Chapter 8

Symmetries

8.1 Quantum dynamics

Let us begin this topic by reviewing some "quantum mechanical" notation (recall Chapter 5). The state, or *ket*, vector $|\psi\rangle$ of a physical system completely characterizes the system at a given instant. The corresponding *bra* vector $\langle\psi|$ is the Hermitian conjugate of $|\psi\rangle$. Properly normalized states satisfy the relation that the "bra-ket" is unity, $\langle\psi|\psi\rangle = 1$. We are imagining that $|\psi\rangle$ is defined in a "vector space" of states that is spanned by some *complete* set of appropriately chosen (see below) *linearly independent* (typically orthogonal) basis states (or vectors), $|\psi_n\rangle$, such that *any* state vector can be expressed as a sum of these basis vectors,

$$|\psi\rangle = \sum_n c_n |\psi_n\rangle \tag{8.1.1}$$

(exactly analogously to the way we think of expressing an ordinary 3-vector as the sum of coefficients times the usual unit vectors $\hat{x}^1, \hat{x}^2, \hat{x}^3$).

Let $|\psi(t)\rangle$ denote the state of a system at time t. Given an initial state $|\psi(0)\rangle$, the goal of quantum dynamics is to predict $|\psi(t)\rangle$ for $t \neq 0$. The superposition principle of quantum mechanics implies that there is a *linear* operator $U(t)$, called the *time-evolution operator*, which maps any state at time zero into the corresponding state at time t,

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle\,. \tag{8.1.2}$$

Time evolution must map any properly normalized state at one time into a *normalized* state at another time, *i.e.*, probability is conserved. This implies that the time evolution operator is *unitary*, *i.e.*, it follows that

$$1 = \langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|U(t)^\dagger U(t)|\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle \Rightarrow U(t)^\dagger = U(t)^{-1}\,. \tag{8.1.3}$$

Here we are assuming that the system under study is "isolated" in the sense that no probability can be added or leak away, *i.e.*, $\langle\psi(t)|\psi(t)\rangle$ is a constant.

It is often convenient to consider a differential form of time evolution. The time derivative of any state must again (by the superposition principle) be given by some linear operator acting on the state. That linear operator, times $i\hbar$, is called the Hamiltonian, denoted H. In other words,

$$i\hbar\,\frac{d}{dt}\,|\psi(t)\rangle = H\,|\psi(t)\rangle\,. \tag{8.1.4}$$

This is the (time dependent) Schrödinger equation. It is a linear first order differential equation, whose solution can be (formally) written immediately (through the wonders of the functions of mathematical physics in Chapter 1) in terms of an exponential (defined as an operator by its power series expansion),

$$|\psi(t)\rangle = e^{-iHt/\hbar}\,|\psi(0)\rangle\,. \tag{8.1.5}$$

ASIDE To make this discussion more concrete we recall the explicit forms of some Hamiltonians of interest. For a *free*, non-relativistic particle of mass m moving in one dimension the classical Hamiltonian is $H = p^2/2m = m\dot{x}^2/2$. Quantum mechanically ($p \to -i\hbar\partial/\partial x$) we have $H = -(\hbar^2/2m)\partial^2/\partial x^2$. For a (non-relativistic) harmonic oscillator ($\omega^2 = k/m$) the classical Hamiltonian is $H = p^2/2m + kx^2/2$, while the quantum mechanical version is $H = -(\hbar^2/2m)\partial^2/\partial x^2 + m\omega^2 x^2/2$.

Comparing (8.1.5) with the definition (8.1.2), one sees that this exponential of the Hamiltonian (times $-it/\hbar$) is precisely the desired time evolution *operator*,

$$U(t) = e^{-iHt/\hbar}\,. \tag{8.1.6}$$

The Hamiltonian must be Hermitian, $H^\dagger = H$, in order for $U(t)$ to be unitary,

$$U(t)^\dagger = e^{+iH^\dagger t/\hbar} \Rightarrow U(t)^\dagger U(t) = 1 = e^{i0} = e^{+i(H^\dagger - H)t/\hbar}\,. \tag{8.1.7}$$

Typically we choose our basis states to be the *eigenstates* of the Hamiltonian,

$$H|\psi_n\rangle = E_n|\psi_n\rangle\,, \tag{8.1.8}$$

as they have particularly trivial time dependence

$$|\psi_n(t)\rangle = e^{-iE_n t/\hbar}\,|\psi_n(0)\rangle\,. \tag{8.1.9}$$

Since H is Hermitian (for our isolated, probability conserving system), its eigenvalues, the E_n, are *real* and the time dependence is simply a *phase* that changes linearly with time. This is what ensures that the normalization measured by $\langle\psi(t)|\psi(t)\rangle$ does not change with time. Such states are often referred to as *stationary* states (since the time dependence is trivial).

In principle, we have just solved all quantum dynamics! Of course, actually evaluating this exponential of the Hamiltonian can be (and usually is) a challenge. A quantum system whose space of states is N-dimensional (*i.e.*, n above runs from 1 to N) will have a Hamiltonian which is an $N \times N$ matrix. Most systems of interest will have a very large, or infinite, dimensional space of states.

ASIDE If we are dealing with states that decay as in our discussion of hadrons in the previous chapter, the energy eigenvalue exhibits a (negative) imaginary part ($-i\Gamma/2$) so that the amplitude squared (in a given state) decays exponentially with time ($\propto e^{-\Gamma t/\hbar} = e^{-t/\tau}$). Thus the pole in the complex energy plane corresponding to this (decaying) state (particle) is not on the real axis.

8.2 Symmetries

We have spent much of this quarter discussing the analysis "power" associated with symmetries. Now we want to develop some of the formalism associated with symmetries. (See also Chapter 10 on Group Theory.) A general linear transformation T (not to be confused with the time reversal operator - see Section 8.12), which maps an arbitrary state $|\psi\rangle$ into some different state $|\widetilde{\psi}\rangle = T|\psi\rangle$,

is called a *symmetry* if T is unitary, $T^\dagger = T^{-1}$, so that probability is conserved, *and* T commutes with the time evolution operator,[1]

$$T\,U(t) = U(t)\,T \text{ or } [\mathrm{T}, \mathrm{U(t)}] = 0\,. \tag{8.2.1}$$

To understand this, consider some arbitrary initial state $|\psi(0)\rangle$, and imagine that you have worked out how this state evolves in time so that you know $|\psi(t)\rangle = U(t)|\psi(0)\rangle$. Applying the transformation T to the initial state $|\psi(0)\rangle$ will, in general, produce a different state $|\widetilde{\psi}(0)\rangle = T|\psi(0)\rangle$. This transformed initial state will evolve in time into $|\widetilde{\psi}(t)\rangle = U(t)|\widetilde{\psi}(0)\rangle = U(t)\,T|\psi(0)\rangle$. But, if condition (8.2.1) is satisfied, then one can interchange $U(t)$ and T and write this result as $|\widetilde{\psi}(t)\rangle = T\,U(t)|\psi(0)\rangle = T|\psi(t)\rangle$. In other words, if T is a symmetry transformation, transforming and then time-evolving any state is the same as first time-evolving, and then applying the symmetry transformation. This is summarized by the diagram[2]

$$\begin{array}{ccc} |\psi(t)\rangle & \xrightarrow[T]{} & |\widetilde{\psi}(t)\rangle \\ {\scriptstyle U(t)}\uparrow & & {\scriptstyle U(t)}\uparrow \\ |\psi(0)\rangle & \xrightarrow[T]{} & |\widetilde{\psi}(0)\rangle \end{array} \tag{8.2.2}$$

showing that $|\widetilde{\psi}(t)\rangle$ can be constructed from $|\psi(0)\rangle$ by following *either* path.

