Chapter 5

QM and Angular Momentum

5.1 Angular Momentum Operators

In your Introductory Quantum Mechanics (QM) course you learned about the basic properties of low spin systems. Here we want to review that knowledge and indicate in more detail how it arises from the basic principles of QM, *i.e.*, that we work with operators, in particular with Hermitian operators, and those operators obey simple commutation relations. To illustrate these ideas let us review the formalism of eigenstates of definite (total) angular momentum. The following analysis applies as well to the spin operator S and, except in one detail noted at the end, to orbital angular momentum L. (Of course, these operators are related via $J = L + S$.) If you have not seen this sort of analysis before, consider it as an introduction to the power of symmetries as expressed in terms of quantum mechanical operators and states. The analysis involves a large number of steps, but, in a very real sense, each of those steps is quite small.

Also be aware of the larger picture, as discussed in Chapter 10, that we are actually discussing the properties of *representations* of the rotation group, $SO(3)$ (for integer angular momentum) and $SU(2)$ (for half-integer angular momentum). The different states in the representation are what you see when you perform rotations on the reference frame. The apparently different states are, in some sense, the same given the underlying symmetry, *i.e.*, we are simply labeling them differently as we change (rotate) the directions of the "axes". Note, in particular, that the possible states of the system must always appear in *complete* representations of the underlying symmetries. So our understanding of symmetries and the associated representations will provide tools to organize our description of physical systems, *e.g.*, the particles of the Standard Model.

We want to work with the (hopefully) familiar (3-vector) total angular momentum operator $\vec{J}$ with three components J_1, J_2 and J_3 (or J_x, J_y and J_z, see, for example, Chapters 7 and 11 in McIntyre). We take all 3 to be Hermitian operators ($J_k^\dagger = J_k$, where $^\dagger = ^{*T}$, *i.e.*, take the complex conjugate and the transpose) and thus to have *real* eigenvalues. An essential feature of the operator nature of J (and of QM) is the fact that these three operators obey the nontrivial commutation relation (*i.e.*, the *algebra* corresponding to $SO(3)$ and $SU(2)$))

$$[J_k, J_l] \equiv J_k J_l - J_l J_k = i\hbar\epsilon_{klm} J_m \quad [k, l, m = 1, 2, 3]\,, \tag{5.1.1}$$

where ϵ_{klm} is the *unique* $3 \times 3 \times 3$ anti-symmetric tensor.

<u>ASIDE</u> The algebra serves to completely define the properties of the group elements, the transformations, near the identity - no change - operator. However, there may still be ambiguity about the properties of the elements "far" from the identity operator, and this point is related to the difference between $SO(3)$ and $SU(2)$ - see Chapter 10.

Next we define J^2, the total angular momentum squared operator,

$$J^2 = J_1^2 + J_2^2 + J_3^3 \,, \tag{5.1.2}$$

which is also a Hermitian operator ($J^{2\dagger} = J^2$) again with real eigenvalues. Actually, since the operators on the RHS of Eq. (5.1.2) are all the squares of Hermitian operators, the corresponding eigenvalues are all positive semi-definite (≥ 0), *i.e.*, the squares of *real* numbers. By the same token the related operator expression

$$J^2 - J_3^2 = J_1^2 + J_2^2 \tag{5.1.3}$$

tells us that the eigenvalues of $J^2 - J_3^2$ are also positive semi-definite, or that the eigenvalues of J^2 are greater than or equal to the eigenvalues of J_3^2. (Actually, as we will see shortly, equality will only occur for the special case of *zero* total angular momentum.)

The next essential fact, following from Eq. (5.1.1), is that J^2 commutes with the individual J_k. For example, we have[1]

$$\begin{aligned}
[J^2, J_3] &= [J_1^2, J_3] + [J_2^2, J_3] + [J_3^2, J_3] \\
&= [J_1, J_3]J_1 + J_1[J_1, J_3] + [J_2, J_3]J_2 + J_2[J_2, j_3] + 0 \\
&= -i\hbar J_2 J_1 - i\hbar J_1 J_2 + i\hbar J_1 J_2 + i\hbar J_2 J_1 = 0 \,.
\end{aligned} \tag{5.1.4}$$

Clearly a similar result holds for $[J^2, J_1]$ and $[J^2, J_2]$. This is simply a specific example of the fact that the length of a vector is *un*changed by a rotation, *i.e.*, the usual (3-D) scalar product is a scalar under rotations.

