# Chapter 2

# Special relativity

## 2.1 Galilean relativity

We start our discussion of symmetries by considering an important example of an invariance, *i.e.* an invariance of the equations of motion under a change of the coordinate system. In particular, Newton's laws of motion,

$$\frac{d\vec{p}}{dt} = \vec{F}\,, \qquad \frac{d\vec{x}}{dt} = \frac{\vec{p}}{m}\,, \tag{2.1.1}$$

retain the same form if one substitutes, *i.e.*, changes coordinates via,

$$\vec{x} \to \vec{x}\,' + \vec{u}\,t\,, \qquad \vec{p} \to \vec{p}\,' + m\,\vec{u}\,, \tag{2.1.2}$$

for any velocity $\vec{u}$ which is constant, *i.e.*, independent of time, $d\vec{u}/dt = 0$. In other words, equations (2.1.1) and (2.1.2) imply that

$$\frac{d\vec{p}\,'}{dt} = \vec{F}\,, \qquad \frac{d\vec{x}\,'}{dt} = \frac{\vec{p}\,'}{m}\,. \tag{2.1.3}$$

This argument indicates that changing coordinates to those of a (relatively) moving (inertial) reference frame does not affect the form of Newton's equations. In other words, there is no *preferred* inertial frame in which Newton's equations are valid; if they hold in one inertial frame, then they hold in *all* inertial frames, *i.e.*, in all frames moving with a constant relative velocity. This is referred to as *Galilean* relativity. Note that an intrinsic feature (assumption) of Galilean relativity is that clocks, *i.e.*, time, are the same in all inertial frames. Once they are synchronized between two different inertial frames, they will remain synchronized.

Consider a particle, or wave moving with velocity $\vec{v}$ when viewed in the unprimed frame. In that frame the position of the particle (or crest of the wave) is given by $\vec{x}(t) = \vec{x}_0 + \vec{v}\,t$. In the primed frame, using (2.1.2), the location of the same particle or wave-crest is given by $\vec{x}\,'(t) = \vec{x}_0 + (\vec{v} - \vec{u})\,t$. Hence, when viewed in the primed frame, the velocity of the particle or wave is given by

$$\vec{v}\,' = \vec{v} - \vec{u}\,. \tag{2.1.4}$$

This *shift* in velocities upon transformation to a moving frame is completely in accord with everyday experience. For example, as illustrated in Figure 2.1 if a person standing on the ground sees a

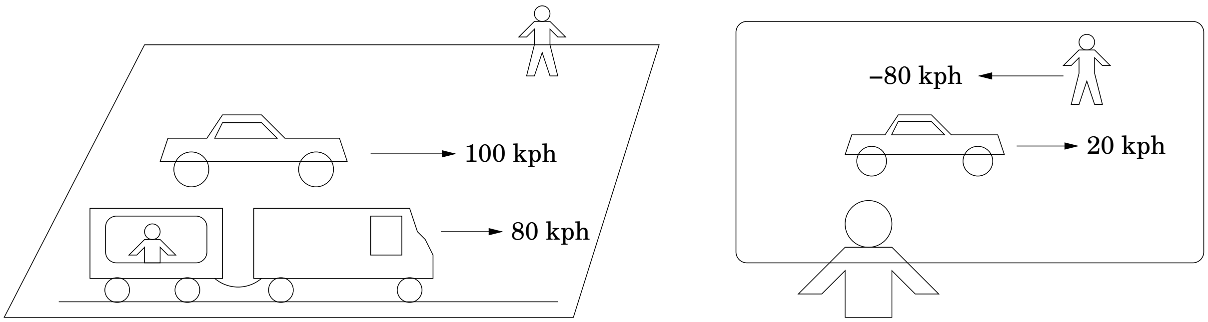


Figure 2.1: A moving train and car, as seen from the ground (left), and from the train (right).