The condition (8.2.1) that the transformation T commutes with the time evolution operator is equivalent to the condition that T commutes with the Hamiltonian,

$$[T, H] \equiv T\,H - H\,T = 0\,. \tag{8.2.3}$$

Symmetries have many useful consequences. One class of applications follows directly from the basic definition embodied in the diagram in Eq. (8.2.2) — if you understand how some state $|\psi\rangle$ evolves in time, you can immediately predict how the transformed state $|\widetilde{\psi}\rangle$ will evolve. For example, we will be discussing a transformation known as charge conjugation which interchanges particles and antiparticles, *e.g.*, turning a proton into an antiproton, a π^+ into a π^-, *etc.* Charge conjugation is a symmetry of the strong and electromagnetic interactions (recall the table at the end of Chapter 7). This symmetry directly implies that the rate at which a Δ^{++} baryon decays to a proton and a π^+, which is a strong interaction decay, is the same as the rate at which the $\bar{\Delta}^{--}$ antibaryon (the antiparticle of the Δ^{++}) decays to an antiproton and a π^-. And it implies that the cross section for π^+ scattering on protons must be the same as the cross section for π^- mesons to scatter on antiprotons. So we have understood a lot about the strong interactions, knowing only one of its symmetry properties, but no other details about the dynamics.

A second category of applications follows from the commutativity (8.2.3) of a symmetry transformation with the Hamiltonian. Recall, from linear algebra, that two matrices (or linear operators) are simultaneously diagonalizable if and only if they commute. Consequently, if T is a symmetry then there exist states $\{|\psi_n\rangle\}$ which are *simultaneous* eigenstates of the Hamiltonian *and* of the transformation T,

$$H\,|\psi_n\rangle = E_n\,|\psi_n\rangle\,, \tag{8.2.4a}$$

$$T\,|\psi_n\rangle = t_n\,|\psi_n\rangle\,. \tag{8.2.4b}$$

[1] This definition applies to time-independent symmetry transformations. A more general formulation is required for Lorentz boosts and time-reversal transformations, which have the effect of changing the meaning of time.

[2] Mathematicians call this a *commutative diagram*.

The eigenvalue E_n of the Hamiltonian is the energy of the state $|\psi_n\rangle$ — as noted above the Hamiltonian eigenstates are called energy eigenstates or *stationary states*. As already noted the time-dependent Schrödinger equation (8.1.4) implies that the time evolution of an energy eigenstate is just $|\psi_n(t)\rangle = e^{-iE_nt/\hbar}|\psi_n(0)\rangle$. Hence, an eigenstate of the Hamiltonian is also an eigenstate of the time evolution operator $U(t)$, with eigenvalue $e^{-iE_nt/\hbar}$, and the state is stationary in time except for the overall *phase*. Finally we have also noted that, since the Hamiltonian is a Hermitian operator, its eigenvalues E_n must be *real* (no decays included yet).

Because the symmetry transformation T is also a unitary operator, its eigenvalues t_n must also be (simply) phase factors, $t_n = e^{i\phi_n}$ for some *real* phase ϕ_n.[3] The simultaneous diagonalizability of H and T implies that energy eigenstates can also be labeled by an additional (quantum) number, t_n (or equivalently ϕ_n), which characterizes the effect of the symmetry transformation T on the state. Phrased differently, the eigenvalues of a symmetry transformation T define a quantum number which distinguishes different classes of eigenstates, and we can make use of this more detailed labeling of the states. Note also that the eigenvalues of T are *constants* of the motion, *i.e.*, are unchanged under the time dependence specified by $U(t)$ and correspond to conserved quantities.

There are many examples of this. A particle moving in a (one dimensional) square well potential, $V(x) = \begin{cases} 0\,, & |x| < L/2\,; \\ \infty\,, & \text{otherwise,} \end{cases}$ is an example of a system in which a parity transformation, $x \to -x$, is a symmetry, since the potential is unchanged by this transformation. Consequently, energy eigenstates in this potential can be labeled by their parity; their wavefunctions must either be even, $\psi_n(x) = \psi_n(-x)$, or odd, $\psi_n(x) = -\psi_n(-x)$, under $x \to -x$.

More generally, real particles (in infinite, empty space) can be labeled by their momentum and energy, as well as their angular momentum (spin), electric charge, baryon number, lepton number and intrinsic parity. As we have discussed several times before (and will again), these are all examples of quantum numbers which are associated with specific symmetries.

A third category of applications of symmetries involves time evolution of states which are eigenstates of some symmetry T but which are not (simple) eigenstates of the Hamiltonian (*i.e.*, can exhibit nontrivial time dependence). Let $|\psi_{\rm in}\rangle$ be some initial state which is an eigenstate of the symmetry T with eigenvalue $t_{\rm in}$. Let $|\psi_{\rm out}\rangle$ be some final state which is an eigenstate of the symmetry T with eigenvalue $t_{\rm out}$. For example, think of $|\psi_{\rm in}\rangle$ as the initial state of some scattering experiment involving two incoming particles of types a and b, while $|\psi_{\rm out}\rangle$ is a final state describing outgoing particles of types c and d. Can the scattering process $a + b \to c + d$ occur? In other words, can the matrix element $\langle\psi_{\rm out}|U(t)|\psi_{\rm in}\rangle$, giving the amplitude for the initial state to evolve into the chosen final state, be non-zero? The answer is no — *unless* the symmetry eigenvalues (*i.e.*, the conserved quantum numbers) of the initial and final states coincide. Written mathematically we have

$$\langle\psi_{\rm out}|U(t)|\psi_{\rm in}\rangle = 0 \quad \text{if} \quad t_{\rm in} \neq t_{\rm out}\,. \tag{8.2.5}$$

To see that Eq. (8.2.5) must be true we use the fact that T is unitary to write $1 = T^\dagger T$. Inserting the identity operator changes nothing, so

$$\langle\psi_{\rm out}|U(t)|\psi_{\rm in}\rangle = \langle\psi_{\rm out}|T^\dagger T U(t)|\psi_{\rm in}\rangle = \langle\psi_{\rm out}|T^\dagger U(t) T|\psi_{\rm in}\rangle\,. \tag{8.2.6}$$

[3] To show this, multiply each side of the eigenvalue condition (8.2.4b) by its Hermitian conjugate to obtain $\langle\psi_n|T^\dagger T|\psi_n\rangle = t_n^* t_n \langle\psi_n|\psi_n\rangle$. The left hand side is just $\langle\psi_n|\psi_n\rangle$ since T is unitary, so this condition can only be satisfied if $|t_n| = 1$.

The last step used the condition that T is a symmetry to interchange T and $U(t)$. By assumption, $|\psi_{\rm in}\rangle$ is an eigenstate of T, $T|\psi_{\rm in}\rangle = t_{\rm in}|\psi_{\rm in}\rangle$, and similarly $T|\psi_{\rm out}\rangle = t_{\rm out}|\psi_{\rm out}\rangle$. Taking the Hermitian conjugate of this last relation gives $\langle\psi_{\rm in}|T^\dagger = \langle\psi_{\rm in}|t^*_{\rm in}$. Use these eigenvalue relations for $|\psi_{\rm in}\rangle$ and $\langle\psi_{\rm out}|$ to simplify $\langle\psi_{\rm out}|T^\dagger U(t)T|\psi_{\rm in}\rangle$. The result is

$$\langle\psi_{\rm out}|U(t)|\psi_{\rm in}\rangle = t^*_{\rm out}\, t_{\rm in}\langle\psi_{\rm out}|U(t)|\psi_{\rm in}\rangle\,. \tag{8.2.7}$$

Note that exactly the same matrix element appears on both sides. To satisfy this equation either $t^*_{\rm out}\, t_{\rm in}$ must equal 1, *or else* the matrix element $\langle\psi_{\rm out}|U(t)|\psi_{\rm in}\rangle$ must vanish. Because the eigenvalues of unitary T are pure phases, it follows that $t^*_{\rm out} = 1/t_{\rm out}$. Hence the condition that $t^*_{\rm out}t_{\rm in} = 1$ is the same as the statement that $t_{\rm in} = t_{\rm out}$. The interested reader may be more familiar with the corresponding statement about the eigenstates of the corresponding Hermitian *generator* Q of the transformation T, $T = e^{iQa}$. Such eigenvectors constitute an orthogonal set of basis states and the eigenvalues of the Hermitian operator Q are *real*. That is precisely what we have just proved. If the eigenvalue of Q is q_n, then $t_n = e^{iq_n a}$ is the required pure phase, $t^*_n = e^{-iq_n a}$. The symmetry respecting physics in $U(t)$ will not change the eigenvalue of either T or Q in going from the "in" state to the "out" state. States corresponding to *different* eigenvalues of the symmetry operators are orthogonal, $\langle\psi_m|\psi_n\rangle = 0$ for $q_m \neq q_n$, and remain so as they evolve in time!