Finally we make the conventional *choice* that our basis states be the simultaneousness eigenstates of of J^2 and J_3 (possible because they commute), $|j, m\rangle$ (where this is the state-vector "ket" familiar from QM and the corresponding "bra", $\langle j, m|$, is the Hermitian conjugate), with eigenvalues, j, m, and defined by

$$J^2|j, m\rangle \equiv (J_1^2 + J_2^2 + J_3^2)|j, m\rangle = j(j+1)\hbar^2|j, m\rangle, \;\; J_3|j, m\rangle = m\hbar|j, m\rangle \,, \tag{5.1.5}$$

where we take these eigenstates to be normalized

$$\langle j.m|j, m\rangle = 1.0 \,. \tag{5.1.6}$$

We are encouraged to think of j as labeling the total angular momentum of the state *independent* of any choice of reference frame, while m labels the component of the angular momentum along the 3-axis in a *specific* choice of reference frame. As we will see in detail below, when we rotate the reference frame (or the state), the value of m changes, but j does not. Thus the states corresponding to a given j value and the possible m values, $-j \leq m \leq j$ comprise a *representation* of the rotation group, *i.e.*, these states are transformed into each other in a specific fashion by the rotations. In Group Theory language (see Chapter 10) the operator J^2 is formally labeled a Casimir operator. It is not an element of the algebra or the group, but does commute with the generators (and thus the

[1] The *signs* in Eq. (5.1.4) follow from the definition of ϵ_{klm}, *i.e.*, $\epsilon_{132} = -1$ while $\epsilon_{231} = +1$.

group elements) and, as noted, its eigenvalues serve to label the specific representation of the group, while m labels the specific element of the $2j+1$ elements in the representation.

These results are presumably familiar from your QM course, including the fact that the allowed values of j are either integer or half-integer and the allowed values of m are the $2j+1$ values in the range $m = -j, -j+1 ... j-1, j$. Here we will see how these results follow from the basic properties of the operators noted above (and the following will serve as an introduction if this was not covered in your 225 class). To that end let us for now define the eigenvalues instead by

$$J^2|j,m\rangle \equiv N^2(j)\hbar^2|j,m\rangle,\ \ J_3|j,m\rangle = m\hbar|j,m\rangle\,, \tag{5.1.7}$$

where we will *derive* below the specific form of $N^2(j)$ and the constraints on the possible values of m.

From the standpoint of the underlying Group theory, we label the J_k as the *generators* of the unitary rotation group ($SO(3)$ and $SU(2)$) in the sense that they "generate" an infinitesimal rotation. Since we want a rotation through a finite angle to be a unitary transformation (*i.e.*, it should conserve probability), the generators are necessarily Hermitian operators, $J_k^\dagger = J_k$ (so that the group elements arising from their exponentiation are Unitary). As already mentioned in Chapter 1 and described in some detail in Chapter 10, the finite rotation corresponds to exponentiating these generators times a continuous parameter (i times the rotation angle over $\hbar$). [2] For example, $e^{iJ_3\theta/\hbar}$ corresponds to a rotation around the 3-axis by an angle θ and is a member of the rotation group. Note that, since

$$\left(e^{iJ_3\theta/\hbar}\right)^\dagger = e^{-iJ_3^\dagger\theta/\hbar} = e^{-iJ_3\theta/\hbar} = \left(e^{iJ_3\theta/\hbar}\right)^{-1}\,, \tag{5.1.8}$$

the operator $e^{iJ_3\theta/\hbar}$ is Unitary (*i.e.*, the Hermitian conjugate is the inverse and this transformation conserves the norm of the state in Eq. 5.1.6) if (and only if) J_3 is Hermitian.

At this point we can also demonstrate that the eigenstates in Eq. (5.1.5) are orthogonal as desired (for different m values). We have (recall that $1 = e^{i0} = e^{-iJ_3\theta/\hbar}e^{+iJ_3\theta/\hbar}$)

$$\langle j,m'|j,m\rangle = \langle j,m'|e^{-iJ_3\theta/\hbar}e^{+iJ_3\theta/\hbar}|j,m\rangle = e^{i(m-m')\theta/\hbar}\langle j,m'|j,m\rangle\,. \tag{5.1.9}$$

There are two ways to satisfy this equation. *Either* the exponential factor is unity because $m = m'$ and these are really both the *same* state, *or* they are different states, the exponential factor is *not* unity, and the solution of this equation is that the matrix element vanishes,

$$\langle j,m'|j,m\rangle = 0, \quad m' \neq m\,. \tag{5.1.10}$$

To proceed we want to make use of the two remaining generators (J_1 and J_2) that do not define our basis eigenstates, and will *change* the states when they operate. This is the part of the analysis that may not be familiar (it appears in Chapter 11 of McIntyre's QM text), but it is illustrative of how we can prove useful results using only the properties of the operators. In particular, we can define the so-called "ladder" (or raising and lowering) operators by

$$J_\pm \equiv J_1 \pm iJ_2\,. \tag{5.1.11}$$

[2] As a familiar example of this exponentiation recall that the form $e^{i\vec{p}\cdot\vec{x}}$ leads to the Taylor series expansion when $\vec{p}$ is replaced by the momentum operator $\hat{p} = -i\hbar\vec{\nabla}$.