car moving at 100 kph (kilometers per hour) parallel to a train moving at 80 kph, then a person sitting in the train will see that car moving with a *relative* velocity of 20 kph = (100 − 80) kph, while the person on the ground recedes from view at a velocity of −80 kph. Similarly, a sound wave propagating at the speed of sound $v_s$ (in a medium), as seen by an observer at rest with respect to the medium, will be seen (or heard) as propagating with speed $v' = v_s - u$ by an observer moving in the same direction as the sound wave with speed $u$ (with respect to the medium). Consequently, the frequency $f' = v'/\lambda$ heard by the moving observer (*i.e.*, the number of wave fronts passing the observer per unit time) will differ from the frequency $f = v_s/\lambda$ heard by the stationary observer,

$$f' = \frac{v_s - u}{\lambda} = f\left(1 - \frac{u}{v_s}\right) . \tag{2.1.5}$$

This is the familiar Doppler shift for the case of a moving observer and stationary source, where both velocities are measured with respect to the medium. Recall from introductory physics that the *medium* plays an important here. If, with respect to medium, it is instead the observer who is stationary and the source that is moving (with the two still separating), the resulting Doppler shift is now

$$f' = \frac{f}{\left(1 + \frac{u}{v_s}\right)} . \tag{2.1.6}$$

Of course, to first order in $u/v_s$ the results are identical, *i.e.*, (2.1.6) approaches (2.1.5), for $u/v_s \ll 1$,

$$f' = \frac{f}{\left(1 + \frac{u}{v_s}\right)} \,|_{u/v_s \ll 1} \simeq f\left(1 - \frac{u}{v_s}\right) . \tag{2.1.7}$$

However, as $u$ approaches $v_s$ ($u/v_s \to 1$) the limits of (2.1.6) and (2.1.5) are quite different,

$$f' = f\left(1 - \frac{u}{v_s}\right) |_{u/v_s \to 1} \to 0 , \tag{2.1.8a}$$

$$f' = \frac{f}{\left(1 + \frac{u}{v_s}\right)} \,|_{u/v_s \to 1} \to \frac{f}{2} . \tag{2.1.8b}$$

Thus there is a *special* reference frame for sound, the rest frame of the medium in which the sound propagates.

## 2.2 Constancy of $c$

When applied to light (*i.e.*, electromagnetic radiation) the Galilean relativity velocity transformation (2.1.4) predicts that observers moving at different speeds will measure different propagation velocities for light coming from a given source (perhaps a distant star). This conclusion is *wrong*. Many experiments, including the famous Michelson-Morley experiment, have looked for, and failed to find, any variation in the speed of light as a function of the velocity of the observer. It has been unequivocally demonstrated that (2.1.4) does *not* apply to light. Note also that, unlike sound, light requires *no* medium to propagate.

Newton's laws, and the associated Galilean relativity relations (2.1.2) and (2.1.4), provide extremely accurate descriptions for the dynamics of particles and waves which move slowly compared to the speed of light $c$. But Newtonian dynamics does not correctly describe the behavior of light or (as we will see) any other particle or wave moving at speeds which are not very small compared to $c$. Our goal is to find a formulation of the dynamics which does not have this limitation.

We will provisionally adopt two postulates:

**Postulate 1** *The speed of light (in a vacuum) is the same in all inertial reference frames.*

**Postulate 2** *There is no preferred reference frame: the laws of physics take the same form in all inertial reference frames.*

We will see that these postulates lead to a fundamentally different view of space and time, as well as to many predictions which have been experimentally tested — successfully. In particular, we *must* view the world as intrinsically 4-dimensional. Not only do velocities change as we move between different inertial frames, but time does also. Yet, as required to match experiment, this new description of space-time must reduce to the familiar Galilean result in the limit $v \ll c$.

It is important to note that Postulate 1 refers to the motion of light in a vacuum. In the case of light propagating in a medium the light will interact with the atoms composing the medium (the indiviudal photons will be absorbed and re-emitted) leading to a (typically small) change in the apparent velocity (as you presumably discussed in your introductory physics class in terms of the index of refraction).

## 2.3 Clocks and rulers

A clock is some construct which produces regular "ticks" that may be counted to quantify the passing of time. An ideal clock is one whose period is perfectly regular and reproducible. Real clocks must be based on some physical phenomenon which is nearly periodic — as close to periodic as possible. Examples include pendula, vibrating crystals, and sun-dials. All of these have limitations. The period of a pendulum depends on its length and the acceleration of Earth's gravity. Changes in temperature will change the length of a pendulum. Moreover, the Earth is not totally rigid: tides, seismic noise, and even changes in weather produce (small) changes in the gravitational acceleration at a given point on the Earth's surface. The frequency (or period) of vibration of a crystal is affected by changes in temperature and changes in mass due to adsorption of impurities on its surface. In addition to practical problems (weather), the length of days as measured by a sun-dial changes with the season and, on much longer time scales, changes due to slowing of the Earth's rotation caused by tidal friction. On the other hand, if we observe the behavior of quantum mechanical systems such as individual atoms, we see much more robust periodic behavior.[1]