The key point here is that symmetries can be used to understand what types of final states can, or cannot, occur in many scattering experiments and decays, even without detailed knowledge of the dynamics. Conservation laws for energy, momentum, angular momentum, electric charge, and baryon and lepton number (and more) can all be viewed as particular cases of this general result.

A final type of application (related to category 2 above) concerns sets of multiple symmetry transformations. Suppose transformations T_1 and T_2 are both symmetries, and hence both commute with the Hamiltonian. But suppose that T_1 and T_2 do *not* commute with each other. (As an example, discussed in detail below, let T_1 be J_3, angular momentum along the $\hat{3}$-axis, and T_2 be J_1 or J_2, or one of the ladder operators $J_\pm$ - recall Chapter 5.) Then one *cannot* simultaneously diagonalize the Hamiltonian and *both* T_1 and T_2, although one can find a basis in which H and, say, T_1 are diagonal. Let $|\psi_n\rangle$ be one of these basis states, so that $H|\psi_n\rangle = E_n|\psi_n\rangle$ and $T_1|\psi_n\rangle = t_{1,n}|\psi_n\rangle$. Applying the symmetry transformation T_2 to the state $|\psi_n\rangle$ will produce some state $|\widetilde{\psi}_n\rangle$, which must also be an eigenstate of the Hamiltonian with exactly the same energy E_n.[4] It may be a linearly independent state, *i.e.*, $|\widetilde{\psi}_n\rangle$ need not be proportional to $|\psi_n\rangle$. Consequently, the existence of symmetry transformations that do not mutually commute can lead to *degenerate* energy levels, *i.e.*, multiple linearly independent states with exactly the same energy. Further these energy degenerate states will comprise *complete* representations of the underlying symmetry group. This is necessarily true because there are (symmetry) transformations that move us around within the representation while simultaneously commuting with the Hamiltonian. Angular momentum eigenstates provide a familiar example of this. In any theory which is rotationally invariant, every energy eigenstate with non-zero angular momentum must be part of a degenerate multiplet. If the (total) angular momentum is $j\hbar$, then the multiplet will contain $(2j+1)$ states, since the projection of the angular momentum along a(n arbitrarily) chosen quantization axis can take any of $2j+1$ values, $\{-j, -j+1, \cdots, j-1, j\}$, but the energy cannot depend on the value of this projection else it would not be rotationally invariant. Thus, knowing something about the symmetries of a given physical system, already tells us a good deal about the possible states of the system. The mathematics of symmetries and the associated

[4]This follows from the given assumption that T_2 is a symmetry, so that it commutes with H. Consequently, $H|\widetilde{\psi}_n\rangle = H\left(T_2|\psi_n\rangle\right) = T_2\left(H|\psi_n\rangle\right) = T_2\left(E_n|\psi_n\rangle\right) = E_n\,|\widetilde{\psi}_n\rangle$.

representations is called Group Theory and we have already seen a brief introduction in Chapter 5. The representation with $2j+1$ elements, which we just described, is a representation of the symmetry group $SO(3)$, and the nearly identical group $SU(2)$. Hopefully you have learned something about these groups either in your quantum or classical mechanics classes or in math methods (or in Chapter 5). Symmetries, and the corresponding conserved quantities, are important for understanding both classical and quantum physics, but probably are more useful in the latter. We will apply the techniques from Chapter 5 at the end of this chapter to the 3-quark spin wave functions that we discussed in Chapter 7.

We can apply similar ideas to the $SU(3)$ (approximate) flavor symmetry. The statement that QCD is flavor symmetric means that the eigenstates of QCD, the hadrons, should appear in *complete* representations of $SU(3)$, the (now) familiar $\underset{\sim}{8}$'s and $\underset{\sim}{10}$'s. If the masses of the u, d, s quarks were identical and we ignored the weak and electromagnetic interactions, all members of the same representation would have the *same* rest mass. However, as we discussed in Chapter 7, the s quark mass is substantially larger than the u and d quark masses (although still much smaller than a typical hadron mass). Thus the full flavor $SU(3)$ symmetry is broken and the masses of hadrons within the same flavor $SU(3)$ multiplet (*i.e.*, representation) are not quite degenerate, but exhibit a small breaking that depends linearly on the number of s quarks. On an even smaller scale, the d quark mass is slightly larger than the u quark mass leading to a tiny breaking of isospin symmetry. Note that in the case of $SU(3)$ the representations are 2-D and the individual states are labeled by the the "total $SU(3)$ quantum number", *e.g.*, $\underset{\sim}{8}$ or $\underset{\sim}{10}$, along with isopsin I, I_3, and the strangeness content S. Also there are 2 linearly independent sets of ladder operators typically chosen as the ones for isospin, $I_\pm$, and one of the sets of operators that move us along the diagonals, typically labeled $U_\pm$ and $V_\pm$, *i.e.*, $SU(3)$ contains 3 $SU(2)$ *subgroups* with ladder operators but only 2 sets are linearly independent. (The existence of the 3 subgroups is related to the 3-fold symmetry of the graphical forms of the representations in Chapter 7.)

8.3 Continuous symmetries

Continuous symmetries are symmetries which depend (continuously!) on some parameter that controls the magnitude of the transformation. Examples include translations and rotations. Let $T(a)$ denote a continuous symmetry transformation depending on the real parameter a. Assume (without loss of generality) that $a = 0$ corresponds to doing nothing, so that $T(0)$ equals the identity (no change) operator. One can always choose to define the parameterization so that $(T(a/2))^2 = T(a)$, or more generally that $(T(a/N))^N = T(a)$ for any N. Here we are appealing to our intuitive expectation that rotating twice through an angle $\theta/2$ $[(T(\theta/2))^2]$ about some axis is equivalent to rotating once through angle θ $[T(\theta)]$ about the same axis. This implies that $T(a)$ depends exponentially on the parameter a (as we have already suggested), so that one can write

$$T(a) = e^{iQa}\,, \tag{8.3.1}$$

for some operator Q, which is called the *generator* of the symmetry $T(a)$. (See also the discussion in Chapters 1, 5 and 10. We are here discussing the structure of the Lie Groups, of which we have discussed $U(1)$, $SU(2)$ and $SU(3)$.) In order for $T(a)$ to be unitary (as required), the generator Q must be Hermitian. Note that the relation between $T(a)$ and Q is completely analogous to the relation between the time evolution operator and the Hamiltonian; the Hamiltonian (divided by $-\hbar$) is the generator of time evolution.

The condition (8.2.3) that $T(a)$ commute with H implies that the generator Q of any continuous symmetry must also commute with the Hamiltonian,

$$[Q, H] = 0\,. \tag{8.3.2}$$

Once again, this implies that Q and H are simultaneously diagonalizable.

Note that, given some continuous symmetry transformation $T(a)$, one can extract the associated generator Q by performing a Taylor series expansion of $T(a)$ about $a = 0$. Keeping just the first non-trivial term gives $T(a) = 1 + iQa + \cdots$, so that

$$Q = -i\,\frac{d}{da}\,T(a)\bigg|_{a=0}\,. \tag{8.3.3}$$

Alternatively, given any Hermitian operator Q which commutes with the Hamiltonian, one can construct a unitary symmetry transformation by exponentiating iQ (times an arbitrary real number), as in (8.3.1). So one can regard either the generator Q, or the finite transformation $T(a)$, as defining a continuous symmetry. In the language of Group Theory, as outlined in Chapter 10, $T(a)$ is an element of the underlying symmetry group (a Lie Group) while Q is an element of the *algebra* corresponding to the group.

8.4 Spacetime symmetries

Spacetime symmetries are symmetries which characterize the underlying geometry of Minkowski space. Translations in both space and time, spatial rotations, and Lorentz boosts are all continuous spacetime symmetries. These are symmetries of the laws of physics, as currently understood. As we have already suggested, associated with those continuous symmetries that commute with the Hamiltonian, there are *additive* conserved quantities or quantum numbers (and corresponding conserved Noether currents - see Chapter 10 and the NYT article on our webpage, `http://courses.washington.edu/partsym/14aut/Noether.pdf`), such as linear momentum, energy and angular momentum.

ASIDE In addition to continuous transformations certain *discrete* transformations are also useful as we have already mentioned, *e.g.*, parity (P:$\vec{x} \to -\vec{x}$), charge conjugation (C:particle $\to$ anti-particle) and time reversal (T:$t \to -t$). These symmetries are associated with *multiplicative* quantum numbers. While these symmetries are *not* exact symmetries (*i.e.*, do *not* commute with *all* of the interactions), they do provide useful approximate symmetries (*i.e.*, commute with "most" of the interactions), as we will discuss later.