Since the J_k are Hermitian, it follows that

$$J_{\pm}^{\dagger} = J_1^{\dagger} \mp iJ_2^{\dagger} = J_1 \mp iJ_2 = J_{\mp} \,. \tag{5.1.12}$$

Using Eqs. (5.1.1) and (5.1.11), and some straightforward algebra, we can evaluate the new commutators

$$\begin{aligned}[J_+, J_-] &= [J_1, J_1] + [J_1, -iJ_2] + [iJ_2, J_1] + [iJ_2, -iJ_2] \\ &= 0 - i(iJ_3\hbar) + i(-iJ_3\hbar) + 0 = 2\hbar J_3\end{aligned} \tag{5.1.13}$$

and

$$\begin{aligned}[J_3, J_{\pm}] &= [J_3, J_1] + [J_3, \pm iJ_2] = i\hbar J_2 + (\pm i)(-i\hbar J_1) \\ &= \hbar(iJ_2 \pm J_1) = \pm\hbar J_{\pm} \,.\end{aligned} \tag{5.1.14}$$

With a little more algebra we can demonstrate (and you should try this at home) that, since the total spin operator J^2 commutes with each of the components (recall Eq. (5.1.4)), it also commutes with the ladder operators,

$$[J^2, J_k] = 0 \Rightarrow [J^2, J_{\pm}] = 0 \,. \tag{5.1.15}$$

Since we eventually want to be able to evaluate the result of operating on the eigenstates with the ladder operators, we want to first evaluate the products in terms of the eigen-operators J^2 and J_3. This may seem unmotivated at first, but the usefulness of this step will be clear shortly. By explicit calculation it follows that

$$J_+J_- = (J_1 + iJ_2)(J_1 - iJ_2) = J_1^2 + J_2^2 - i[J_1, J_2] = J^2 - J_3^2 + \hbar J_3 \,, \tag{5.1.16}$$

and

$$J_-J_+ = (J_1 - iJ_2)(J_1 + iJ_2) = J_1^2 + J_2^2 + i[J_1, J_2] = J^2 - J_3^2 - \hbar J_3 \,, \tag{5.1.17}$$

Note that the difference between these two equations is $2\hbar J_3$ as expected from Eq. (5.1.13). The content of these relations is that the ladder operators move us around in a given representation of $SO(3)$ or $SU(2)$ (hence the label), but do *not* change the representation, *i.e.*, do not change the eigenvalue of J^2 (recall Eq. (5.1.2)). To see this explicitly we first note the operator relation (recall Eq. (5.1.14))

$$J_3J_{\pm} = J_{\pm}J_3 + [J_3, J_{\pm}] = J_{\pm}J_3 \pm \hbar J_{\pm} \,. \tag{5.1.18}$$

Thus, when we apply this operator to an eigenstate, we obtain

$$J_3J_{\pm}|j,m\rangle = J_{\pm}J_3|j,m\rangle + [J_3, J_{\pm}]|j,m\rangle = m\hbar J_{\pm}|j,m\rangle \pm \hbar J_{\pm}|j,m\rangle = (m \pm 1)\hbar J_{\pm}|j,m\rangle \,, \tag{5.1.19}$$

clearly indicating that the operator $J_{\pm}$ raises/lowers the J_3 eigenvalue by one (explaining the "ladder" label),

$$J_{\pm}|j,m\rangle \propto |j, m \pm 1\rangle \,, \tag{5.1.20}$$

or, including an explicit coefficient,

$$J_{\pm}|j,m\rangle \equiv A_{\pm}(j, m \pm 1)|j, m \pm 1\rangle \,. \tag{5.1.21}$$

We will determine the coefficient $A_{\pm}(j, m \pm 1)$ shortly. From Eqs. (5.1.15) and (5.1.7) we have

$$J^2J_{\pm}|j,m\rangle = J_{\pm}J^2|j,m\rangle = N^2(j)\hbar^2J_{\pm}|j,m\rangle = N^2(j)\hbar^2A_{\pm}(j, m \pm 1)|j, m \pm 1\rangle \,, \tag{5.1.22}$$

confirming that $J_\pm$ does *not* change the J^2 eigenvalue.

So, as already noted, these last equations tell us to interpret the operator $J_\pm$ as stepping us through the 1-D representation labeled by total angular momentum j.

To determine the coefficient $A_\pm(j,m)$ we perform the following manipulations, which follow from the definitions above. First, from the definition of the coefficient in Eq. (5.1.21) and the unit normalization of the eigenstates, we have

$$\langle j,m|J_+J_-|j,m\rangle = \langle j,m|J_-^\dagger J_-|j,m\rangle = |A_-(j,m-1)|^2\langle j,m-1|j,m-1\rangle = |A_-(j,m-1)|^2\,, \quad (5.1.23)$$

and

$$\langle j,m|J_-J_+|j,m\rangle = \langle j,m|J_+^\dagger J_+|j,m\rangle = |A_+(j,m+1)|^2\langle j,m+1|j,m+1\rangle = |A_+(j,m+1)|^2\,. \quad (5.1.24)$$