An idealized clock, which is particularly simple to analyze, is shown in Fig. 2.2. A short pulse of light repeatedly bounces back and forth (in a vacuum) between two parallel mirrors. Each time the light pulse reflects off one of the mirrors constitutes a "tick" of this clock.[2] If $L$ is the distance between the mirrors, then the period (round-trip light travel time) of this clock is $\Delta t = 2L/c$.

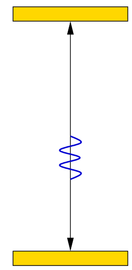

Figure 2.2: An idealized clock in which a pulse of light repeatedly bounces between two mirrors.

Now consider this same clock as seen by an observer moving to the left (perpendicular to the direction of the bouncing light) at velocity $-u$. In the observer's frame, the clock moves to the right at velocity $u$, as shown in Fig. 2.3. Let $\Delta t'$ be the period of the clock as viewed in this frame, so that the pulse of light travels from the lower mirror to the upper mirror and back to the lower mirror in time $\Delta t'$. The upper reflection takes place halfway through this interval, when the upper mirror has moved a distance $u\,\Delta t'/2$ to the right, and the light returns to the lower mirror after it has moved a distance $u\,\Delta t'$. Hence the light must follow the oblique path shown in the figure. The distance the light travels in one period is twice the hypotenuse, $D = 2\sqrt{L^2 + (u\,\Delta t'/2)^2} = \sqrt{4L^2 + (u\,\Delta t')^2}$. Now use the first postulate: the speed of light in this frame is $c$, exactly the same as in the original frame. This means that the distance $D$ and the period $\Delta t'$ must be related via $D = c\,\Delta t'$. Combining these two expressions gives $c\,\Delta t' = \sqrt{4L^2 + (u\,\Delta t')^2}$ and solving for $\Delta t'$ yields $\Delta t' = 2L/\sqrt{c^2 - u^2}$. Inserting $2L = c\,\Delta t$ and simplifying produces

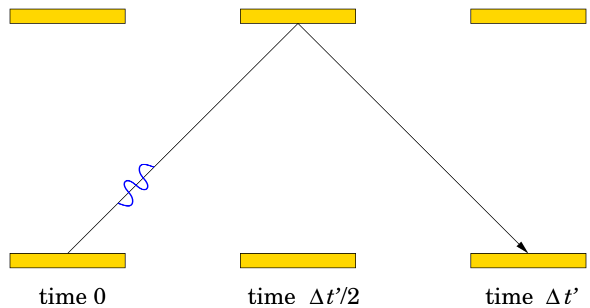


Figure 2.3: Three snapshots of the same clock viewed from a moving frame.

$$\Delta t' = \frac{\Delta t}{\sqrt{1-(u/c)^2}}\,. \tag{2.3.1}$$

This is a remarkable result. It shows that the period of a clock, when viewed in a frame in which the clock is moving, is different, and *longer*, than the period of the clock as viewed in its rest frame. This phenomena is known as *time dilation*. It is an inescapable consequence of the constancy of the speed of light. Although we have analyzed a particularly simple model of a clock to deduce the existence of time dilation, the result is equally valid for *any* good clock.[3] In other words, moving clocks run *slower* than when at rest, by a factor of

$$\gamma \equiv \frac{1}{\sqrt{1-(u/c)^2}}\,, \tag{2.3.2}$$

[1] Atomic clocks can now provide a very high time standard indeed as described in this Wikipedia article.

[2] To actually build such a clock, one would make one of the mirrors partially reflecting so that a tiny part of each light pulse is transmitted and measured by a photo-detector. These practical aspects are inessential for our purposes.

[3] After all, if some other good clock remains synchronized with our idealized clock when viewed in their common rest frame, then the same synchronization between the two clocks must also be present when the two clocks are viewed in a moving frame.

where $u$ is the speed with which the clock is moving. Note that $\gamma > 1$ for any non-zero speed $u$ which is less (in magnitude) than $c$ (also note that $\gamma$ diverges as $u$ approaches $c$ from below and assumes a non-physical imaginary value for $u > c$).