Returning to continuous spacetime symmetries, the total momentum operator $\vec{P}$ (divided by $\hbar$) is the generator of spatial translations ($-i\nabla$). Hence, the unitary operator $T_{\rm trans}(\Delta\vec{x})$, which has the effect of performing a spatial translation through a displacement $\Delta\vec{x}$, is an exponential of momentum,[5]

$$T_{\rm trans}(\Delta\vec{x}) = e^{i\vec{P}\cdot\Delta\vec{x}/\hbar}\,. \tag{8.4.1}$$

[5]Recall that in single particle quantum mechanics in, for simplicity, one dimension, the coordinate representation of the momentum operator is $\frac{\hbar}{i}\frac{\partial}{\partial x}$. So acting with $\exp(iP\Delta x/\hbar) = 1 + iP\Delta x/\hbar - \frac{1}{2}(iP\Delta x/\hbar)^2 + \cdots$ on an arbitrary state $|\Psi\rangle$ is the same as acting with $\exp(\Delta x\frac{\partial}{\partial x}) = 1 + \Delta x\frac{\partial}{\partial x} + \frac{1}{2}(\Delta x\frac{\partial}{\partial x})^2 + \cdots$ on the wavefunction $\Psi(x)$. This produces $\Psi(x) + \Delta x\,\Psi'(x) + \frac{1}{2}(\Delta x)^2\Psi''(x) + \cdots$ which is the Taylor series expansion of the translated wavefunction $\Psi(x + \Delta x)$.

In any translationally invariant theory, the total momentum $\vec{P}$ commutes with the Hamiltonian (and hence with the time evolution operator). Therefore, conservation of momentum is a direct consequence of spatial translation invariance.

The Hamiltonian H (divided by $-\hbar$) is the generator of time translations, and the associated unitary operator which has the effect of performing a time translation through an interval Δt is precisely the time evolution operator

$$U(\Delta t) = e^{-iH\Delta t/\hbar}\,. \tag{8.4.2}$$

The Hamiltonian commutes with itself, and therefore it satisfies the conditions defining the generator of a symmetry. Since the Hamiltonian is the operator which measures energy, this shows that conservation of energy is a direct consequence of time translation invariance.

A general *spacetime* translation with displacement $\Delta x = (\Delta x^0, \Delta\vec{x})$ is just a combination of a spatial translation through $\Delta\vec{x}$ and a time translation through $\Delta t = \Delta x^0/c$. The unitary operator which implements this spacetime translation is the product of $T_{\rm trans}(\Delta\vec{x})$ and $U(\Delta t)$.[6] Defining $P^0 \equiv H/c$ allows one to write this as a single exponential of a Minkowski space dot product,

$$T_{\rm trans}(\Delta x) = T_{\rm trans}(\Delta\vec{x}) \times U(\Delta x^0/c) = e^{-iP^\mu \Delta x_\mu/\hbar}\,. \tag{8.4.3}$$

The total angular momentum $\vec{J}$ (divided by $\hbar$) is the generator of rotations. The unitary operator which implements a rotation through an angle θ about an axis defined by a unit vector $\hat{n}$ is an exponential of the component of angular momentum along $\hat{n}$,

$$T_{\rm rot}(\theta, \hat{n}) = e^{i\theta\,\hat{n}\cdot\vec{J}/\hbar}\,. \tag{8.4.4}$$

The total angular momentum $\vec{J}$ commutes with the Hamiltonian in any rotationally invariant theory. Hence, conservation of angular momentum is a direct consequence of spatial rotation invariance.

One can also define operators $\vec{G}$ which are the generators of Lorentz boosts, so that the unitary operator which implements a boost along some direction $\hat{n}$ can be written as an exponential,

$$T_{\rm boost}(y, \hat{n}) = e^{iy\,\hat{n}\cdot\vec{G}/\hbar}\,. \tag{8.4.5}$$

The parameter y, which, as we have noted earlier, is called *rapidity*, is not the velocity of the boost, but it determines the boost velocity via $v/c = \tanh y$. If we consider a particle whose motion (energy E, momentum p) is characterized by this boost, we have $\tanh y = pc/E$. In contrast to the situation with rotations and translations, the boost generators $\vec{G}$ do not commute with H because Lorentz boosts change the meaning of time.[7] Because of this, invariance under Lorentz boosts does not lead to any *additional* conserved quantities analogous to momentum or angular momentum.

ASIDE: Since in this class we often discuss physics as it is being studied at particle colliders, *e.g.*, the LHC, we should note that, in the context of such experiments, the term rapidity is typically used for boosts *only* along the direction of the beams, say the $\hat{x}^3$ direction. Thus one will often see (as in

[6] Because $T_{\rm trans}(\Delta\vec{x})$ commutes with $U(\Delta t)$ (or equivalently, because $\vec{P}$ commutes with H), the order in which one performs this product doesn't matter.

[7] The boost generators $\vec{G}$ depend explicitly on time, and the required condition that they must satisfy turns out to be $\frac{\partial}{\partial t}\vec{G} + i[H, \vec{G}] = 0$.

the HW) rapidity defined via

$$\begin{aligned} p^\mu =& (E/c, p_1, p_2, p_3) \\ p_T =& \sqrt{p_1^2 + p_2^2} \\ E =& \sqrt{m^2c^4 + p_T^2 c^2}\cosh y \\ p_3 =& \sqrt{m^2c^2 + p_T^2}\sinh y \\ \tanh y =& p_3 c/E, \quad y = 0.5 \ln\left[(E + p_3 c)/(E - p_3 c)\right]. \end{aligned} \tag{8.4.6}$$

Thus the 3-momenta of particles are characterized by the "transverse" momentum (transverse to the beam) p_T and the rapidity y, where the 2-vector direction of $\vec{p}_T$ is described by the azimuthal angle ϕ. Note that making the $\hat{x}^3$ direction "special" in this sense is reasonable as the initial state (beams colliding along the $\hat{x}^3$ direction) clearly breaks the overall rotational symmetry of the initial state and the final states exhibit this lack of symmetry. The many particles produced at the LHC are observed to be distributed approximately uniformly in the rapidity defined in Eq. (8.4.6) but are restricted to small values of the transverse momentum, p_T (of order a few GeV/c). In those rare events which produce particles at large p_T, the large p_T particles are observed to cluster into localized (in rapidity and azimuthal angle ϕ) "jets" of particles and are interpreted to arise from the large angle scattering of the underlying quarks and gluons. (These jets as a characteristic feature of high energy collisions is a topic I have studied for much of my scientific career.)

8.5 Charge, lepton, and baryon number

The electric charge Q is an operator which, when acting on any state containing particles with individual charges $\{q_i\}$ $(i = 1, \cdots, N)$, measures the sum of all these charges,

$$Q|\Psi\rangle = q_{\rm tot}|\Psi\rangle \tag{8.5.1}$$

with $q_{\rm tot} \equiv \sum_{i=1}^N q_i$. So, as its name suggests, Q measures the total electric charge of any state. More precisely, each q_i should be understood as the charge of a particle in units of $|e|$. The electric charges of all known particles which can be produced in isolation at distances large compared to a fermi (*i.e.*, not including quarks) are integer multiples of $|e|$; this is known as *charge quantization*. Hence the operator Q will always have integer eigenvalues.

Maxwell's equations are inconsistent if electric charge is not conserved. Therefore, Q must commute with the Hamiltonian (or with the time evolution operator), and hence Q is the generator of a continuous symmetry,

$$T_Q(\alpha) \equiv e^{i\alpha Q}. \tag{8.5.2}$$

Applying this transformation to any state (of definite charge) multiplies the state by a *phase* proportional to its electric charge, $T_Q(\alpha)|\Psi\rangle = e^{i\alpha q_{\rm tot}}|\Psi\rangle$. Note that this is also a continuous symmetry (characterized by the continuous parameter α) that operates not in spacetime like translations, but simply in the space of complex numbers. In the group theory language of Chapter 10, the invariance of electromagnetism is described by the group $U(1)$. The corresponding conserved additive quantum number is just the charge $q_{\rm tot}$.