Now we can use Eqs. (5.1.16) and (5.1.17) to explicitly evaluate these matrix elements and find

$$\begin{aligned}\langle j,m|J_+J_-|j,m\rangle &= \langle j,m|J^2 - J_3^2 + \hbar J_3\ |j,m\rangle = N^2(j)\hbar^2 - m^2\hbar^2 + m\hbar^2 \\ &= \hbar^2(N^2(j) - m^2 + m) = |A_-(j,m-1)|^2\,,\end{aligned} \quad (5.1.25)$$

and

$$\begin{aligned}\langle j,m|J_-J_+|j,m\rangle &= \langle j,m|J^2 - J_3^2 - \hbar J_3\ |j,m\rangle = N^2(j)\hbar^2 - m^2\hbar^2 - m\hbar^2 \\ &= \hbar^2(N^2(j) - m^2 - m) = |A_+(j,m+1)|^2\,.\end{aligned} \quad (5.1.26)$$

We can choose the phases of the eigenstates so that both coefficients are positive, real (without any impact on the quantum physics) and define the coefficients in the operation of the ladder operators to be (keeping the still to be evaluated parameter $N^2(j)$)

$$A_\pm(j,m\pm 1) = \hbar\sqrt{N^2(j) - m^2 \mp m}\,, \quad (5.1.27)$$

Now we return to the discussion surrounding Eq. (5.1.3). We have seen that the ladder operators raise and lower the eigenvalue m without changing the eigenvalue $N^2(j)$. However, Eq. (5.1.3) tells us that $N^2(j) - m^2 \geq 0$ for all allowed values of $N^2(j)$ (*i.e.*, allowed values of j) and m. These two results can *both* be true if and only if the raising and lowering process truncates, since otherwise we will eventually obtain an m^2 value greater than any (fixed) $N^2(j)$ value. Thus there must be maximum and minimum values of m, m_{max} and m_{min}, such that

$$J_+|j,m_{max}\rangle = 0,\ \ J_-|j,m_{min}\rangle = 0\,. \quad (5.1.28)$$

These results can be rewritten as the statements that

$$A_+(j,m_{max}+1) = 0,\ \ A_-(j,m_{min}-1) = 0\,. \quad (5.1.29)$$

Combining with Eq. (5.1.27) we have

$$N^2(j) - m_{max}^2 - m_{max} = 0\,, \quad N^2(j) - m_{min}^2 + m_{min} = 0\,. \quad (5.1.30)$$

Since the raising and lowering is *always* by a unit step (of $\hbar$, this is QM after all), we know that $m_{max} - m_{min} =$ an integer, which we label n. Using the difference of the two results in Eq. (5.1.30) to eliminate $N^2(j)$ and substituting $m_{max} = m_{min} + n$, we find

$$m_{min} = -\frac{n}{2}, \quad m_{max} = \frac{n}{2}\,. \quad (5.1.31)$$

Returning to Eq. (5.1.30) we find also that

$$N^2(j) = \frac{n}{2}\left(\frac{n}{2}+1\right) . \tag{5.1.32}$$

So now we can make the standard identification for the eigenvalue j,

$$\begin{aligned} j \equiv \frac{n}{2} &\Rightarrow -j \leq m \leq j \\ N^2(j) &= j(j+1) . \end{aligned} \tag{5.1.33}$$

Substituting in Eq. (5.1.27) we have

$$A_{\pm}(j, m \pm 1) = \hbar\sqrt{j(j+1) - m(m \pm 1)} = \hbar\sqrt{(j \mp m)(j \pm m + 1)} , \tag{5.1.34}$$

These expressions for the coefficients explicitly verify the truncation results of Eq. (5.1.29), *i.e.*, $A_+(j, m_{max} + 1 = j + 1) = 0$ and $A_-(j, m_{min} - 1 = -j - 1) = 0$.

Since n is an integer, there are two possibilities corresponding to odd or even n. If n is an odd integer, then the "total angular momentum" eigenvalue $j = n/2$ is half-integer, while if n is an even integer, then $j = n/2$ is integer. In either case the number of distinct values of m is the familiar $2j + 1$ corresponding to the values $m = -j$ to $m = j$.[3] Since the same arithmetic applies to the spin of an individual particle, we see that both integer spin particles, *i.e.*, bosons, and half-integer spin particles, *i.e.*, fermions, are possible. On the other hand, *orbital* angular momentum arises from the $\vec{L} = (\vec{r} \times \vec{p})$ operator (with a clear classical connection) and assumes *only* integer values.

To recap, we have used only the facts that the 3 components of the total angular momentum operator are Hermitian operators (and so have *real* eigenvalues) and that these operators satisfy the commutation relation of Eq. (5.1.1), to derive that the possible eigenvalues of J^2 and J_3 are specified by a single parameter j. Further this parameter is either half-integer (1/2, 3/2, ...) or integer (0, 1, 2 ...). The eigenvalue of J^2 is given by $j(j+1)\hbar^2$ corresponding to the $2j+1$ m values in the range $-j \leq m \leq j$ with J_3 eigenvalues $m\hbar$ in the range $-j\hbar$ to $+j\hbar$. Note that only for the "trivial" case $j = 0$ is the eigenvalue of J^2 equal to the eigenvalue of J_3 (both are 0). For j greater than 0 we have $j(j+1)$ greater than m^2, as we should expect for QMical systems where the other components besides J_3, which do *not* commute with J_3, will exhibit nonzero, if indeterminate, values in an eigenstate of J_3.