In the above discussion, we examined the case where the axis of our idealized clock was perpendicular to the direction of motion. What if the axis of the clock is parallel to the direction of motion? This situation is shown in Fig. 2.4. Analyzing this case is also instructive.

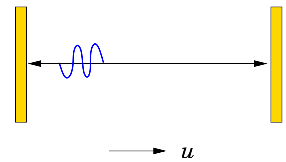

Figure 2.4: Our idealized clock, now rotated so that its axis is parallel to the direction of motion.

The round-trip light travel time (or period) must again be $\Delta t' = \gamma\, \Delta t$, because time dilation applies to *any* clock.[4] Let $L'$ be the distance between the mirrors, as viewed in the primed frame. The mirrors are moving to the right at velocity $u$, as shown in the figure. Suppose the light reflects off the right-hand mirror at time $\delta t'$ after leaving the left-hand mirror. During this time the right-hand mirror will have moved a distance $u\, \delta t'$ and therefore the distance light travels on this leg is $L' + u\, \delta t'$, longer than $L'$ due to the motion of the mirror. Since $\Delta t'$ is the round-trip time, the light travel time for the return leg must be $\Delta t' - \delta t'$. On the way back, the light travel distance is $L' - u\,(\Delta t' - \delta t')$, since the motion of the left-hand mirror is *decreasing* the distance the light must travel.

Now use Postulate 1. For the first leg, the light travel distance $L' + u\, \delta t'$ must equal $c\, \delta t'$, since the speed of light in any (inertial) frame is $c$. Hence $\delta t' = L'/(c - u)$. And for the second leg, equating the distance $L' - u\,(\Delta t' - \delta t')$ with $c\,(\Delta t' - \delta t')$ implies that $\Delta t' - \delta t' = L'/(c + u)$. Substituting in $\delta t'$ gives

$$\Delta t' = \frac{L'}{c+u} + \frac{L'}{c-u} = \frac{2c\, L'}{c^2 - u^2} = \gamma^2\,(2L'/c)\,. \tag{2.3.3}$$

But we already know that $\Delta t' = \gamma\, \Delta t = \gamma\,(2L/c)$. The only way these two results for $\Delta t'$ can be consistent is if the distance $L'$ between the mirrors, as seen in the frame in which the clock is moving parallel to its axis, is *smaller* than $L$ by a factor of $\gamma$,

$$L' = \frac{L}{\gamma} = L\,\sqrt{1 - (u/c)^2}\,. \tag{2.3.4}$$

This phenomena is known as *Lorentz contraction*. We have deduced it by using an ideal clock to convert a measurement of distance (the separation between mirrors) into a measurement of time. But the same result must apply to the measurement of any length which is parallel to the direction of motion. In other words, a ruler whose length is $L$, as measured in its rest frame, will have a length of $L' = L/\gamma$ when viewed in a frame in which the ruler is moving with a velocity parallel to itself (*i.e.*, parallel to the long axis of the ruler).

## 2.4 Observational tests

As we have seen, both time dilation and Lorentz contraction are direct, logical consequences of the frame-*independence* of the speed of light. Therefore every experimental test of the frame indepen-

[4] To expand on this, imagine constructing two identical copies of our idealized clock. In their common rest frame, orient the axis of one clock perpendicular to the axis of the other clock. Since these two ideal clocks remain synchronized when viewed in their rest frame, they must also be synchronized when viewed from a moving frame whose velocity is parallel to one clock and perpendicular to the other.

dence of $c$ is a test of the existence of both time dilation and Lorentz contraction. Nevertheless, it is interesting to ask how these effects can be directly observed.

One place where time dilation has a "real world" impact is in the functioning of the global positioning system (GPS). Time-dilation, due to the orbital motion of GPS satellites, slows the atomic clocks carried in these satellites by about 7 microseconds per day. This is easily measurable, and is a huge effect compared to the tens of nanosecond (per day) timing accuracy which can be achieved using GPS signals.[5]

A different observable phenomena where time dilation plays a key role involves muons produced in cosmic ray showers. When a high energy cosmic ray (typically a proton or atomic nucleus) strikes an air molecule in the upper reaches of the atmosphere (typically above 50 km $= 5 \times 10^4$ m), this can create a particle shower containing many elementary particles of various types (which we will be discussing later) including electrons, positrons, pions, and muons. Muons are unstable particles; their lifetime $\tau$ is 2.2 microseconds. Moving at almost the speed of light, a high energy muon will travel a distance of about $c\tau \approx (3 \times 10^8 \text{ m/s}) \times (2 \times 10^{-6} \text{ s}) = 600$ m in time $\tau$. This is small compared to the height of the atmosphere, and yet muons produced in showers originating in the upper atmosphere are easily observed on the ground. How can this be, if muons decay after merely a couple of microseconds?