In precisely the same fashion, one may regard baryon number B and lepton number L [8] as quantum operators which measure the total baryon number or lepton number, respectively, with underlying $U(1)$ symmetries. And one may exponentiate either of these operators to form continuous symmetry transformations generated by B and L,

$$T_B(\alpha) \equiv e^{i\alpha B}\,, \qquad T_L(\alpha) \equiv e^{i\alpha L}\,. \tag{8.5.3}$$

8.6 Approximate symmetries

There are many circumstances where it is useful to consider transformations which are almost, but not quite, symmetries of a theory. Consider, for example, a hydrogen atom in a weak background magnetic field. If the magnetic field were zero, then the Hamiltonian describing the dynamics of the atom would be rotationally invariant. As noted above, this implies that energy eigenstates with non-zero angular momentum must form *degenerate* energy levels. Turning on a magnetic field breaks three dimensional rotation invariance, since the Hamiltonian will now contain terms which depend on the direction of the background magnetic field. (More precisely, turning on a magnetic field reduces the symmetry from three dimensional rotation invariance down to one dimensional rotation invariance with respect to rotations about the direction of the magnetic field.) The presence of the magnetic field will perturb the energy levels of the atom, and lift the degeneracy of energy eigenstates with differing angular momentum projections along the direction of the field. But, if the magnetic field is sufficiently weak, the energy splitting induced by the field will be small (compared to the spacings between the non-degenerate energy levels in the absence of the field). In this circumstance, it makes sense to regard the Hamiltonian of the system as the sum of a "large" rotationally invariant piece H_0, which describes the atom in the absence of a magnetic field, plus a "small" perturbation ΔH, which describes the interaction with the weak magnetic field,

$$H = H_0 + \Delta H\,. \tag{8.6.1}$$

One can systematically calculate properties of the atom as a series expansion in the size of ΔH, or more correctly in the size of ΔH divided by the appropriate eigenvalue of H_0, since the expansion parameter must be dimensionless. The starting point involves ignoring ΔH altogether and understanding the properties of H_0, including the properties of its eigenstates. And when studying the physics of H_0 alone, one can use full three dimensional rotation symmetry to characterize the corresponding energy eigenstates. These eigenstates, in turn, provide a complete, orthogonal set of basis states useful for analyzing this system *even* when the perturbation, ΔH, is included.

Exactly the same approach can be applied to particle physics to separate the effects of weak and electromagnetic interactions (small) from those of strong interactions (large). Similarly we can study the effects of the (small) symmetry breaking due to the quark masses on the flavor $SU(3)$ symmetric limit. In fact, this approach of using "perturbation theory" to analyze a system in terms of perturbations around a simple, completely analyzable symmetric limit, is an extremely important tool to have in your physics toolbox!

[8] Note that we are (perhaps unfortunately) using the same symbol L for both lepton number and orbital angular momentum. The meaning in any specific situation will (hopefully) be obvious from the context.

8.7 Flavor symmetries

Strong interactions, as described by quantum chromodynamics, preserve the net number of quarks of each flavor. Strong interactions can cause the creation or annihilation of quark-antiquark pairs of any given flavor, but this does not change the number of quarks minus antiquarks of each flavor. This is also true of electromagnetic interactions, but not weak interactions. Consequently, in a hypothetical world in which weak interactions are turned off, operators which measure the number of quarks minus antiquarks of each flavor,

$$N_u = (\#\ u \text{ quarks}) - (\#\ \bar{u} \text{ quarks})\,, \tag{8.7.1a}$$
$$N_d = (\#\ d \text{ quarks}) - (\#\ \bar{d} \text{ quarks})\,, \tag{8.7.1b}$$
$$N_s = (\#\ s \text{ quarks}) - (\#\ \bar{s} \text{ quarks})\,, \tag{8.7.1c}$$
$$N_c = (\#\ c \text{ quarks}) - (\#\ \bar{c} \text{ quarks})\,, \tag{8.7.1d}$$
$$N_b = (\#\ b \text{ quarks}) - (\#\ \bar{b} \text{ quarks})\,, \tag{8.7.1e}$$
$$N_t = (\#\ t \text{ quarks}) - (\#\ \bar{t} \text{ quarks})\,, \tag{8.7.1f}$$

all commute with the QCD (and EM) Hamiltonian. Therefore, all these operators may be regarded as generators of continuous symmetries. Note that baryon number equals the total number of quarks minus antiquarks, divided by three,

$$B = \tfrac{1}{3}\left[(\#\text{ quarks}) - (\#\text{ antiquarks})\right] = \tfrac{1}{3}\sum_{f=u,d,s,c,b,t} N_f\,, \tag{8.7.2}$$

since baryons contain three quarks, while antibaryons contain three antiquarks. For historical reasons, it is conventional to refer to *strangeness* as the number of strange *anti*-quarks minus quarks,

$$S \equiv -N_s = (\#\ \bar{s} \text{ quarks}) - (\#\ s \text{ quarks})\,. \tag{8.7.3}$$

This definition assigns strangeness +1 to the (originally observed) K^+ meson (with a $\bar{s}$ antiquark), and this convention predates the development of QCD and the quark model of hadrons.

In a world without weak interactions, π mesons would be absolutely stable because there are no lighter strongly interacting hadrons into which pions could decay. Kaons (K mesons) would also be stable, even through they are over three times heavier than pions, because K mesons are the lightest hadron with nonzero strangeness (and strangeness is conserved by strong interactions). Similarly, the Ω^- baryon, containing three strange quarks, would be stable because there is no other combination of hadrons it might decay into that has lower energy, together with baryon number one and strangeness minus three.

Completely analogous arguments apply to hadrons containing the heavier charm and bottom quarks. In the absence of weak interactions, there would be many additional stable hadrons containing nonzero net "charmness", or net "bottomness" (or "topness").

8.8 Isospin

Figure 8.1 graphically displays the mass spectrum of light mesons and baryons. Looking at this figure, or the tables containing information about hadrons in the previous chapter, many degeneracies or near-degeneracies are immediately apparent. For example, the masses of the π^+ and π^- mesons are

