Chapter 4

Relativistic dynamics

We have seen in the previous lectures that our relativity postulates suggest that the most efficient (lazy but smart) approach to relativistic physics is in terms of 4-vectors, and that velocities never exceed c in magnitude. In this chapter we will see how this 4-vector approach works for dynamics, *i.e.*, for the interplay between motion and forces.

A particle subject to forces will undergo non-inertial motion. According to Newton, there is a simple (3-vector) relation between force and acceleration,

$$\vec{f} = m\,\vec{a}\,, \tag{4.0.1}$$

where acceleration is the second time derivative of position,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2}\,. \tag{4.0.2}$$

There is just one problem with these relations — they are *wrong*! Newtonian dynamics is a good *approximation* when velocities are very small compared to c, but outside of this regime the relation (4.0.1) is simply incorrect. In particular, these relations are inconsistent with our relativity postulates. To see this, it is sufficient to note that Newton's equations (4.0.1) and (4.0.2) predict that a particle subject to a constant force (and initially at rest) will acquire a velocity which can become arbitrarily large,

$$\vec{v}(t) = \int_0^t \frac{d\vec{v}}{dt'}dt' = \frac{\vec{f}}{m}t \to \infty \quad \text{as } t \to \infty\,. \tag{4.0.3}$$

This flatly contradicts the prediction of special relativity (and causality) that no signal can propagate faster than c. Our task is to understand how to formulate the dynamics of non-inertial particles in a manner which is consistent with our relativity postulates (and then verify that it matches observation, including in the non-relativistic regime).

4.1 Proper time

The result of solving for the dynamics of some object subject to known forces should be a prediction for its position as a function of time. But whose time? One can adopt a particular reference frame, and then ask to find the spacetime position of the object as a function of coordinate time t in the chosen frame, $x^\mu(t)$, where, as always, $x^0 \equiv c\,t$. There is nothing wrong with this, but it is a frame-*dependent* description of the object's motion.

For many purposes, a more useful description of the object's motion is provided by using a choice of time which is directly associated with the object in a frame-*independent* manner. Simply imagine that the object carries with it its own (good) clock. Time as measured by a clock whose worldline is the same as the worldline of the object of interest is called the *proper time* of the object. To distinguish proper time from coordinate time in some inertial reference frame, proper time is usually denoted as τ (instead of t).

Imagine drawing ticks on the worldline of the object at equal intervals of proper time, as illustrated in Figure 4.1 . In the limit of a very fine proper time spacing $\Delta\tau$, the invariant interval between neighboring ticks is constant, $s^2 = (c\Delta\tau)^2$. In the figure, note how the tick spacing, as measured by the coordinate time x^0, varies depending on the instantaneous velocity of the particle. When the particle is nearly at rest in the chosen reference frame (*i.e.*, when the worldline is nearly vertical), then the proper time clock runs at nearly the same rate as coordinate time clocks, but when the particle is moving fast then its proper time clock runs more slowly that coordinate time clocks due to time dilation.

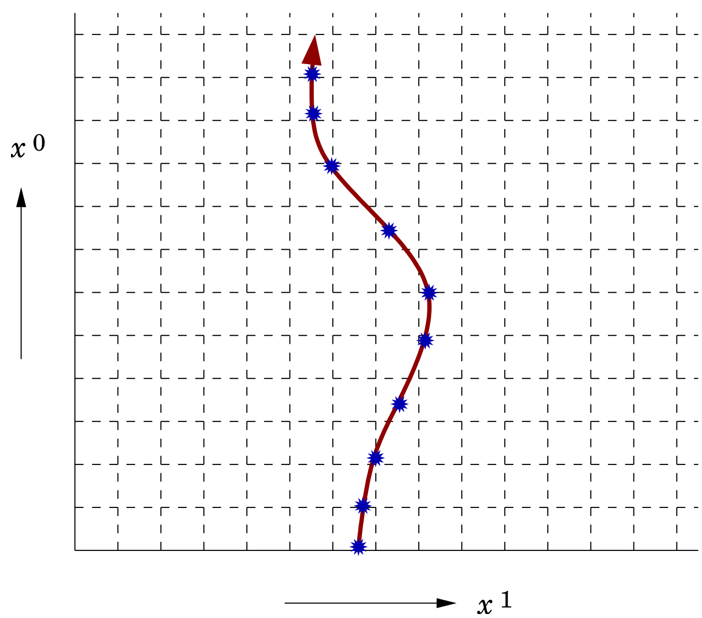

Figure 4.1: The worldline of a non-inertial particle, with tick marks at constant intervals of proper time.

4.2 4-velocity

Using the proper time to label points on the spacetime trajectory of a particle means that its space-time position is some function of proper time, $x(\tau)$. The time component of x (in a chosen reference frame) gives the relation between coordinate time and proper time of events on the worldline,

$$c\,t = x^0(\tau)\,. \tag{4.2.1}$$

The corresponding *four-velocity* of a particle is the derivative of its spacetime position with respect to proper time, (note that both u and x are 4-vectors)

$$u \equiv \frac{dx(\tau)}{d\tau}\,. \tag{4.2.2}$$

Since $x^0 = ct$, the time component of the 4-velocity gives the rate of change of coordinate time with respect to proper time,

$$u^0 = c\,\frac{dt}{d\tau}\,. \tag{4.2.3}$$

The spatial components of the 4-velocity give the rate of change of the spatial position with respect to proper time, $u^k = dx^k/d\tau$. This is *not* the same as the ordinary 3-velocity $\vec{v}$, which is the rate of change of position with respect to coordinate time, $v^k = dx^k/dt$. But we can relate the two using calculus,

$$u^k = \frac{dx^k}{d\tau} = \frac{dt}{d\tau}\,\frac{dx^k}{dt} = \frac{u^0}{c}\,v^k\,. \tag{4.2.4}$$

From our discussion of time dilation, we already know that moving clocks run slower than clocks at rest in the chosen reference frame by a factor of γ. In other words, it must be the case that

$$\frac{u^0}{c} = \frac{dt}{d\tau} = \gamma = \left[1 - \frac{\vec{v}^{\,2}}{c^2}\right]^{-1/2}\,. \tag{4.2.5}$$

Combined with Eq. (4.2.4), this shows that the spatial components of the 4-velocity equal the 3-velocity times a factor of γ,

$$u^k = \gamma\, v^k = \frac{v^k}{\sqrt{1-\vec{v}^{\,2}/c^2}}\,. \tag{4.2.6}$$

We can now use Eqs. (4.2.5) and (4.2.6) to evaluate the square of the 4-velocity,

$$u^2 = (u^0)^2 - (u^k)^2 = \gamma^2\left(c^2 - \vec{v}^2\right) = c^2\,. \tag{4.2.7}$$

So a 4-velocity vector *always* squares to $+c^2$, regardless of the value of the 3-velocity. (Recall that the plus sign here corresponds to our choice of metric; the East Coast metric yields $u^2 = -c^2$, but still a constant.)