The resolution of this apparent paradox is time dilation. Two microseconds is the lifetime of a muon *in its rest frame*. One may view a muon, or a bunch of muons moving together, as a type of clock. If there are $N_0$ muons initially, then after some time $t$ (as measured in the rest frame of the muons) on average all but $N_1 = N_0\, e^{-t/\tau}$ muons will have decayed. Turning this around, if all but some fraction $N_1/N_0$ of muons decay after some interval of time, then the length of this interval equals $\tau\, \ln(N_0/N_1)$ — as measured in the muons' rest frame. But as we have seen above, a moving clock (any moving clock!), runs slower by a factor of $\gamma$. Therefore, fast moving muons decay more slowly than do muons at rest. This means that muons produced in the upper atmosphere at a height $H$ (typically tens of kilometers) will have a substantial probability of reaching the ground before decaying provided they are moving fast enough so that $\gamma\, c\,\tau > H$.

Muons produced in the upper atmosphere and reaching the earth before decaying also illustrate Lorentz contraction — if one considers what's happening from the muon's perspective. Imagine riding along with a muon produced in an atmospheric shower. Or, as one says more formally, consider the co-moving reference frame of the muon. In this frame, the muon is at rest but the Earth is racing toward the muon at nearly the speed of light. The muon will decay, on average, in two microseconds. But the thickness of the atmosphere, in this frame, is reduced by Lorentz contraction. Therefore, the surface of the Earth will reach the muon before it (typically) decays if $(H/\gamma)/c < \tau$. This is the same condition obtained above by considering physics in the frame of an observer on the ground. This example nicely illustrates the second relativity postulate: because the laws of physics are frame independent, one may use whatever frame is most convenient to analyze some particular phenomena. In this example, whether one regards time dilation or Lorentz contraction as being responsible for allowing muons produced in the upper atmosphere to reach the ground depends on the frame one chooses to use. However, both approaches agree with the observed fact that high

[5]However, this is only part of the story regarding relative clock rates in GPS satellites. The difference in gravitational potential between the satellites' orbits and the Earth's surface also produces a change in clock rates due to a general relativistic effect known as *gravitational redshift*. This effect goes in the opposite direction (speeding orbiting clocks relative to Earth-bound ones) and is larger in magnitude, 45 microseconds per day. So GPS clocks actually run faster than clocks on the ground by $45 - 7 = 38$ microseconds per day.

energy muons *can* reach the ground from the upper atmosphere.

## 2.5 Superluminal motion?

The expressions for time dilation (2.3.1) and length contraction (2.3.4) make sense (*i.e.*, yield real, not imaginary, results) only for $u \leq c$. As we will discuss more explicitly in Chapter 3, a basic feature of Special Relativity is that nothing (no signal, no particle, no information) can travel *faster* than the speed of light $c$. Thus there was considerable excitement in autumn 2011 when the OPERA neutrino detector at the Gran Sasso Laboratory in Italy (see link below) reported that neutrinos (which are thought to have a very small but nonzero mass) had seemed to travel to the detector from CERN in Geneva, Switzerland at a speed that exceeded $c$. (see, *e.g.*, this Science Daily story.) Such a measurement requires the realization of the typical (introductory physics) picture of a reference frame densely populated by synchronized identical clocks. In particular, the clocks in Geneva and in Gran Sasso need to be synchronized with a precision of better than 50 nanoseconds. As the discussion above suggests, this is a daunting challenge indeed, but possible using the GPS system.

Hence the OPERA result fundamentally conflicted with Special Relativity. Either our postulates, or the experimental measurement, had to be in error. All indications are that the original measurement was in error. The OPERA team reported in early 2012 that this original measurement likely suffered from a synchronization error caused by a loose connection in a cable relaying the GPS signals to the experiment's clocks. Subsequent results reported by the companion experiment ICARUS confirmed that the speed of neutrinos is indeed bounded above by the speed of light.