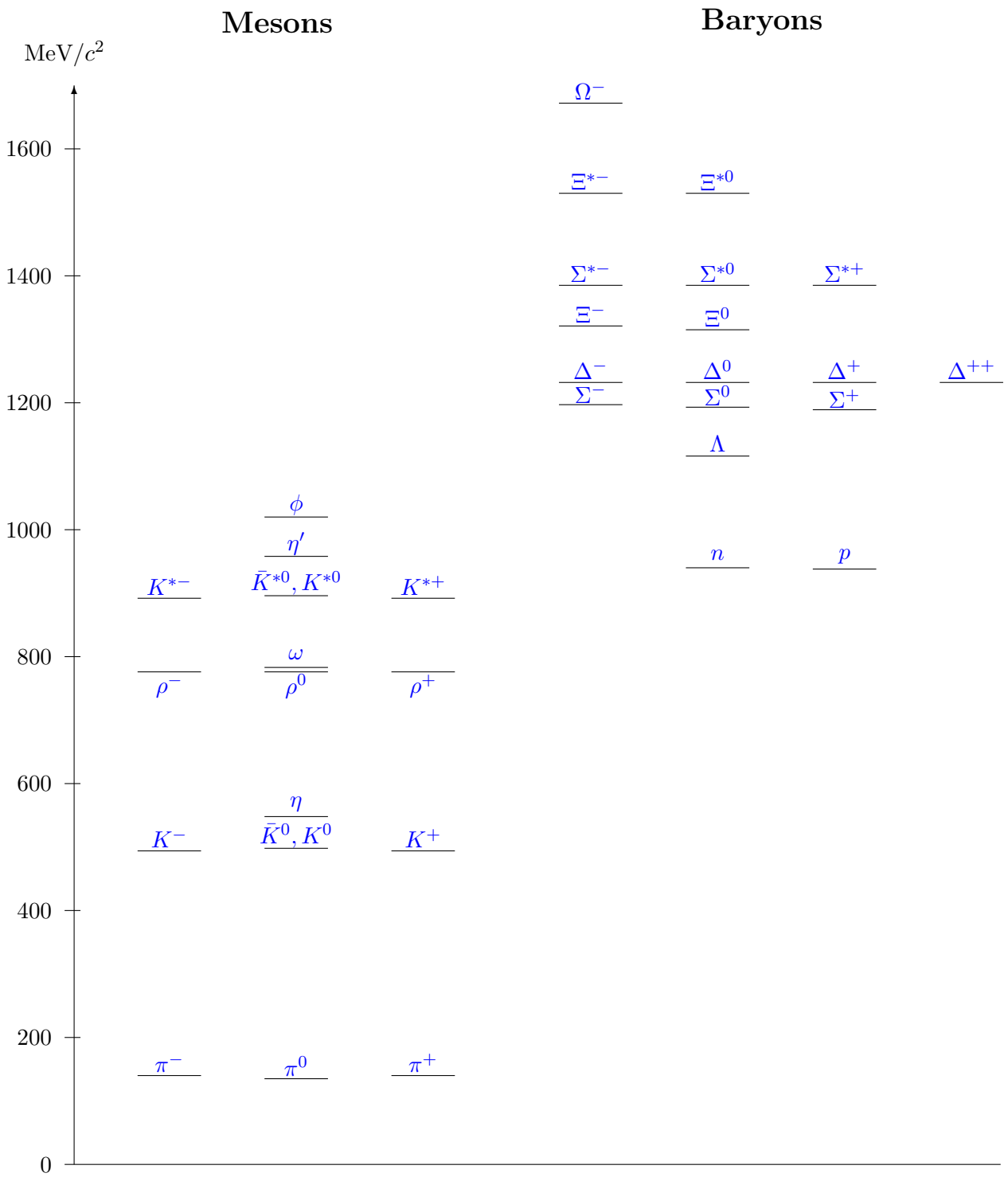

Figure 8.1: The mass spectrum of light mesons and baryons. Each column show mesons or baryons with a particular charge. (Only the lightest mesons and baryons with given quark content and spin are shown. Many additional excited states with higher masses are known, but are not displayed in this plot.)

the same, while the mass of the π^0 meson is only slightly different. The mass of the neutron is quite close to that of the proton. The masses of the Σ^+, Σ^0 and Σ^- baryons are all different from the other hadrons, but very close to each other. And likewise for many other "multiplets" of mesons and baryons. As we have noted several times earlier, the most obvious "near degeneracy" is that due to isospin symmetry, a topic which we discuss again and more completely in the following.

Comparing the quark content of various hadrons (and referring to Tables 7.7 - 7.10 as needed), one sees that the near-degeneracies are all associated with substitutions of u for d quarks, or vice-versa. For example, the Σ^+ baryon has two up and one strange quark. Replacing one up quark by a down converts the Σ^+ into a Σ^0, whose mass is larger than that of the Σ^+ by 3.3 MeV/c^2, which is less than 0.3% of the Σ^+ mass (a true "near degenarcy"). Replacing the remaining up quark by a down converts the Σ^0 into a Σ^-, whose mass is an additional 4.8 MeV/c^2 larger.

The mass differences among the three Σ baryons, the three π mesons, between the neutron and proton, or within any of the other nearly degenerate multiples, must arise from some combination of two effects. First, the masses of up and down quarks are not quite the same. The mass of a down quark (*c.f.* Table 7.1) is a few MeV/c^2 larger than that of an up quark. This mass difference is tiny compared to the masses of hadrons, but it is comparable to the few MeV/c^2 mass splittings within the various near-degenerate multiplets.

Second, while the color interactions of up and down quarks are identical, the other interactions are different. They have differing electric charges (2/3 for u, and $-1/3$ for d), which means that their electromagnetic interactions are not the same. Their weak interactions also differ. But, as far as hadronic masses are concerned, the effects of weak and electromagnetic interactions are numerically small perturbations on top of the dominant effects due to strong interactions, and strong interactions are flavor-blind. In a hypothetical world in which weak and electromagnetic interactions are absent, and in which up and down quarks have the same mass, these near-degeneracies would all become *exact* degeneracies.

As we have discussed many times, you are encouraged to interpret this situation as being similar to angular momentum multiplets (because it is very similar - the *same* equations have the *same* solutions). In any rotationally invariant theory, every state with angular momentum $J\hbar$ is part of a multiplet containing $2J+1$ degenerate states. A rotation transforms the different states in the multiplet into linear combinations of each other. The simplest non-trivial case is $J = \frac{1}{2}\hbar$, the so-called *fundamental* representation, whose multiplet contains two (linearly independent) states conventionally chosen to have angular momentum up or down along some given axis, and represented by $|\uparrow\rangle$ and $|\downarrow\rangle$. The action of a rotation corresponds to a linear transformation,

$$\begin{pmatrix} |\uparrow'\rangle \\ |\downarrow'\rangle \end{pmatrix} = M \begin{pmatrix} |\uparrow\rangle \\ |\downarrow\rangle \end{pmatrix} . \tag{8.8.1}$$

For a rotation about an axis $\hat{n}$ through an angle θ, the matrix M has the form (note especially the factors of 1/2 in the angle and the representation of the operator $\vec{I} = \vec{\sigma}/2$)

$$M = e^{i\theta\hat{n}\cdot\vec{I}} = e^{i(\theta/2)\hat{n}\cdot\vec{\sigma}} = \left(\cos\tfrac{\theta}{2} + i\hat{n}\cdot\vec{\sigma}\,\sin\tfrac{\theta}{2}\right) , \tag{8.8.2}$$

with $\vec{\sigma}$ denoting the Pauli matrices, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.[9] You can easily (and should) check that M is a unitary matrix with determinant equal to one. As noted earlier, the

[9] Of special interest are the combinations $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$ which provide a realization of the ladder operators of Eq. (**??**): $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. The former turns $|\downarrow\rangle$ into $|\uparrow\rangle$ and the latter does the opposite.

space (or group) of such 2×2 matrices is called $SU(2)$, the Special Unitary Group in 2 dimensions. [10]

In the limit that the u and d quarks are degenerate in mass (and weak and electromagnetic interactions are turned off), there is an analogous symmetry which transforms up and down quark states into new linear combinations of the two flavors,

$$\begin{pmatrix} |u'\rangle \\ |d'\rangle \end{pmatrix} = M \begin{pmatrix} |u\rangle \\ |d\rangle \end{pmatrix}, \tag{8.8.3}$$

where M is any 2×2 unitary matrix with determinant one. This is the formal definition of the symmetry called *isospin* (or isotopic spin) with M an isospin "rotation".

The mathematical structure of isospin rotations is completely analogous to spatial rotations (although isospin has nothing to do with ordinary spatial rotations, but, as usual, "the same equations have the same solutions"). There are three generators of isospin rotations, I_1, I_2 and I_3. Their commutation relations have the same form as the commutation relations of angular momentum operators (which are the generators of spatial rotations, [11]

$$[I_a, I_b] = i\epsilon_{abc}\, I_c\,. \tag{8.8.4}$$

Total isospin is denoted by I, and can have either integer or half-integer values. An up quark has $I_3 = +1/2$, while a down quark has $I_3 = -1/2$. Hence an up quark behaves (with respect to isospin rotations) just like an up spin does (with respect to spatial rotations). This parallel is the origin of the names 'up' and 'down' for the two lightest quarks. For antiquarks, the assignments are reversed, a $\bar{u}$ quark has $I_3 = -1/2$ while a $\bar{d}$ has $I_3 = +1/2$.[12]

When we build states containing multiple up and down quarks (or antiquarks), the addition of isospin works just like adding angular momentum. For example, combining two isospin one-half objects can yield either isospin 0 or isospin 1, *i.e.*, $\underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\sim}{3} \oplus \underset{\sim}{1}$. An antisymmetric combination of u and d quarks,

$$(ud - du)\,, \tag{8.8.5}$$

gives $I = 0$ isospin singlet, while a symmetric combination gives the isospin one triplet state(s). Hence, the three $I = 1$ flavor states of two quarks are

$$(uu),\ (ud + du),\ (dd)\,, \tag{8.8.6}$$

with I_3 for these states equaling $+1$, 0, and -1, respectively. Similarly, when three u or d quarks are combined (as in a baryon), the resulting isospin can be either 1/2 or 3/2, *i.e.*, $\underset{\sim}{2} \otimes \underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\sim}{4} \oplus \underset{\sim}{2} \oplus \underset{\sim}{2}$.