Let's summarize what we've learned a bit more geometrically. The worldline $x(\tau)$ describes some trajectory through spacetime. At every event along this worldline, the four-velocity $u = dx/d\tau$ is a 4-vector which is *tangent* to the worldline. When one uses proper time to parametrize the worldline, the tangent vector u has a constant square, $u^2 = c^2$. So you can think of u/c as a tangent 4-vector which has unit "length" everywhere along the worldline. The fact that u^2 is positive (in our metric choice) shows that the 4-velocity is always a timelike vector. (Note that it is a timelike vector in both metrics, but with appropriately differing signs for the square.)

Having picked a specific reference frame in which to evaluate the components of the 4-velocity u, Eqs. (4.2.5) and (4.2.6) show that the components of u are completely determined by the ordinary 3-velocity $\vec{v}$, so the information contained in u is precisely the same as the information contained in $\vec{v}$. You might then ask "why bother with 4-velocity?" The answer is that the 4-velocity u is a more natural quantity to use — it has geometric meaning which is *independent* of any specific choice of reference frame. Moreover, the components u^μ of the 4-velocity transform linearly under a Lorentz boost in exactly the *same* fashion as any other 4-vector. [See Eq. (3.6.5)]. In contrast, under a Lorentz boost the components of the 3-velocity v transform in a somewhat messy fashion, but we can use the four-velocity to analyze this question.

4.3 Relativistic Addition of Velocities

Consider a point particle moving with 3-velocity v' in the x^1 direction in the S' frame (to match our previous convention) such that $(u')^T = (\gamma_{v'}c, \gamma_{v'}v', 0, 0)$, with $\gamma_{v'} = 1/\sqrt{1 - v'^2/c^2}$. Now view the motion of this particle in the S frame, which is defined such that, in the S frame, the S' frame is moving in the $+x^1$ direction with velocity v_0. Thus the boost between the two frames is (recall Eq. (3.5.7))

$$\Lambda(v_0) = \begin{pmatrix} \gamma_0 & \gamma_0\,(v_0/c) & 0 & 0 \\ \gamma_0\,(v_0/c) & \gamma_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \gamma_0 = \frac{1}{\sqrt{1 - v_0^2/c^2}}\,, \tag{4.3.1}$$

so that

$$u = \Lambda(v_0)u' = \begin{pmatrix} c\gamma_0\gamma_{v'}(1 + v'v_0/c^2) \\ \gamma_0\gamma_{v'}(v' + v_0) \\ 0 \\ 0 \end{pmatrix}. \tag{4.3.2}$$

These results allow us to obtain the ordinary 3-velocity in the S frame from the 4-velocity via

$$v = |\vec{v}| = v^1 = c\frac{u^1}{u^0} = \frac{v_0 + v'}{1 + v_0 v'/c^2}\,,\ \left(v^2 = v^3 = 0\right)\,. \tag{4.3.3}$$

While the numerator is the familiar Galilean result for velocity addition (and reduces to this result for velocities small compared to c), the denominator is new to the *relativistic* addition of 3-velocities. (Note that the plus signs in the numerator and denominator correspond to the two 3-velocities being in the *same* direction. The signs would be negative for velocities in opposite directions.) This expression has the interesting feature, required by our relativistic Postulates, that, if either (or both) of the initial three-velocities approach c, v also approaches but *never* exceeds c. The reader is encouraged to complete this analysis and carry out the algebra necessary to obtain the following results,

$$\begin{aligned} (u')^2 &= u^2 = c^2, \\ \gamma_0\gamma_{v'}(1 + v'v_0/c^2) &= \gamma_v = \frac{1}{\sqrt{1 - v^2/c^2}}\,. \end{aligned} \tag{4.3.4}$$

The first equation confirms the Lorentz invariance of the four-velocity squared, while the second equation confirms that, in the S frame, the four-velocity can be written in the standard from $u^T = (\gamma_v c, \gamma_v v, 0, 0)$.

4.4 4-momentum

To discuss momentum we should first be explicit concerning what we mean by the symbol m. The *rest mass* m of any object is the mass of the object as measured in its rest frame. The *4-momentum* of a particle (or any other object) with rest mass m is defined to be m times the object's 4-velocity,

$$p = m\,u\,. \tag{4.4.1}$$

For systems of interacting particles, this is the quantity to which conservation of momentum will apply. Spatial momentum components (in a given reference frame) are just the spatial components of the 4-momentum. The definition of 3-momentum which you learned in introductory physics, $\vec{p} = m\,\vec{v}$, is, at best, a non-relativistic approximation. This is important, so let us repeat,

$$\vec{p} \neq m\,\vec{v}\,. \tag{4.4.2}$$

From now on do *not* think of momentum as mass times 3-velocity. Rather, think 4-dimensionally with momentum as mass times 4-velocity.[1]

If the spatial components of the 4-momentum are the (properly defined) spatial momentum, what is the time component p^0? There is only one possible answer — it must be related to energy.[2] In fact, the total energy E of an object equals the time component of its four-momentum times c, or

$$p^0 = E/c\,. \tag{4.4.3}$$

Using the relation (4.4.1) between 4-momentum and 4-velocity, plus the result (4.2.5) for u^0, allows one to express the the total energy E of an object in terms of its rest mass and its velocity,

$$E = c\,p^0 = mc\,u^0 = mc^2\gamma = \frac{mc^2}{\sqrt{1-\vec{v}^{\,2}/c^2}} = mc^2\cosh y\,. \tag{4.4.4}$$

In other words, the relativistic gamma factor of any object is equal to the ratio of its total energy to its rest energy (recall Eq. (3.5.8)),

$$\gamma = \frac{E}{mc^2} = \cosh y\,. \tag{4.4.5}$$

When the object is at rest, its kinetic energy (or energy due to motion) vanishes, but its *rest energy*, given by Einstein's famous expression mc^2, remains. If the object is moving slowly (compared to c), then it is appropriate to expand the relativistic energy (4.4.4) in powers of $\vec{v}^{\,2}/c^2$. This gives

$$E = mc^2 + \tfrac{1}{2}m\vec{v}^{\,2} + \cdots\,. \tag{4.4.6}$$

In other words, for velocities small compared to c, the total energy E equals the rest energy mc^2 plus the usual non-relativistic kinetic energy, $\frac{1}{2}m\vec{v}^{\,2}$, up to higher order corrections which, relative to the non-relativistic kinetic energy, are suppressed by additional powers of $\vec{v}^{\,2}/c^2$. We can, of course, define the *relativistic* kinetic energy via $K = E - mc^2 = mc^2(\gamma-1)$, which reduces to the non-relativistic form $\frac{1}{2}m\vec{v}^{\,2}$ for $\vec{v}^{\,2}/c^2 \ll 1$.