5.2 Spin 1/2 in Vector/Matrix Notation

Here we will study a spin 1/2 system as an example of a 2-state system as you studied in your QM class, which will provide a (Unitary) fundamental representation of the group $SU(2)$. The resulting matrices are Unitary ($T^{-1} = T^{\dagger}$), unlike the Orthogonal matrices ($R^{-1} = R^T$) we saw in our discussion of the rotation group, $SO(3)$, back in Chapter 1, and, as the fundamental representation

[3] The underlying physics point here is the *quantization* of angular momentum in units of $\hbar$, *i.e.*, changes in angular momentum (spin) can only occur in *integer* steps with magnitude $\hbar$. We don't notice this in ordinary life because a typical (classical) angular momentum has magnitude of order kg m^2/s or Joule * second. In units of $\hbar$ this is of order 10^{34} $\hbar$ and changes in the angular momentum of magnitude $\hbar$ will appear to be continuous changes.

of $SU(2)$, will provide a fully faithful description of the group elements. We can represent the corresponding states as (in a variety of notations)

$$|\frac{1}{2},\frac{1}{2}\rangle = |\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}\,, \quad |\frac{1}{2},-\frac{1}{2}\rangle = |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}\,, \tag{5.2.1}$$

where the up and down arrow notation is specific to the spin 1/2 interpretation and corresponds to the component of spin along the 3 axis. In this final basis we can use the so-called Pauli matrices, which you likely learned about in Physics 225 or 227 as a basis set of 2×2 matrices. They are defined by

$$\sigma_1 = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \sigma_2 = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}, \sigma_3 = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \tag{5.2.2}$$

and are Hermitian. Also they obey the commutation relation (note the factors of 1/2)

$$\left[\frac{\sigma_j}{2},\frac{\sigma_k}{2}\right] = i\epsilon_{jkl}\frac{\sigma_l}{2}\,. \tag{5.2.3}$$

Thus we can define the (Hermitian) representation of the spin operator for our spin 1/2 system as (recall Eqs. (1.5.4) and (1.5.5))

$$S_k = \hbar\frac{\sigma_k}{2}\,, \quad [S_j, S_k] = i\epsilon_{jkl}\hbar S_l\,. \tag{5.2.4}$$

Note, in particular, that

$$S_3 = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\,, \tag{5.2.5}$$

as is appropriate for basis states that are eigenstates of S_3 with eigenvalues $\pm\hbar/2$. It follows that

$$\begin{aligned}
S_3|\frac{1}{2},\frac{1}{2}\rangle &= \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}|\frac{1}{2},\frac{1}{2}\rangle\,,\\
S_3|\frac{1}{2},-\frac{1}{2}\rangle &= \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix} = -\frac{\hbar}{2}|\frac{1}{2},-\frac{1}{2}\rangle\,,\\
S^2|\frac{1}{2},\pm\frac{1}{2}\rangle &= \frac{3}{4}\hbar^2|\frac{1}{2},\pm\frac{1}{2}\rangle\,.
\end{aligned} \tag{5.2.6}$$

Now consider the raising and lowering (ladder) operators. The representations for the raising and lowering operators are

$$S_+ = S_1 + iS_2 = \hbar\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, S_- = S_1 - iS_2 = \hbar\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\,, \tag{5.2.7}$$

which clearly perform the following transformations

$$\begin{aligned}
S_+|\uparrow\rangle &= \hbar\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 0\,, \quad S_-|\uparrow\rangle = \hbar\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \hbar\begin{pmatrix}0\\1\end{pmatrix} = \hbar|\downarrow\rangle\,,\\
S_+|\downarrow\rangle &= \hbar\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \hbar\begin{pmatrix}1\\0\end{pmatrix} = \hbar|\uparrow\rangle\,, \quad S_-|\downarrow\rangle = \hbar\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 0\,.
\end{aligned} \tag{5.2.8}$$

This checks with Eq. (5.1.34) that yields for the spin 1/2 coefficients

$$
\begin{aligned}
A_+\left(\frac{1}{2},\frac{3}{2}\right) &= \hbar\sqrt{\frac{1}{2}\left(\frac{3}{2}\right)-\frac{1}{2}\left(\frac{3}{2}\right)} = 0\,, \\
A_-\left(\frac{1}{2},-\frac{1}{2}\right) &= \hbar\sqrt{\frac{1}{2}\left(\frac{3}{2}\right)-\frac{1}{2}\left(-\frac{1}{2}\right)} = \hbar\,, \\
A_+\left(\frac{1}{2},\frac{1}{2}\right) &= \hbar\sqrt{\frac{1}{2}\left(\frac{3}{2}\right)-\left(-\frac{1}{2}\right)\frac{1}{2}} = \hbar\,, \\
A_-\left(\frac{1}{2},-\frac{3}{2}\right) &= \hbar\sqrt{\frac{1}{2}\left(\frac{3}{2}\right)-\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)} = 0\,.
\end{aligned} \tag{5.2.9}
$$

(Be certain to verify that you understand how these results arise.) The raising and lowering operators are simply flipping the spin component along the 3 axis, or producing zero if this spin component cannot be raised or lowered further.