Looking back at the nearly degenerate set of particles shown in Figure 8.1 , the π^+, π^0 and π^- mesons form an $I = 1$ multiplet, whose masses would be exactly equal were it not for the perturbing effects

[10] As discussed in the Group Theory Lecture in Chapter 10, the two groups $SO(3)$ and $SU(2)$ are intimately related with exactly the same "algebra". Thus it is no surprise that the current discussion looks like the (hopefully) more familiar case of angular momentum.

[11] This is just the statement that the two groups have the same algebra as already noted in the previous footnote.

[12] It is (note the now familiar minus sign) $\begin{pmatrix} -|\bar{d}\rangle \\ |\bar{u}\rangle \end{pmatrix}$ which transforms in the same manner as $\begin{pmatrix} |u\rangle \\ |d\rangle \end{pmatrix}$, namely $\begin{pmatrix} -|\bar{d}'\rangle \\ |\bar{u}'\rangle \end{pmatrix} = M \begin{pmatrix} -|\bar{d}\rangle \\ |\bar{u}\rangle \end{pmatrix}$.

of weak and electromagnetic interactions and the up and down quark mass difference. Similarly, the K^+ and K^0 mesons (whose quark contents are $u\bar{s}$ and $d\bar{s}$, respectively) form an isospin 1/2 multiplet with strangeness one, while the K^- and $\bar{K}^0$ mesons (with quark content $s\bar{u}$ and $s\bar{d}$) form another $I = 1/2$ multiplet with $S = -1$. The three rho mesons form another $I = 1$ multiplet. Turning to the baryons, the two nucleons (*i.e.*, the proton and the neutron) form an $I = 1/2$ multiplet (as do the Ξ baryons), while the three Σ baryons have $I = 1$ and the four Δ baryons have $I = 3/2$.

Conservation of isospin (by the strong interactions) can also be used to explain a variety of more detailed hadronic properties, including the fraction of Δ^+ decays which yield $p\pi^0$ versus $n\pi^+$, or the fraction of different pion pairs produced by ρ decays. While the total rate of Δ^+ or ρ^0 decays will depend on the details of the QFT version of the strong interactions, the *relative* branching ratios into differing particle-pair states, *e.g.*, $p\pi^0$ versus $n\pi^+$ or $\pi^0\pi^0$ versus $\pi^+\pi^-$, will (in the limit of conserved isospin) be determined by the corresponding Clebsch-Gordan coefficient. Recall that a table of C-G coefficients was included at the end of Chapter 7. We will work through some detailed examples in the HW.

Isospin conservation can also be used to explain the absence of many unseen decay modes. For example, the $\Lambda(1690)$ is an excited state of the $\Lambda(1116)$ baryon (still $I = 0$, but $J^P = \frac{3}{2}^-$, *i.e.*, 1 unit of orbital angular momentum), with 1690 MeV rest energy and quark content uds. Roughly 25% of the time, a $\Lambda(1690)$ decays to a $\Lambda(1116)$ plus two pions. But it *never* decays to a $\Lambda(1116)$ plus a single pion, despite that fact that more energy would be available for conversion into kinetic energy if only a single pion were produced. To understand why decays to a $\Lambda(1116)$ plus two pions are favored, note that the $\Lambda(1116)$ baryon, and its excited states like the $\Lambda(1690)$ have $I = 0$, while pions have $I = 1$. So the decay $\Lambda(1690) \to \Lambda(1116) + \pi$ would have $\Delta I = 1$ — an initial state with isospin zero and a final state of isospin one. Thus this strong interaction decay is not allowed, since the strong interactions conserve isospin. But in the final state of the observed decay $\Lambda(1690) \to \Lambda(1116) + \pi + \pi$, the total isospin is the combination of two $I = 1$ pions plus the $I = 0$ $\Lambda(1116)$ baryon. Adding two isospin one objects can yield isospin two, one, or zero. So, if the final pions combine to form zero isospin, then isospin will be conserved in this decay, and a strong interaction decay is allowed. The two pion decay can conserve isospin, while the single pion case cannot. Similarly we can understand why the $\Lambda(1690)$ can decay into a Σ ($I = 1$) and one or two pions.

Because isospin is only an approximate symmetry, predictions one can make using isospin invariance are not exact results in the real world. However, because the up and down quark mass difference is so small, and weak and electromagnetic interactions are *much* weaker than strong interactions, predictions which follow from isospin invariance are quite accurate — violations are typically at or below the 1% level.

8.9 Parity

The most familiar of the relevant *discrete* symmetries with *multiplicative* eigenvalues is parity. A parity transformation, denoted P, has the effect of reversing all spatial coordinate axes. Therefore, a parity transformation acting on a state of a single particle located at some spatial position $\vec{x}$ produces a state in which the particle is located at $-\vec{x}$. Fourier transforming to the momentum representation, one can equally well say that a parity transformation acting on a single particle state with momentum $\vec{p}$ will produce a state with momentum $-\vec{p}$. Written symbolically, this suggests that if $|\vec{p}\rangle$ represents a state of some particle with momentum $\vec{p}$, then the parity transformed state should be $P\,|\vec{p}\rangle = |-\vec{p}\rangle$. This is not quite right, however, as the unitary transformation P can also produce

a change in the overall phase of the state. Therefore, in general one must write

$$P\,|\vec{p}\rangle = \eta_P\,|-\vec{p}\rangle\,, \tag{8.9.1}$$

where η_P is some phase factor which can depend on the type of particle under consideration. A parity transformation does *not* change the spin or angular momentum of a particle.[13]

Applying two parity transformations amounts to reversing the directions of all spatial coordinate axes, and then reversing them all over again. This overall transformation has no net effect. Hence, as an operator, parity must square to the identity, $P^2 = 1$. This implies that the phase η_P appearing in Eq. (8.9.1) must square to one, $\eta_P^2 = 1$, so either $\eta_P = +1$ or $\eta_P = -1$. This sign is called the *intrinsic parity* of a particle. Some particles (such as protons and neutrons) have positive intrinsic parity, while others (such as pions and photons) have negative parity. One can show (from relativistic quantum mechanics) that for particles which are bosons, the intrinsic parities of antiparticles are the same as the corresponding particles, while for fermions, antiparticles have intrinsic parities which are *opposite* to the corresponding particle.

For multiparticle states, the form of the wavefunction describing the relative motion of the particles also affects the behavior of the state under a parity transformation. If two particles A and B (viewed in their mutual center-of-momentum frame) have orbital angular momentum L, then an additional factor of $(-1)^L$ appears in the result of a parity transformation,[14]

$$P\,|\Psi_L^{A+B}\rangle = \eta_P^A\,\eta_P^B\,(-1)^L\,|\Psi_L^{A+B}\rangle\,, \tag{8.9.2}$$

where η_P^A and η_P^B are the intrinsic parities of the individual particles. Note, as claimed earlier, that the resulting total response to the parity transformation is the *product* of the individual bits.

Intrinsic parities can be assigned to particles in such a way that parity is a symmetry of the strong and electromagnetic interactions. In particular, the light mesons in Tables 5.7 and 5.8 are all parity-odd (*i.e.*, they have negative intrinsic parity).[15] The photon is also parity-odd. The baryons listed in Tables 5.9 and 5.10 are all parity-even corresponding to the *conventional* choice that the lowest mass baryons and the quarks have positive intrinsic parity, while the lowest mass antibaryons and the antiquarks have negative intrinsic parity. The excited baryons, with nonzero internal orbital angular momentum, can have negative parity (*e.g.*, the $J^P = \frac{3}{2}^-$, isospin $\frac{1}{2}$ baryon with mass 1520 MeV/c^2 or the $\Lambda(1690)$ mentioned above).

Physics became much more interesting when it was realized that parity is *not* respected by all interactions. In particular, parity is *not* a symmetry of the weak interactions, and the discovery that all interactions do not respect parity was a BIG deal historically (the 1957 Nobel Prize in Physics). This will be discussed further in the next chapter. Thus parity is an approximate symmetry, useful for understanding strong or electromagnetic processes, but is not a true symmetry of all nature.

[13] Recall that $\vec{L} = \vec{r} \times \vec{p}$. Since a parity transformation reverses both $\vec{r}$ and $\vec{p}$, the (orbital) angular momentum $\vec{L}$ does not change. The intrinsic spin transforms in the same fashion as $\vec{L}$. We label vectors like $\vec{L}$ that do *not* change sign under parity as axial or pseudo-vectors (in contrast to the more familiar polar vectors that do change sign).