The corresponding spatial component of the four-momentum is then (recall Eqs. (4.4.2) and (3.5.8))

$$\vec{p} = m\vec{v}\gamma = \frac{m\vec{v}}{\sqrt{1-\vec{v}^{\,2}/c^2}} = mc\hat{v}\sinh y\,, \tag{4.4.7}$$

(where $\hat{v}$ is the spatial unit vector in the direction of $\vec{v}$) which, as advertised, approaches the non-relativistic definition for $v \ll c$, $\gamma \to 1$ and $\sinh y \to v/c$.

We saw above that (with our choice of metric) 4-velocities square to c^2. Since 4-momentum is just mass times 4-velocity, the 4-momentum of any object with (rest) mass m satisfies

$$p^2 = m^2c^2\left(\cosh^2 y - \sinh^2 y\right) = m^2c^2\,. \tag{4.4.8}$$

[1] Many introductory relativity books introduce a velocity-dependent mass $m(v) \equiv m\,\gamma$, in order to write $\vec{p} = m(v)\,\vec{v}$, and thereby avoid ever introducing four-velocity (or any other 4-vector). This is pedagogically terrible and offers no benefit whatsoever. If you have previously seen this use of a velocity-dependent mass, erase it from your memory banks!

[2] To see why, recall from mechanics (quantum or classical) that translation invariance in space is related to the existence of conserved spatial momentum, and translation invariance in time is related to the existence of a conserved energy. We will discuss this in more detail later. Since Lorentz transformations mix space and time, it should be no surprise that the four-momentum, which transforms linearly under Lorentz transformations, must characterize both the energy *and* the spatial momentum.

Since $p^2 = (p^0)^2 - (p^k)^2$, and $p^0 = E/c$, this may rewritten (in any chosen inertial reference frame) as

$$E^2 = c^2\, \vec{p}^{\,2} + (mc^2)^2 \,. \tag{4.4.9}$$

So, if you know the spatial momentum $\vec{p}$ and mass m of some object, you can directly compute its energy E without first having to evaluate the object's velocity. Note, in particular, that Eq. (4.4.9) is true for *either* choice of metric!

But what if you want to find the ordinary 3-velocity? Return to the relation $u^k = \gamma\, v^k$ [Eq. (4.2.6)] between 3-velocity and 4-velocity, and multiply both sides by m to rewrite this result in terms of four-momentum. Since spatial momentum $p^k = mu^k$, and total energy $E = mc^2\gamma$, we have $p^k = (E/c^2)\, v^k$ or

$$v^k = \frac{p^k}{E/c^2}\,, \quad |\vec{v}| = c \tanh y \,. \tag{4.4.10}$$

Three-velocity is *not* equal to momentum divided by mass. Rather, the ordinary 3-velocity equals the spatial momentum divided by the total energy (over c^2). And its magnitude *never* exceeds c.

4.5 4-force

In the absence of any forces, the momentum of an object remains constant. In the presence of forces, an object's momentum will change. In fact, force is just the time rate of change of momentum. But what time and what momentum? Newtonian (non-relativistic) dynamics says that $d\vec{p}/dt = \vec{F}$ along with $d\vec{x}/dt = \vec{p}/m$, where $\vec{p}$ is 3-momentum and t is coordinate time. This is wrong — inconsistent with our relativity postulates. A frame-independent formulation of dynamics must involve quantities which have intrinsic frame-independent meaning — such as 4-momentum and proper time. The appropriate generalization of Newtonian dynamics which is consistent with our relativity postulates is

$$\frac{dx}{d\tau} = \frac{p}{m}\,, \tag{4.5.1a}$$

$$\frac{dp}{d\tau} = f\,. \tag{4.5.1b}$$

Eq. (4.5.1a) is just the definition (4.2.2) of 4-velocity rewritten in terms of 4-momentum, while Eq. (4.5.1b) is the *definition* of force as a four-vector. The only difference in these equations, relative to Newtonian dynamics, is the replacement of 3-vectors by 4-vectors and coordinate time by proper time.

Equations (4.5.1) are written in a form which emphasizes the role of momentum. If you prefer, you can work with 4-velocity instead of 4-momentum and rewrite these equations as $dx/d\tau = u$ and $du/d\tau = f/m$. Defining the *4-acceleration* $a \equiv du/d\tau = d^2x/d\tau^2$, this last equation is just $f = m\,a$. This is the relativistic generalization of Newton's $\vec{f} = m\,\vec{a}$, with force and acceleration now defined as spacetime vectors.[3]

In non-relativistic dynamics, if you know the initial position and velocity of a particle, and you know the force $\vec{f}(t)$ which subsequently acts on the particle, you can integrate Newton's equations to find

[3] Eq. (4.5.1b) is equivalent to $f = m\,a$ *provided* the mass m of the object is constant. For problems involving objects whose mass can change, such as a rocket which loses mass as it burns fuel, these two equations are not equivalent and one must use the more fundamental $dp/d\tau = f$.

the trajectory $\vec{x}(t)$ of the particle. Initial conditions plus a three-vector $\vec{f}(t)$ completely determine the resulting motion. To integrate the relativistic equations (4.5.1), you need initial conditions plus a four-vector force $f(\tau)$. This would appear to be more information (four components instead of three), and yet relativistic dynamics must reduce to non-relativistic dynamics when velocities are small compared to c.

The resolution of this apparent puzzle is that the four-force *cannot* be a completely arbitrary four-vector. We already know that, for any object with mass m, its four-momentum must satisfy $p^2 = (mc)^2$ [Eq. (4.4.8)]. Take the derivative of both sides with respect to proper time. The right hand side is constant in time (provided that the object in question is some stable entity with a fixed rest mass), so its proper time derivative vanishes. The derivative of the left hand side gives twice the dot product of p with f, and hence the 4-force must always be *orthogonal* to the 4-momentum,

$$p \cdot f = 0\,. \tag{4.5.2}$$

Written out in components, this says that $p^0 f^0 = p^i f^i$, or

$$f^0 = \frac{p^i f^i}{p^0} = \frac{\vec{v}}{c} \cdot \vec{f}\,, \tag{4.5.3}$$

showing that the time component of the force is just a particular linear combination of the spatial components, *i.e.*, the four components of the force cannot vary freely, but rather must satisfy this constraint.

