angle α about these two axes. For the rotation about the 1 axis we have

$$\begin{aligned} R_1(\alpha) \equiv e^{i\alpha J_1/\hbar} &= \mathbf{1} + \sum_{n=1}^{\infty} \frac{(i\alpha J_1/\hbar)^n}{n!} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cos\alpha + i(J_1/\hbar)\sin\alpha \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cos\alpha + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \sin\alpha \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}. \end{aligned} \tag{1.6.3}$$

To obtain an intuitive picture (or check) of the "signs in front of the sines" consider a vector in the 2-3 plane where both the 2- and the 3-components are positive ($x^2 > 0$ and $x^3 > 0$) and the vector lies in the 2-3 plane oriented between the positive 2- and 3-axes. Then a positive rotation about the 1 axis (your right hand is essential here) rotates the 2-axis *towards* the fixed location vector and the 3-axis *away* from the fixed location vector. Hence we expect that the 2-component in the new, rotated frame to be *larger* (than in the old frame), while the 3-component will be smaller. This result is precisely what the "signs" in Eq. (1.6.3) tell us, *i.e.*, addition occurs for the 2-component while subtraction occurs for the 3-component.

Finally the result for a rotation about the 2 axis is clear except, perhaps, the signs. We have

$$
\begin{aligned}
R_2(\alpha) \equiv e^{i\alpha J_2/\hbar} &= \mathbf{1} + \sum_{n=1}^{\infty} \frac{(i\alpha J_2/\hbar)^n}{n!} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cos\alpha + i(J_2/\hbar)\sin\alpha \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cos\alpha + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \sin\alpha \qquad (1.6.4) \\
&= \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}.
\end{aligned}
$$

Again these are the expected Orthogonal matrices with unit determinant. To check the signs we can perform a similar exercise to that above, but now in the 1-3 plane. A positive rotation about the 2-axis moves the 3-axis towards the fixed location vector and the 1-axis away. Hence we expect the "+" sign in the 3-row of the matrix and the "-" sign in the 1-row as seen in Eq. (1.6.4).