## 2.6 Further resources

*Michelson-Morley experiment*, Wikipedia

*GPS and Relativity*, R. Pogge

*Relativity in the Global Positioning System*, N. Ashby

*GPS*, Wikipedia

*Introduction to Cosmic Rays*, VVC SLAC

*Do-it-yourself Cosmic Ray Muon Detector*

*Cosmic ray*, Wikipedia

*OPERA*, Wikipedia

*ICARUS*, Wikipedia

## 2.7 Example Problems

### Kogut 2-1

This is a typical "Star Trek" style multi-frame problem. We have 2 frames of reference: the Earth (frame $S$) and the spaceship (frame $S'$) moving with velocity $v = 0.6\,c$ with respect to each other. Everything is synchronized at event 1 ($t_1 = t'_1 = 0$, $x_1 = x'_1 = 0$), as the (small) spaceship passes the earth. Event 2 marks the emission of a pulse of light from the earth towards the spaceship at

$t_2 = 10$ minutes $= 600$ seconds. Event 3 marks the detection of the light pulse by our friends on the spaceship. We want to use the concept of *proper* time, which is the time measured by a clock for events occuring *at* the location of the clock, *i.e.*, for events occurring at the *same* point in the (rest) frame of the clock.

(a) Q: Is the time interval between events 1 and 2 a proper time interval in the spaceship frame? In the Earth frame?

A: Events 1 and 2 occur at the same point in frame $S$ (*i.e.*, on the Earth), but not at the same point in frame $S'$ on the spaceship. Hence the time interval between events 1 and 2 is a *proper* time interval on the Earth, but not on the spaceship.

(b) Q: Is the time interval between events 2 and 3 a proper time interval in the spaceship frame? In the Earth frame?

A: Events 2 and 3 occur at different points in both frames. Hence the time interval between events 2 and 3 is *not* a *proper* time interval in *either* frame.

(c) Q: Is the time interval between events 1 and 3 a proper time interval in the spaceship frame? In the Earth frame?

A: Events 1 and 3 occur at the same point on the spaceship (frame $S'$) (*e.g.*, at the center of the small ship), but not at the same point on the Earth. Hence the time interval between events 1 and 3 is a *proper* time interval on the spaceship but not on the Earth.

(d) Q: What is the time of event 2 as measured on the spaceship?

A: We want to determine the time of the light emission in frame $S'$, $t'_2$. This time interval (from event 1) is not a proper time interval in the $S'$ frame and we must account for time dilation (with respect to the proper time interval in frame $S$). We have

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-(0.6)^2}} = 1.25\,, \tag{2.7.1a}$$

$$t'_2 = \gamma t_2 = 1.25(600\,\text{seconds}) = 750\,\text{seconds} = 12.5\,\text{minutes}. \tag{2.7.1b}$$

(e) Q: According to the spaceship, how far away is the Earth when the light pulse is emitted?

A: We want to determine the distance to the earth from the spaceship at the time of the emission in frame $S'$. This is just the distance traveled by the earth as viewed by spaceship at velocity $v$ during the time interval determined in part (d). We have

$$l'_2 = vt'_2 = (0.6) \times 3.0 \times 10^8\,\text{m/s} \times 750\,\text{s} = 1.35 \times 10^{11}\,\text{m}\,. \tag{2.7.2}$$

(f) Q: From your answers in parts (d) and (e), what does the spaceship clock read when the light pulse arrives?

A: So we want to determine the time of event 3 in frame $S'$ on the spaceship. We have determined both the time of emission and the distance to earth at emission in this frame, and we know that light travels at speed $c$ in *all* frames (plus the fact that the spaceship is not moving in its own rest frame, $S'$). Thus we need only calculate the time for light to travel from the earth to the spaceship and add it to the time of the emission (all in the $S'$ frame)

$$t'_3 = t'_2 + l'_2/c = 750\,\text{s} + 1.35 \times 10^{11}\,\text{m}/(3.0 \times 10^8\,\text{m/s}) = (750+450)\,\text{s} = 1200\,\text{s} = 20\,\text{minutes}\,. \tag{2.7.3}$$

(g) Q: Find the time of event 3 according to the Earth's clock by analyzing everything from the Earth's perspective.