4.6 Constant acceleration

Let us put this formalism into action by examining the case of motion under the influence of a constant force. But what is a "constant" force? We have just seen that the 4-force must always be orthogonal to the 4-momentum. So it is *impossible* for the 4-force $f(\tau)$ to be a fixed four-vector in an arbitrary frame, independent of τ. However, it is possible for the force to be constant when viewed in a frame which is *instantaneously* co-moving with the accelerating object.

Suppose a particle begins at the spacetime origin with vanishing 3-velocity (or 3-momentum) at proper time $\tau = 0$, and a (3-)force of magnitude F, pointing in the x^1 direction, acts on the particle. Hence the components of the initial spacetime position, four-velocity, and four-force are $x_0^\mu = (0, 0, 0, 0)$, $u_0^\mu = (c, 0, 0, 0)$, and $f_0^\mu = (0, F, 0, 0)$, respectively. The 4-velocity at later times may be written as some time-dependent Lorentz boost acting on the initial 4-velocity,

$$u(\tau) = \Lambda_{\rm boost}(\tau)\, u_0\,. \tag{4.6.1}$$

The condition that the force is constant (f_0) in a co-moving frame amounts to the statement that the *same* Lorentz boost (as in Eq. 4.6.1)) relates the 4-force at any time τ (in the frame where the velocity is $u(\tau)$) to the initial force,

$$f(\tau) = \Lambda_{\rm boost}(\tau)\, f_0\,. \tag{4.6.2}$$

At all times, $u^2 = c^2$ (because u is a 4-velocity), and $f^2 = -F^2$, because the magnitude of the force is assumed to be constant.

Since the initial force points in the x^1 direction, the particle will acquire some velocity in this direction, but the x^2 and x^3 components of the velocity will always remain zero. Hence the boost

$\Lambda_{\rm boost}(\tau)$ will always be some boost in the x^1 direction, and the force $f(\tau)$ will likewise always have vanishing x^2 and x^3 components. In other words, the 4-velocity and 4-force will have the form

$$u^\mu(\tau) = \big(u^0(\tau), u^1(\tau), 0, 0\big) \ , \qquad f^\mu(\tau) = \big(f^0(\tau), f^1(\tau), 0, 0\big) \ , \tag{4.6.3}$$

with $u^0(0) = c$, $u^1(0) = 0$ and $f^0(0) = 0$, $f^1(0) = F$. From (4.5.2) the dot product $f \cdot u = f^0 u^0 - f^1 u^1$ must vanish, implying that $f^0/f^1 = u^1/u^0$. So the components of the 4-force must be given by

$$f^\mu(\tau) = \frac{F}{c}\ \big(u^1(\tau), u^0(\tau), 0, 0\big) \ . \tag{4.6.4}$$

(Do you see why? This is the *only* form for which $f \cdot u = 0$ and $f^2 = -F^2$.)

Now we want to solve $m\, du/d\tau = f(\tau)$. Writing out the components explicitly (and dividing by m) gives

$$\frac{du^0(\tau)}{d\tau} = \frac{F}{mc}\ u^1(\tau)\,, \qquad \frac{du^1(\tau)}{d\tau} = \frac{F}{mc}\ u^0(\tau)\,. \tag{4.6.5}$$

This is easy to solve if you remember some basic mathematical physics (from Chapter 1) - $\frac{d}{dz}\sinh z = \cosh z$ and $\frac{d}{dz}\cosh z = \sinh z$ (recall Eq. (1.3.3)). To satisfy Eq. (4.6.5), and our initial conditions, we simply choose

$$u^0(\tau) = c\,\cosh\frac{F\tau}{mc}\,, \qquad u^1(\tau) = c\,\sinh\frac{F\tau}{mc}\,. \tag{4.6.6}$$

The ordinary velocity is given by $v^k = u^k\,(c/u^0)$ [Eq. (4.2.4)], so the speed of this particle subject to a constant force is

$$v(\tau) = c\,\tanh\frac{F\tau}{mc}\,. \tag{4.6.7}$$

Since $\tanh z \sim z$ for small values of the argument (recall Eq. (1.3.10)), the speed grows linearly with time at early times, $v(\tau) \sim (F/m)\,\tau$. This is precisely the expected *non-relativistic* behavior. But this approximation is only valid when $\tau \ll mc/F$ and the speed is small compared to c. The argument of the tanh becomes large compared to unity when $\tau \gg mc/F$, and $\tanh z \to 1$ as $z \to \infty$. So the speed of the accelerating particle asymptotically approaches, but never reaches, the speed of light. In fact, we see from our previous definitions of the 4-momentum in terms of the rapidity y that it is the rapidity that grows linearly with τ in the case of "constant acceleration",

$$\tanh y = \frac{v(\tau)}{c} = \tanh\frac{F\tau}{mc} \Rightarrow y = \frac{F\tau}{mc}\,. \tag{4.6.8}$$

At this point, we have determined how the velocity of the particle grows with time, but we need to integrate $dx/d\tau = u$ to find its spacetime position. Due to the properties of the hyperbolic functions the integrals are elementary,

$$x^0(\tau) = \int_0^\tau d\tau'\,u^0(\tau') = c\int_0^\tau d\tau'\ \cosh\frac{F\tau'}{mc} = \frac{mc^2}{F}\ \sinh\frac{F\tau}{mc}\,, \tag{4.6.9a}$$

$$x^1(\tau) = \int_0^\tau d\tau'\,u^1(\tau') = c\int_0^\tau d\tau'\ \sinh\frac{F\tau'}{mc} = \frac{mc^2}{F}\left[\cosh\frac{F\tau}{mc} - 1\right]. \tag{4.6.9b}$$

Note that the hyperbolic sines and cosines grow exponentially for large arguments, $\sinh z \sim \cosh z \sim \frac{1}{2}\,e^z$ when $z \gg 1$. Hence, when $\tau \gg mc/F$ the coordinates $x^0(\tau)$ and $x^1(\tau)$ both grow like $e^{F\tau/mc}$ with increasing proper time. But the accelerating particle becomes ever more time-dilated; the rate of change of proper time with respect to coordinate time, $d\tau/dt = c/u^0 = 1/\cosh\frac{F\tau}{mc}$, behaves as $2\,e^{-F\tau/mc} \sim mc/(Ft)$.

4.7 Plane waves

Next we want to discuss the very important question of how *waves* are described in relativistic notation. Consider some wave (any type of wave) with wave-vector $\vec{k}$ and frequency ω (3-vector notation), as measured in some inertial frame. The amplitude of the wave is described by a complex exponential, $\mathcal{A}\, e^{i\vec{k}\cdot\vec{x}-i\omega t}$, with the usual understanding that it is the *real* part of this function which describes the physical amplitude. Such a wave has a wavelength $\lambda = 2\pi/|\vec{k}|$ and planar wave-fronts orthogonal to the wave-vector that move at speed $v = \omega/|\vec{k}|$ in the direction of $\vec{k}$.