Next we look at finite transformations in the underlying (Unitary Group) $SU(2)$. We proceed much as we did when we studied the group of rotations in 3-D in Chapter 1 (*i.e.*, the group SO(3)). To proceed it is useful to have the analogue of Eq. (1.5.8) for the Pauli matrices,

$$
\sigma_k^{2n} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \mathbf{1}, \quad \sigma_k^{2n+1} = \sigma_k\,. \tag{5.2.10}
$$

<u>ASIDE</u> This result allows us to easily verify the last result in Eq. (5.2.6),

$$
S_k^2 = \frac{\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad S^2 = S_1^2 + S_2^2 + S_3^2 = \frac{3\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\,. \tag{5.2.11}
$$

Proceeding as we did in Chapter 1, we find results similar to, but simpler than Eqs. (1.5.9) and (1.5.10) (see also Chapter 10). For the finite transformation generated by S_3 or σ_3 we have

$$
\begin{aligned}
T_3(\alpha) \equiv e^{i\alpha S_3/\hbar} &= e^{i\sigma_3\alpha/2} = \mathbf{1} + \sum_{n=1}^{\infty}\frac{(i\alpha\sigma_3)^n}{2^n n!} \\
&= \mathbf{1}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha)^{2n}}{2^{2n}(2n)!} + i\sigma_3\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha)^{2n+1}}{2^{2n+1}(2n+1)!}\,.
\end{aligned} \tag{5.2.12}
$$

Using what we know about the sinusoidal functions, especially Eqs. (1.3.2) and (1.3.3), we can write this transformation in the compact form

$$
T_3(\alpha) = \begin{pmatrix}\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2} & 0\\ 0 & \cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\end{pmatrix} = \begin{pmatrix}e^{i\alpha/2} & 0\\ 0 & e^{-i\alpha/2}\end{pmatrix} \tag{5.2.13}
$$

(see also Eq. (10.3.23)). This $SU(2)$ transformation generated by S_3 or σ_3 in the basis of eigenstates of S_3 is simply a change of phase by $\pm\alpha/2$ where the sign depends on the sign of the eigenvalue, and it is an Unitary matrix. In particular, the transformation is diagonal and does *not* involve any mixing of the two eigenstates. In fact, since our basis states are eigenstates of S_3, we could have evaluated this transformation directly,

$$
T_3(\alpha)|\frac{1}{2},m\rangle = e^{i\alpha S_3/\hbar}|\frac{1}{2},m\rangle = e^{i\alpha m}|\frac{1}{2},m\rangle\,, \tag{5.2.14}
$$

where $m = \pm 1/2$. This is the *same* result as the matrix form in Eq. (5.2.13).

5.3 Spin 1 in Vector/Matrix Notation

To further strengthen our understanding of spin systems, we want to consider a spin 1 system[4] where again we use the simultaneous eigenstates of S^2 and S_3 as the basis states. The possible eigenvalues of S_3 are now $+1, 0-1$. So our vectors will have 3 components similar to the discussion of ordinary rotations, $SO(3)$, in Chapter 1, but is important to remember that here we are talking about $SU(2)$ Unitary transformations of a 3 state QMical system (where phases can matter) and not ordinary location vectors in 3 dimensions. For this case the basis states are

$$|s.m\rangle = |1,1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad |1,0\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad |1,-1\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{5.3.1}$$

Next we want representations of S_1, S_2 and S_3 in this basis. As just noted, this is not the same basis as in Chapter 1.5 (*i.e.*, not ordinary location 3-vectors) and we do not expect the same representation as in Eq. (1.5.5). In particular, in the basis of its *own* eigenstates, S_3 should be represented by

$$S_3 = \hbar \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}, \tag{5.3.2}$$

so that

$$S_3|1,1\rangle = \hbar|1,1\rangle, \quad S_3|1,0\rangle = 0, \quad S_3|1,-1\rangle = -\hbar|1,-1\rangle. \tag{5.3.3}$$

A little algebra (or a good book) yields the corresponding representations of S_1 and S_2 to be

$$S_1 = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}, \quad S_2 = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}. \tag{5.3.4}$$

The reader is *strongly* encouraged to verify that this set of representations (matrices) are Hermitian and satisfy the required commutator,

$$[S_j, S_k] = i\epsilon_{jkl}\hbar S_l. \tag{5.3.5}$$

(The reader is also encouraged to compare with and understand the differences from the matrices in Eq. (1.5.5) for $SO(3)$.) With a little more algebra we find that

$$S_1^2 = \frac{\hbar^2}{2}\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix}, \quad S_2^2 = \frac{\hbar^2}{2}\begin{pmatrix}1&0&-1\\0&2&0\\-1&0&1\end{pmatrix}, \quad S_3^2 = \hbar^2\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}, \tag{5.3.6}$$

(it may be informative to compare these results with Eqs. (1.5.8), (1.6.1) and (1.6.2)) so that

$$S^2 = S_1^2 + S_2^2 + S_3^2 = 2\hbar^2\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \tag{5.3.7}$$

[4] In the individual particle language we will use shortly spin 1 particles are labeled vector particles in an obvious reference to the more familiar, but distinct, 3-component location vectors in 3-D space.