A: We return to frame $S$ and find the time of event 3. On the Earth we have that, at the time of event 2, the distance to the rocket ship is

$$l_2 = vt_2 = (0.6) \times 3.0 \times 10^8\,\mathrm{m/s} \times 600\,\mathrm{s} = 1.08 \times 10^{11}\,\mathrm{m}\,. \tag{2.7.4}$$

So the time interval (in this frame) between events 2 and 3 is given by (note that the spaceship continues to move in the $S$ frame)

$$c(t_3 - t_2) = l_2 + v(t_3 - t_2) \to t_3 - t_2 = \frac{l_2}{c - v} = \frac{1.08 \times 10^{11}\,\mathrm{m}}{1.2 \times 10^8\,\mathrm{m/s}} = 900\,\mathrm{s}\,. \tag{2.7.5}$$

So finally we obtain

$$t_3 = t_2 + (t_3 - t_2) = (600 + 900)\,\mathrm{s} = 1500\,\mathrm{s} = 25\,\mathrm{minutes}\,. \tag{2.7.6}$$

(h) Q: Are your answers to parts (f) and (g) consistent with your conclusions from parts (a), (b) and (c)?

A: We learned in (c) that the time interval between events 1 and 3 is a proper time interval in frame $S'$, but not $S$, where the interval is dilated. We can check this point via

$$t'_3\gamma = (20\,\mathrm{minutes})(1.25) = 25\,\mathrm{minutes} = t_3\,, \tag{2.7.7}$$

which checks with our result in (g).

**Kogut 2-2**

Here we consider two rockets, $A$ and $B$, to define two reference frames, and let the rockets have identical proper lengths (*i.e.*, lengths in their respective rest frames) of 100 m. The two rockets pass each other moving in opposite directions and we consider two events defined in frame $A$ by the passing of the front of rocket $B$. Event 1 is when the front of $B$ passes the front end of $A$ and event 2 is when the front of $B$ passes the back end of $A$. The time interval in frame $A$ between the two events is $1.5 \times 10^{-6}$ s.

(a) Q: What is the relative velocity of the rockets?

A: Since we know the length of rocket $A$ (in its rest frame) and the time interval for the front of rocket $B$ to travel the length of $A$, all measured in frame $A$, we can find the relative velocity from

$$v_{\mathrm{rel}} = \frac{100\,\mathrm{m}}{1.5 \times 10^{-6}\,\mathrm{s}} = 6.67 \times 10^7\,\mathrm{m/s}\,. \tag{2.7.8}$$

(b) Q: According to the clocks on rocket B, how long does the front end of A take to pass the entire length of rocket B?

A: The passing of rocket $A$ viewed from $B$ will be exactly equivalent to the passing of $B$ viewed in $A$ (by symmetry, the relative speed is the same of both frames). Hence the time interval for these 2 events in $B$ is again $1.5 \times 10^{-6}$ s,

$$t_B = \frac{100\,\mathrm{m}}{v_{\mathrm{rel}}} = \frac{100\,\mathrm{m}}{6.67 \times 10^7\,\mathrm{m/s}} = 1.5 \times 10^{-6}\,\mathrm{s} = t_A\,. \tag{2.7.9}$$

(c) Q: According to the clocks on rocket B, how much time passes between the time when the front end of A passes the front end of B and the time when the rear end of A passes the front end of B? Does this time interval agree with your answer to (b)? Should it?

A: Now consider a third event in $B$ (event 1 was the front of $A$ passing the front of $B$ and event 2 was the front of $A$ passing the back of $B$) defined by when the back of $A$ passes the front of $B$. The important point now is that in frame $B$ the length of rocket $A$ is (relativistically) contracted. We have

$$\gamma = \frac{1}{\sqrt{1-(v_{\rm rel}/c)^2}} = 1.0257\,, \tag{2.7.10a}$$

$$L_{A{\rm in}B} = \frac{100\,{\rm m}}{\gamma} = 97.50\,{\rm m}\,, \tag{2.7.10b}$$

$$t_C = \frac{L_{A{\rm in}B}}{v_{\rm rel}} = \frac{t_B}{\gamma} = 1.46\times 10^{-6}\,{\rm seconds}\,. \tag{2.7.10c}$$

This is a time interval defined by a *new* pair of events, not equivalent to the previous pair, and the new time interval does not and should not agree with the result in (b).