As mentioned earlier (Eq. (3.6.12)), it is natural to combine ω and $\vec{k}$ into a spacetime wave-vector (*i.e.*, 4-vector) k with components

$$k^\mu = (\omega/c,\, k^1, k^2, k^3)\,, \tag{4.7.1}$$

so that $\omega = c\, k^0$ and we have simply $e^{i\vec{k}\cdot\vec{x}-i\omega t} = e^{-ik\cdot x}$ (with our choice of metric).

The virtue of this formulation is that it is frame-*independent*. The spacetime position x and wave-vector k are geometric entities which you should think of as existing independent of any particular choice of coordinates. The value of the amplitude, $\mathcal{A}\, e^{-ik\cdot x}$, depends on the position x and the wave-vector k, but one may use whatever reference frame is most convenient to evaluate the dot product of these 4-vectors since $k\cdot x$ is the *same* in all frames. (This fact gives us the opportunity to be both lazy and smart, and is the real *power* of the 4-vector notation.)

Just as surfaces of simultaneity are observer-dependent, so is the frequency of a wave. After all, measuring the frequency of a wave involves counting the number of wave crests which pass some detector (or observer) in a given length of time. The time component of the wave-vector gives (by construction) the frequency of the wave as measured by observers who are at rest in the frame in which the components k^μ are defined. Such observers have 4-velocities whose components are just $(c, 0, 0, 0)$ (in that frame, *i.e.*, in their rest frame). Consequently, for these observers the frequency of the wave may be written as a dot product of the observer's 4-velocity and the wave-vector,

$$\omega_{\rm obs} = u_{\rm obs}\cdot k\,. \tag{4.7.2}$$

This expression is now written in a completely general fashion that is observer-*dependent* but frame-*independent*. That is, the expression (4.7.2) depends explicitly on the observer's 4-velocity u, but is independent of the frame used to evaluate the dot product between u and k (*i.e.*, the dot product must be the same in every frame). Therefore, the frequency which is measured by *any* observer will be given by the dot product of the observer's 4-velocity u and the wave-vector k. The dot product can be evaluated in *any* convenient (lazy but smart) frame, but, of course, u and k must *both* be evaluated in that *same* frame.

As we will discuss a bit more below, it should be no surprise that this approach is particularly useful for the discussion of light waves, where $v = c$, $\omega = c|\vec{k}|$ and

$$k^\mu_{\rm light} = \frac{\omega}{c}\left(1, \hat{k}\right)\,,\; k^2_{\rm light} = 0\,. \tag{4.7.3}$$

As a simple first application of Eqs. (4.7.2) and (4.7.3) we can derive the basic form of the relativistic Doppler shift of light. Consider a source of plane wave light, which emits light with frequency $\nu_0 = \omega_0/2\pi$ as measured in the rest frame of the source (note that ω has units of radians per second, while ν is measured in cycles per second - NOT the same units!). Further we take the light waves to be moving in the x^1 direction ($\hat{k} = \hat{e}_1$). Consider an observer moving *away* from the source (the *receding* case) also in the x^1 direction with velocity v, as measured in the source rest frame. Clearly the lazy but smart choice of frame is the source frame, where we have

$$k^\mu = \frac{\omega_0}{c}\,(1,1,0,0)\ ,\ u_{\rm obs} = \frac{c}{\sqrt{1-v^2/c^2}}\,(1, v/c, 0, 0)\ , \tag{4.7.4a}$$

$$\frac{\nu_{\rm obs}}{\nu_0} = \frac{\omega_{\rm obs}}{\omega_0} = \frac{u_{\rm obs}\cdot k}{\omega_0} = \sqrt{\frac{1-v/c}{1+v/c}} \le 1\ [\text{receding}]\ . \tag{4.7.4b}$$

In this case (see Eq. (3.3.7) in Kogut), with the source and observer *receding* from each other, the observer sees a *smaller* frequency than the source emits (the light is red-shifted, $\nu_{\rm obs}/\nu_0 = \sqrt{(1-v/c)/(1+v/c)} \le 1$). For an observer *approaching* a source, we simply change the sign of v in Eq. (4.7.4) and the light is blue-shifted to a larger frequency,[4]

$$\frac{\nu_{\rm obs}}{\nu_0} = \frac{\omega_{\rm obs}}{\omega_0} = \frac{u_{\rm obs}\cdot k}{\omega_0} = \sqrt{\frac{1+v/c}{1-v/c}} \ge 1\ [\text{approaching}]\ . \tag{4.7.5}$$

A more sophisticated application of Eq. (4.7.2), demonstrating the value of writing physical quantities in frame independent form, is illustrated in Figure 4.2.[5] Mounted on the inner surface of a centrifuge, which is rotating at angular frequency Ω, is an emitter of light at one point, and a receiver at a different point. Let ϕ be the angle between emitter and receiver, relative to the center of the centrifuge, as measured in the inertial lab frame. The (inner) radius of the centrifuge is R. The frequency of the light as measured by an observer who is instantaneously at rest relative to the emitter is $\nu_{\rm e}$. The frequency of the light as measured by an observer who is instantaneously at rest relative to the receiver is $\nu_{\rm r}$. What is the fractional difference $(\nu_{\rm r} - \nu_{\rm e})/\nu_{\rm e}$? How does this frequency shift depend on the angle ϕ and the rotation frequency Ω?

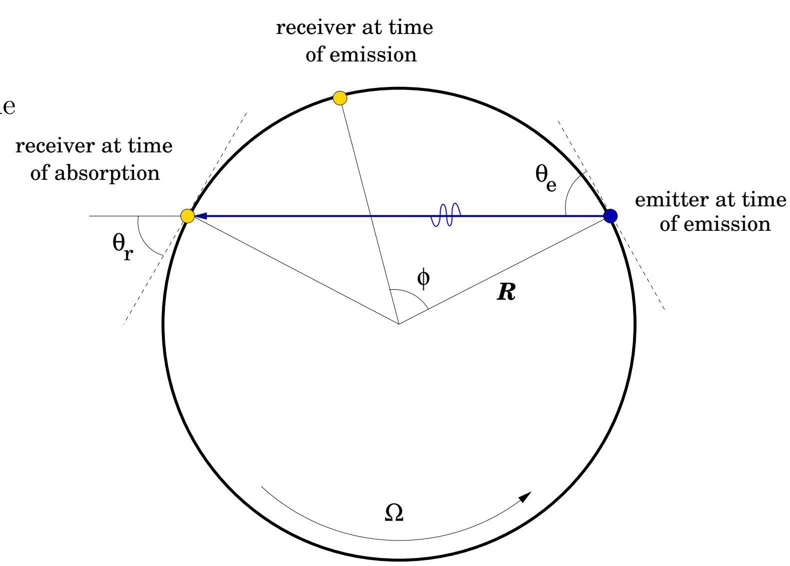

Figure 4.2: Inside a rotating centrifuge, light is emitted at one point and later received at another point. Is there a Doppler shift between the frequencies of emission and reception?