So we have confirmed that in a spin $s = 1$ system the shared eigenvalue of S^2 is $\hbar^2 s(s+1) = 2\hbar^2$.

Finally with this representation we can construct the corresponding raising and lowering operators,

$$S_+ = S_1 + iS_2 = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} ,$$
$$S_- = S_1 - iS_2 = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} = \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} . \tag{5.3.8}$$

These matrices perform the expected transformations,

$$S_+|1,1\rangle = 0\,, \quad S_+|1,0\rangle = \sqrt{2}\hbar|1,1\rangle\,, \quad S_+|1,-1\rangle = \sqrt{2}\hbar|1,0\rangle$$
$$S_-|1,1\rangle = \sqrt{2}\hbar|1,0\rangle\,, \quad S_-|1,0\rangle = \sqrt{2}\hbar|1,-1\rangle\,, \quad S_-|1,-1\rangle = 0\,. \tag{5.3.9}$$

Recall from Eq. (5.1.34) that $\sqrt{2}\hbar$ is the expected nonzero coefficient.

5.4 Examples

To practice using the techniques described above let us consider the form of the $SU(2)$ transformation generated by S_1 and S_2 when operating on the eigenstates of S_3 as the basis states. Since the structure of Eq. (5.2.3) still obtains, the procedure follows much as it did for S_3. For S_1 we have

$$T_1(\alpha) \equiv e^{i\alpha S_1/\hbar} = e^{i\sigma_1 \alpha/2} = \mathbf{1} + \sum_{n=1}^{\infty} \frac{(i\alpha\sigma_1)^n}{2^n n!}$$
$$= \mathbf{1} \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha)^{2n}}{2^{2n}(2n)!} + i\sigma_1 \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha)^{2n+1}}{2^{2n+1}(2n+1)!} . \tag{5.4.1}$$

Again we recognize the sums and rewrite in the compact form

$$T_1(\alpha) = \begin{pmatrix} \cos\frac{\alpha}{2} & i\sin\frac{\alpha}{2} \\ i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} . \tag{5.4.2}$$

Note that in this case the transformation is no longer diagonal, *i.e.*, this transformation does "rotate" the eigenstates of S_3 into one another (with some extra phases). This is intuitively reasonable as the transformation is about an axis orthogonal to the direction along which we quantized the spin component. Note also that

$$T_1(\pi) = \begin{pmatrix} \cos\frac{\pi}{2} & i\sin\frac{\pi}{2} \\ i\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} , \tag{5.4.3}$$

i.e., a "rotation" by π flips spin up to spin down and conversely (along with adding a phase of $\pi/2$), which is intuitively expected.

The story for S_2 is quite similar leading to

$$T_2(\alpha) \equiv e^{i\alpha S_2/\hbar} = e^{i\sigma_2 \alpha/2} = \mathbf{1} + \sum_{n=1}^{\infty} \frac{(i\alpha\sigma_2)^n}{2^n n!}$$
$$= \mathbf{1} \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha)^{2n}}{2^{2n}(2n)!} + i\sigma_2 \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha)^{2n+1}}{2^{2n+1}(2n+1)!} . \tag{5.4.4}$$

Again we recognize the sums and rewrite in the compact form

$$T_2(\alpha) = \begin{pmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} . \tag{5.4.5}$$

This really does look like an ordinary rotation (recall Eq. (1.5.10)) except for the factor of 1/2 in the arguments of the sines and cosines, which is the residue of spin 1/2. It is also important to remember that this 2-state system is in terms of the eigenstates of S_3 and not in terms of the two components of an ordinary location 2-vector.

Note that a rotation through 2π about any of the 3 axes, which you might naively expect to bring us back to to where we started (*i.e.*, yield the unit matrix) as happens in $SO(3)$, is given instead by the negative of the unit matrix (see Eqs. (5.4.2), (5.4.5) and (5.2.13))

$$T_1(2\pi) = T_2(2\pi) = T_3(2\pi) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{5.4.6}$$

This illustrates a fundamental difference between half-integer spin particles (fermions) and the more classically behaved integer-spin particles (bosons - see below). This difference plays an essential role in our understanding of how the fundamental particles behave.

As a final example let us evaluate finite $SU(2)$ transformations for vector particles, *i.e.*, in the basis of the previous section. The products of the (representations of the) generators satisfy slightly different relations than the more familiar form in Eq. (5.2.10) (also recall Eq. (5.3.6))

$$S_1^{2n} = \frac{\hbar^{2n}}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} , \quad S_1^{2n+1} = \hbar^{2n} S_1 , \tag{5.4.7}$$

$$S_2^{2n} = \frac{\hbar^{2n}}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} , \quad S_2^{2n+1} = \hbar^{2n} S_2 , \tag{5.4.8}$$

and

$$S_3^{2n} = \hbar^{2n} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad S_3^{2n+1} = \hbar^{2n} S_3 . \tag{5.4.9}$$

Thus, following a path similar to the one above, we find that, in the vector $SU(2)$ representation, the finite transformation generated by S_1 is given by

$$\begin{aligned} T_1(\alpha) \equiv e^{i\alpha S_1/\hbar} &= \mathbf{1} + \sum_{n=1}^{\infty} \frac{(i\alpha S_1/\hbar)^n}{n!} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha)^{2n}}{(2n)!} + iS_1 \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha)^{2n+1}}{(2n+1)!} , \end{aligned} \tag{5.4.10}$$

where in the second step we split apart the unit matrix to provide the $n = 0$ term in the first sum. As in the earlier analyses we can now rewrite the two sums as the cosine and sine functions. We have

$$T_1(\alpha) = \frac{1}{2} \begin{pmatrix} 1+\cos\alpha & i\sqrt{2}\sin\alpha & -1+\cos\alpha \\ i\sqrt{2}\sin\alpha & 2\cos\alpha & i\sqrt{2}\sin\alpha \\ -1+\cos\alpha & i\sqrt{2}\sin\alpha & 1+\cos\alpha \end{pmatrix} . \tag{5.4.11}$$

This matrix clearly reduces to the unit matrix for $\alpha = 0$ and it is straight forward to demonstrate that it is a Unitary matrix ($T_1^{-1} = T_1^\dagger$) for real α (and distinct from the Orthogonal matrices we saw for $SO(3)$ in Chapter 1). The form of the matrix for general α values is less intuitive. However, for $\alpha = \pi$, which we discussed above for the spin 1/2 case, and where we expect to be exchanging the spin up and spin down states we find

$$T_1(\pi) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} . \tag{5.4.12}$$

This matrix does indeed exchange the $m = \pm 1$ states with each other, and introduces a phase of -1 everywhere.

The same analysis for the transformation generated by S_2 yields

$$\begin{aligned} T_2(\alpha) \equiv e^{i\alpha S_2/\hbar} &= \mathbf{1} + \sum_{n=1}^{\infty} \frac{(i\alpha S_2/\hbar)^n}{n!} \\ &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \sum_{n=0}^{\infty} \frac{(-1)^n(\alpha)^{2n}}{(2n)!} + iS_2 \sum_{n=0}^{\infty} \frac{(-1)^n(\alpha)^{2n+1}}{(2n+1)!} , \end{aligned} \tag{5.4.13}$$

where in the second step we again split apart the unit matrix to provide the $n = 0$ term in the first sum. As in the previous analyses we can now rewrite the two sums as the cosine and sine functions. We have

$$T_2(\alpha) = \frac{1}{2}\begin{pmatrix} 1+\cos\alpha & \sqrt{2}\sin\alpha & 1-\cos\alpha \\ -\sqrt{2}\sin\alpha & 2\cos\alpha & \sqrt{2}\sin\alpha \\ 1-\cos\alpha & -\sqrt{2}\sin\alpha & 1+\cos\alpha \end{pmatrix} . \tag{5.4.14}$$

Except for the factors of i this transformation is very similar to T_1. T_2 is also a Unitary matrix. $T_2(0)$ is again the unit matrix, while

$$T_2(\pi) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \tag{5.4.15}$$

which again exchanges the $m = \pm 1$ states, but with somewhat different phases.

Finally consider the transformation generated by S_3, which, as noted earlier for the spin 1/2 case, is particularly simple to evaluate when the basis states are eigenstates of S_3 as here. We have

$$\begin{aligned} T_3(\alpha) \equiv e^{i\alpha S_3/\hbar} &= \mathbf{1} + \sum_{n=1}^{\infty} \frac{(i\alpha S_3/\hbar)^n}{n!} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sum_{n=0}^{\infty} \frac{(-1)^n(\alpha)^{2n}}{(2n)!} + iS_3 \sum_{n=0}^{\infty} \frac{(-1)^n(\alpha)^{2n+1}}{(2n+1)!} \\ &= \begin{pmatrix} \cos\alpha + i\sin\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\alpha - i\sin\alpha \end{pmatrix} = \begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\alpha} \end{pmatrix} , \end{aligned} \tag{5.4.16}$$

which is, as expected, quite similar to Eq. (5.2.13)

To contrast with the result for spin 1/2 in Eq. (5.4.6) we can evaluate Eqs. (5.4.16), (5.4.11) and (5.4.14) for a rotation through 2π applied to a spin 1 system,

$$T_1(2\pi) = T_2(2\pi) = T_3(2\pi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{5.4.17}$$

For integer spin we obtain the expected "classical" result of returning to where we started, similar to what happens with a 2π $SO(3)$ transformation. Note that the representations of $SU(2)$ provided by different spin states need not be identical. It is the spin 1/2 representation that is the fundamental representation and gives us a faithful description of the properties all of the group elements in $SU(2)$.

The reader is encouraged to practice using the concepts described above by reproducung the above expressions for the various transformations.