One approach for solving this problem would involve explicitly constructing the Lorentz transformations which relate the lab frame to the instantaneous rest frames of the emitter and receiver, and then combining these two transformations to determine the net transformation which directly connects

[4]A familiar application of this result in the context of astronomy and cosmology is the the redshift z, defined as the *fractional* change in the frequency (or wavelength). For a radially expanding (receding) universe (with radial velocity v), this leads to the relation $1 + z = \sqrt{(1+v/c)/(1-v/c)}$.

[5]This discussion is an adaptation of an example in *Gravitation* by Misner, Thorne and Wheeler.

emitter and receiver. Given the three-dimensional geometry involved, this is rather involved (and would not correspond to the lazy but smart approach).

A much better approach is to choose a convenient single frame, namely the lab frame, in which to evaluate the components of the four-vectors appearing in the frame-independent expression (4.7.2) for the frequency. We need to compute

$$\frac{\nu_{\rm r}}{\nu_{\rm e}} = \frac{u_{\rm r} \cdot k}{u_{\rm e} \cdot k} = \frac{u_{\rm r}^0\, k^0 - \vec{u}_{\rm r} \cdot \vec{k}}{u_{\rm e}^0\, k^0 - \vec{u}_{\rm e} \cdot \vec{k}}\,. \tag{4.7.6}$$

Here $u_{\rm e}$ is the four-velocity of the emitter at the moment it emits light, and $u_{\rm r}$ is the four-velocity of the receiver at the moment when it receives the light.

If $\theta_{\rm e}$ denotes the angle between the spatial wavevector and the direction of motion of the emitter (at the time of emission), and $\theta_{\rm r}$ denotes the angle between $\vec{k}$ and receiver's direction (at the time of reception) (all as indicated in Figure 4.2), then we can express the spatial dot products in terms of cosines of these angles,

$$\frac{\nu_{\rm r}}{\nu_{\rm e}} = \frac{u_{\rm r}^0\, k^0 - |\vec{u}_{\rm r}||\vec{k}| \cos\theta_{\rm r}}{u_{\rm e}^0\, k^0 - |\vec{u}_{\rm e}||\vec{k}| \cos\theta_{\rm e}}\,. \tag{4.7.7}$$

The speed of the inner surface of the centrifuge is constant, $v = \Omega\, R$, and hence the *speeds* of the emitter and receiver, as measured in the lab frame, are identical — even though their velocity vectors are different. The time component of a 4-velocity, $u^0/c = (1 - v^2/c^2)^{-1/2}$, only depends on the *magnitude* of the velocity $\vec{v}$, and hence $u_{\rm r}^0 = u_{\rm e}^0$. The equality of the emitter and receiver speeds also implies that the magnitudes of the spatial parts of the 4-velocities coincide, $|\vec{u}_{\rm r}| = |\vec{u}_{\rm e}|$. So using expression (4.7.7) for the frequency ratio, the only remaining question is how does $\theta_{\rm r}$ compare to $\theta_{\rm e}$?

This just involves ordinary geometry. Looking at the figure, notice that $\theta_{\rm e}$ and $\theta_{\rm r}$ are the angles between the path of the light, which is a chord of the circle, and tangents to the circle at the endpoints of the chord. But the angle a chord makes with these tangents is the same at either end, implying that $\theta_{\rm e} = \theta_{\rm r}$. And this means $\nu_{\rm r} = \nu_{\rm e}$ — there is *no* Doppler shift no matter how fast the centrifuge rotates (or what the values of ϕ, $\theta_{\rm e}$ or R are)! (Try obtaining this result directly using boosts and more complex trigonometry.)

4.8 Electromagnetism

As noted earlier, it should be no surprise that the technology we are developing is especially useful for "objects" that travel at the speed of light, such as light itself. Unfortunately we do not have time here for an extensive explorations of the relativistic aspects of electromagnetism, which will be left for other classes. But one aspect, how to represent the Lorentz force in the framework we have been discussing, is natural to describe here.

As we have seen above, generalizations from non-relativistic to relativistic dynamics are mostly a matter of replacing 3-vectors by 4-vectors (and coordinate time by proper time). But what about electric and magnetic fields? Both are (apparently) 3-vectors, and there is no sensible way to turn them into 4-vectors. Recall that $\vec{E}$ is an ordinary "polar" 3-vector, which changes sign under reflection, while $\vec{B}$ is an "axial" 3-vector, which does *not* change sign under reflection. It turns out that what *is* sensible (and natural) is to package the components of $\vec{E}$ and $\vec{B}$ together into a 4×4 matrix (a 4-tensor) called the *field strength tensor*, whose components are

$$\|F^{\mu}{}_{\nu}\| = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix} . \tag{4.8.1}$$

It is important to realize that this explicit form for the field strength tensor (*i.e.*, the explicit signs and factors of c) corresponds to our specific choices of units (here SI, with E measured in Newtons/Coulomb and B in Tesla) and the choice to write down the form with one superscript and one subscript.[6] Different choices for these details yield slightly different expressions.

With this repackaging of electric and magnetic fields, the Lorentz force (as a 4-vector) has a remarkably simple form,

$$f^{\mu}_{\text{Lorentz}} = \frac{q}{c}\, F^{\mu}{}_{\nu}\, u^{\nu} . \tag{4.8.2}$$

Verifying that this 4-force leads to exactly the same rate of change of energy and momentum as does the traditional form of writing the Lorentz force, $\vec{f} = q\,(\vec{E} + \vec{v} \times \vec{B})$, is an instructive and recommended exercise.

4.9 Scattering

When objects (elementary particles, molecules, automobiles, ...) collide, the results of the collision can differ markedly from the initial objects. Composite objects can fall apart or change form (leading to large insurance premiums). Interestingly, dramatic changes during collisions can also occur for elementary particles. In fact, studying the collisions of elementary particles is a primary method used to investigate fundamental interactions and explains the existence of large energy particle colliders like the LHC. These machines are really just (large!) microscopes with *very fine* resolution and with the capability to produce particles, like the Higgs boson, that we do not observe in everyday life.

A complete description of what emerges from a collision (or 'scattering event') depends on microscopic details of the interaction between the incident objects. But certain general principles constrain the possibilities, most importantly, the conservation of energy and momentum. As discussed in section 4.4, the total energy E and spatial momentum $\vec{p}$ of any object may be combined to form the 4-momentum $p^{\mu} = (E/c, \vec{p})$. Consequently, energy and momentum conservation may be rephrased as the conservation of 4-momentum: in the absence of any external forces, the total 4-momentum of any system cannot change,

$$\frac{d}{dt}\, p_{\,\text{tot}}(t) = 0\, . \tag{4.9.1}$$

[6]To appreciate the field strength tensor in its full 4-glory, we recognize that it is defined in terms of a 4-vector potential, $A^{\mu} = (\phi, c\vec{A})$, where ϕ is the usual electric scalar potential (in volts) and $\vec{A}$ is the usual 3-vector potential, which you may have seen in previous courses. Then the field strength tensor is the 4-curl, $F^{\mu}{}_{\nu} = \partial^{\mu} A_{\nu} - \partial_{\nu} A^{\mu}$, with $\partial^{\mu} = \partial/\partial x_{\mu} = (\partial_t/c, -\vec{\nabla})$ and $\partial_{\mu} = \partial/\partial x^{\mu} = (\partial_t/c, \vec{\nabla})$ (in our metric). Finally we need the 3-vector definitions $\vec{E} = -\partial\vec{A}/\partial t - \vec{\nabla}\phi$ and $\vec{B} = \vec{\nabla} \times \vec{A}$, where this cross product explains why $\vec{B}$ is an axial or pseudo-vector. The interested reader is encouraged to work out the terms is Eq. (4.8.1) using these definitions. Note that with both indices either up or down the resulting tensor is fully anti-symmetric instead of the mixed symmetry (metric independent) form in Eq. (4.8.1).

In a scattering process two or more objects, initially far apart, come together and interact in some manner (which may be very complicated), thereby producing some number of objects that subsequently fly apart. When the incoming objects are far apart and not yet interacting, the total 4-momentum is just the sum of the 4-momentum of each object,

$$p_{\,\rm in} = \sum_{a=1}^{N_{\rm in}} p_a \,, \tag{4.9.2}$$

where $N_{\rm in}$ is the number of incoming objects (and the index a labels particles, not spacetime directions). Similarly, when the outgoing objects are arbitrarily well separated, they are no longer interacting and the total 4-momentum is (again) the sum of the individual 4-momenta of all of the outgoing objects,

$$p_{\,\rm out} = \sum_{b=1}^{N_{\rm out}} p_b \,. \tag{4.9.3}$$

Hence, for any scattering processes, conservation of energy and momentum implies that the total incident 4-momentum equals the total outgoing 4-momentum (independent of the values of $N_{\rm in}$ and $N_{\rm out}$),

$$p_{\,\rm in} = p_{\,\rm out} \,. \tag{4.9.4}$$

As with any 4-vector equation, one may choose to write out the components of this equation in whatever reference frame is most convenient (as long as we use the *same* frame for both $p_{\,\rm in}$ and $p_{\,\rm out}$). For analyzing scattering processes, sometimes it is natural to work in the rest frame of one of the initial objects (the 'target'); this is commonly called the *lab frame* and experiments of this variety are called "fixed target" experiments (the frame of the actual lab is the target frame). Alternatively, one may choose to work in the reference frame in which the total spatial momentum vanishes. In this frame, commonly called the *CM frame*,[7] the components of the total 4-momentum are

$$p^\mu_{\rm CM} = (E_{\rm CM}/c,\, 0,\, 0,\, 0)\,, \tag{4.9.5}$$

where $E_{\rm CM}$ is the total energy of the system in the CM frame. In the early days of particle physics, where only a single beam of accelerated particles was available, fixed target experiments were the norm and the CM frame was an intellectual construct. For the kinematic reasons we are about to discuss, it became clear that moving the actual lab to the CM frame would provide an enormous increase in efficiency for particle production. As a result, we now live in the era of particle *colliders*, where two beams of accelerated particles, moving in opposite directions, are caused to collide essentially head-on.

As an application of these ideas, consider first the scattering of protons of energy $E_{\rm in} = 1\,\mathrm{TeV}$ on protons at rest (in ordinary matter). The proton rest energy $m_p\,c^2$ is a bit less than $1\,\mathrm{GeV}$. Using Eq. (4.4.5), one sees that a proton with $1\,\mathrm{TeV}$ energy is ultrarelativistic, $\gamma = E_{\rm in}/(m_p\,c^2) \approx 10^3$. When an ultrarelativistic proton strikes a target proton at rest, both protons can be disrupted and new particles may be created. Schematically,

$$p + p \to X\,,$$

[7]'CM' means 'center of mass', but this historical name is really quite inappropriate for relativistic systems, which may include massless particles that carry momentum but have no rest mass. The widely used 'CM' label should always be understood as referring to the zero (spatial) momentum frame.

where X stands for one or more outgoing particles. What is the largest mass of a particle which could be produced in such a collision?

The total energy of the incident particles (in the rest frame of the target) is $E_{\rm tot} = E_{\rm in} + m_p c^2 \approx 1.001\,\text{TeV}$. If all of this energy is converted into the rest energy of one or more outgoing particles, then you might conclude that these collisions could produce particles with mass up to $E_{\rm tot}/c^2 \approx 10^3\, m_p$. This would be consistent with conservation of energy. But this is *wrong*, as it completely ignores conservation of 3-momentum. In the rest frame of the target, the total spatial momentum $\vec{p}_{\rm tot}$ is non-zero (and equal to the momentum $\vec{p}_{\rm in}$ of the projectile proton). If there is a single outgoing particle X, it cannot be produced at rest — it must emerge from the collision with a non-zero spatial momentum equal to $\vec{p}_{\rm tot}$. That means its energy will be greater than its rest energy.

To determine the largest mass of a particle which can be produced in this collision, one must simultaneously take into account conservation of both energy and momentum. That is, one must satisfy the 4-vector conservation equation (4.9.4). In the lab frame, if we orient coordinates so that the 3-axis is the collision axis, then

$$p_{\rm in} = p_{\rm projectile} + p_{\rm target} = \begin{pmatrix} E_{\rm in}/c \\ 0 \\ 0 \\ p_{\rm in} \end{pmatrix} + \begin{pmatrix} m_p\, c \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.9.6}$$

If a single particle X emerges, then its four-momentum is the total outgoing four-momentum,

$$p_{\rm out} = p_X = \begin{pmatrix} E_X \\ p_X^1 \\ p_X^2 \\ p_X^3 \end{pmatrix}. \tag{4.9.7}$$

Demanding that $p_{\rm in}$ coincide with $p_{\rm out}$ determines $\vec{p}_X = p_{\rm in}\,\hat{e}_3$ and $E_X = E_{\rm in} + m_p c^2$. Eq. (4.4.9), applied to the projectile proton (with known mass), may be used to relate the incident spatial momentum and energy, $\vec{p}_{\rm in}^{\,2} = (E_{\rm in}/c)^2 - (m_p\, c)^2$. The same relation (4.4.9), applied to the outgoing particle X, connects its energy E_X and momentum $\vec{p}_X$ to the desired maximum mass m_X, $(m_X c^2)^2 = E_X^2 - (c\,\vec{p}_X)^2$. Inserting numbers and computing E_X, $|\vec{p}_X| = p_{\rm in}$, and finally m_X is straightforward. But even less work is required if one recalls [from Eq. (4.4.8)] that the square of any four-momentum directly gives the rest mass of the object, $p^2 = m^2 c^2$. Hence

$$\begin{aligned} m_X^2\, c^2 = p_X^2 = p_{\rm out}^2 = p_{\rm in}^2 &= (p_{\rm projectile} + p_{\rm target})^2 \\ &= p_{\rm projectile}^2 + p_{\rm target}^2 + 2\, p_{\rm projectile} \cdot p_{\rm target} \\ &= 2\, m_p^2\, c^2 + 2 E_{\rm in}\, m_p\, . \end{aligned} \tag{4.9.8}$$

Consequently, $m_X = \sqrt{2m_p(m_p + E_{\rm in}/c^2)} = m_p\sqrt{2 + 2E_{\rm in}/(m_p c^2)} \approx \sqrt{2002}\, m_p \approx 45\, m_p$. Even though the projectile proton has an energy a thousand times greater than its rest energy, the maximum mass particle which can be created in this collision is only 45 times heavier than a proton. The rest of the energy most provide the kinetic energy associated with the conserved spatial momentum. More generally, the maximum mass that can be produced grows (only) like the *square root* of the lab frame energy, $m_X \sim \sqrt{2E_{\rm in} m_p/c^2}$, when $E_{\rm in} \gg m_p c^2$. This is why "colliders", where the lab and CM frames coincide with both the "beam" and "target" particles racing towards each other, are most efficient when hunting for new particles. In particular, if we collide two particles with the same mass (*e.g.*, either identical particles or particle and antiparticle), the same energy (E_{in}) but opposite momenta, the largest rest mass (particle) we can produce is $m_X = 2E_{\text{in}}$, which in this case increases *linearly* with the beam energy E_{in}.

4.10 Example Problems

Kogut 4-3

In the S' frame we have an event at the 4-vector point $x' = \left(c \times 9 \times 10^{-8}\,\text{s}, 100\,\text{m}, 0, 0\right)$. We want to determine the location of this event in the S frame, where the S' frame moves with velocity $v/c = 4/5$ along the x axis with respect to the S frame and, for convenience (we are free to be lazy but smart), we assume that the origins (in space and time) of the 2 frames are synchronized. The boost factor between the two frames is $\gamma = 1/\sqrt{1-(v/c)^2} = 5/3$. Thus the corresponding Lorentz boost gives us

$$x = \Lambda(v)x' = \begin{pmatrix} 5/3 & 4/3 & 0 & 0 \\ 4/3 & 5/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 27\,\text{m} \\ 100\,\text{m} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 178.33\,\text{m} \\ 202.67\,\text{m} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} c \times 59.44 \times 10^{-8}\,\text{s} \\ 202.67\,\text{m} \\ 0 \\ 0 \end{pmatrix}.$$

Kogut 4-4

In frame S we are given two events defined by the 4-vectors $x_1 = (L, L, 0, 0)$ and $x_2 = (L/2, 2L, 0, 0)$, or $\Delta x^0 = L/2$ and $\Delta x^1 = -L$. We want to boost to a frame S' where the events (appear to) occur at the same time. Thus we want to solve (note v is the velocity of S' in S so using Λ^{-1} or $\Lambda(-v)$

$$\Delta x'^0 = \gamma\left(\Delta x^0 + (-v/c)\Delta x^1\right) = 0 \Rightarrow L/2 + (-v/c)(-L) = 0 \Rightarrow v/c = -1/2,$$

corresponding to the S' frame moving towards negative x values. So we find the common time in the S' frame is

$$t' = \gamma\left(ct_1 + (-v/c)x_1\right)/c = \left(2/\sqrt{3}\right)\left(L + (1/2)L\right)/c = \sqrt{3}L/c\,.$$

As a check, note that we obtain the same result if we use instead t_2 and x_2.

Kogut 6-11

Consider a relativistic particle whose (relativistic) kinetic energy is twice its rest energy, *i.e.*, its total energy is *three* times its rest energy. Thus we have

$$K = 2mc^2 \Rightarrow E = 3mc^2 \Rightarrow \gamma = 3 \Rightarrow \frac{v}{c} = \sqrt{1-\frac{1}{9}} = 0.943\,.$$

Thus the magnitude of this particle's momentum is

$$p = \gamma m v = 3(v/c)mc = 2.83\,mc\,.$$

If the kinetic energy is $5mc^2$, we have instead

$$\gamma = 6 \Rightarrow \frac{v}{c} = \sqrt{1-\frac{1}{36}} = 0.986\,,\; p = 5.92\,mc\,.$$

Kogut 6-16

Here we have the opportunity to consider the most fuel-efficient rocket exhaust - photons (the fastest exit velocity for any given energy) in a problem with an explicitly time *dependent* mass for the rocket. We are given only the rocket's initial and final masses, M_i and M_f, and want to calculate its final velocity (starting at rest). Being smart but lazy we do NOT integrate Newton's law! Instead we simply use 4-momentum (energy-momentum) conservation. We image that a certain fraction of the initial mass of the rocket, $\Delta M = M_i - M_f$, is instantaneously converted into a photon (or several collinear photons). To conserve momentum the rocket must recoil in the direction opposite to the photon(s). We have (in the initial rest frame of the rocket)

$$E_f = E_i \Rightarrow M_i c^2 = E_{\text{photon(s)}} + M_f \gamma c^2 \,,$$

$$\vec{p}_{\text{total}} = 0 \Rightarrow M_f v\gamma = p_{\text{photon(s)}} = E_{\text{photon(s)}}/c = M_i c - M_f c\gamma \Rightarrow M_i = M_f \gamma \frac{c+v}{c} \tag{4.10.1}$$

$$\Rightarrow \frac{M_i}{M_f} = \gamma(1 + v/c) = \sqrt{\frac{1+v/c}{1-v/c}} \quad \text{or} \quad \frac{v}{c} = \frac{(M_i/M_f)^2 - 1}{(M_i/M_f)^2 + 1} \,. \tag{4.10.2}$$

So the price of going very fast, $v \to c$, is that the final mass (including the astronaut) must be very much smaller than the initial mass ($M_f \ll M_i$), *i.e.*, accelerating rockets to near light-speed is an expensive activity (as has always been known to NASA).