[14] This factor comes from the behavior of spherical harmonics, which describe states of definite orbital angular momentum, under the transformation $\vec{x} \to -\vec{x}$, namely $Y^{LM}(-\hat{x}) = (-1)^L\,Y^{LM}(\hat{x})$.

[15] All these mesons are s-wave quark-antiquark bound states, so they have no orbital angular momentum. Their negative parity reflects the *opposite* intrinsic parities of fermions and antifermions, here applied to quarks and antiquarks. Higher energy even-parity mesons do exist; these may be understood as bound states with non-zero orbital angular momentum.

8.10 Charge conjugation

A charge conjugation transformation, denoted C, has the effect of interchanging particles and antiparticles, $C|A\rangle = |\bar{A}\rangle$. So, for example, charge conjugation turns a proton into an antiproton, an electron into a positron, and a π^+ into a π^-. For particles which are their own antiparticles ("self-conjugate" particles), such as the photon and π^0, there can also be an overall (eigenvalue) phase factor,

$$C\,|A\rangle = \eta_C\,|A\rangle \qquad \text{(self-conjugate particles)}. \tag{8.10.1}$$

Similarly to the parity transformation, applying charge conjugation twice takes us back to where we started so that $\eta_C^2 = 1$, $\eta_c = \pm 1$. These phases, which again depend on particle type, can be defined in such a way that charge conjugation is a symmetry of the strong and electromagnetic interactions (and match conventional choices for the phases - recall the minus sign in our definition of the antiquark doublet). However, charge conjugation is not an invariance of weak interactions. So charge conjugation is only an approximate symmetry, like parity, but is very useful when considering strong or electromagnetic processes.

Charge conjugation has no effect on momenta or spins of particles, but the electric charge and other (additive) flavor quantum numbers (B, L, S, I_3) all have their signs changed by the charge conjugation transformation. Hence, only particles which are neutral (and whose strangeness, I_3, baryon, and lepton numbers all vanish) can be self-conjugate.

The photon is charge-conjugation odd (*i.e.*, its phase $\eta_C = -1$). To understand why, consider a classical electromagnetic field produced by some charge or current density. A charge conjugation transformation would change the electrically charged particles which are the source of the electromagnetic field into their oppositely charged antiparticles. In other words, the charge and current densities appearing in Maxwell's equations would change sign. Since Maxwell's equations are linear, this implies that the electromagnetic field itself would change sign. The photon is a quantized excitation in the electromagnetic field. Its behavior under charge conjugation reflects the behavior of a classical EM field: it changes sign.

Since a single photon is charge-conjugation odd, a multi-photon state containing N photons is charge-conjugation even if N is even, and charge-conjugation odd if N is odd. The neutral pion (dominantly) decays to two photons, while π^0 decay to three photons has not been observed. Neutral pion decay is an electromagnetic process, for which charge conjugation is a symmetry. Hence, the neutral pion is charge-conjugation *even*.

As an example of the utility of charge conjugation symmetry, consider positronium (as we did in the HW, but here using slightly different tools). This is the name given to bound states of an electron and a positron. Since an electron and positron have opposite electric charges, they have an attractive Coulomb interaction, and consequently form Coulombic bound states — just like the electron and proton in a hydrogen atom. Relative to hydrogen, there are two noteworthy differences. First, because the positron mass equals the electron mass (instead of being much much heavier like a proton), spacings between energy levels in positronium are half the corresponding spacings in hydrogen (the reduced mass is smaller by a factor of 2). More importantly, positronium is not stable. Unlike a hydrogen atom, the electron and positron in positronium can (and eventually will) annihilate into photons.

Consider positronium in its $1s$ ground state. How many photons will be produced when it decays? Answering this requires a consideration of symmetries, not hard calculations. Energy and momentum conservation forbid decay into a single photon since the photon has zero rest mass and positronium

does not. To understand whether decay into two photons is possible, we need to specify the initial state more carefully. Since the electron and positron each have spin 1/2, the total spin of positronium can be either $S = 0$ or $S = 1$. Since a $1s$ state has no orbital angular momentum, the total angular momentum is the same as the spin. The singlet ($S = 0$) state of positronium is known as *para-positronium*, while the triplet ($S = 1$) state is called *ortho-positronium*. Recall that when two spins are combined to form $S = 1$, the spin wavefunction [either $\uparrow\uparrow$, $(\uparrow\downarrow + \downarrow\uparrow)$, or $\downarrow\downarrow$, depending on the value of S_3] is *symmetric* under interchange of the two spins. But the singlet spin wavefunction, $(\uparrow\downarrow - \downarrow\uparrow)$, is *antisymmetric* under interchange of spins.

Charge conjugation operating on positronium interchanges the types of the two fermions, without affecting the spins or positions of the two particles. This means that C interchanges the positions of the electron and positron flipping their relative separation, $\vec{r} \to -\vec{r}$ (*i.e.*,physical quantities will depend on the positions of the physically distinct electron and positron). Since a $1s$ state has a rotationally invariant spatial wavefunction, swapping the positions of the electron and positron does not change the value of the spatial wavefunction. In the $S = 1$ spin triplet (ortho-positronium), swapping the two spins also does not change the value of the spin wavefunction, since the spin wavefunction is symmetric. Hence, the action of charge conjugation on ortho-positronium is the same as completely interchanging the two particles (because the state is symmetric under interchange of positions and spins). Finally, similarly to our discussion of the intrinsic parities of fermions and antifermions, the anticommutation properties of the corresponding creation (and annihilation) *operators* introduces an extra factor of (-1). Consequently, ortho-positronium must be charge-conjugation *odd*. In contrast (but by the same reasoning), in the $S = 0$ spin singlet (para-positronium), the action of charge conjugation differs from that of a complete interchange of the two fermions by an extra minus sign coming from the antisymmetry of the spin wavefunction.

More generally a fermion-antifermion pair with definite values of orbital angular momentum (L) and total spin (S) has the following property under C,

$$C|f\bar{f}\text{pair}, L, S\rangle = (-1)^L(-1)^{S+1}(-1)|f\bar{f}\text{pair}, L, S\rangle = (-1)^{L+S}|f\bar{f}\text{pair}, L, S\rangle\,, \tag{8.10.2}$$

where the middle expression explicitly displays the corresponding behavior of the spatial wavefunction, the spin wavefunction and the intrinsic phase, respectively. Thus, as already noted, the $1s$ ortho-positronium state ($L = 0$, $S = 1$) is C odd, while para-positronium ($L = 0$, $S = 0$) is C even. Therefore, para-positronium is charge-conjugation *even*. Correspondingly the ground state mesons are C even for spin zero and C odd for spin one.

We noted above that a multi-photon state is charge conjugation even or odd depending on whether the number of photons is even or odd. Hence, charge conjugation invariance (of electromagnetic interactions) implies that para-positronium must decay to an *even* number of photons, while ortho-positronium must decay to an *odd* number of photons. Every additional photon in the final state *decreases* the rate of decay (by at least one factor of the fine structure constant α), *i.e.*, *increases* the lifetime. Therefore, singlet positronium should decay to two photons, while triplet positronium should decay, more slowly, to three photons. This is precisely what is observed. The lifetime of spin singlet para-positronium is 125 ps $= 125 \times 10^{-12}$ s, while the lifetime of spin triplet ortho-positronium has the much larger value 142 ns $= 142 \times 10^{-9}$ s. Recall that we found this same distinction between the decays of the positronium states in our analysis in the HW, although there we made use of detailed considerations of angular momentum conservation.

Similarly applying Eq. 8.10.2 to the neutral (quark-antiquark) π^0 ($L = S = 0$) tells us that the π^0 is an eigenstate of C with eigenvalue +1, consistent with the fact that it decays into 2 photons (via the C conserving electromagnetic interactions).

Finally we return briefly to the topic of G-parity introduced in the previous chapter. This transformation is the product of both C and an isospin rotation of π radians about the 2-axis,

$$G = Ce^{i\pi I_2} \,. \tag{8.10.3}$$

Since the strong interactions respect both C and I, G is a symmetry of the strong interactions and is helpful in organizing multi-pion states. The analog of Eq. 8.10.2 is

$$G|f\bar{f}\text{pair}, L, S, I\rangle = (-1)^{L+S+I}|f\bar{f}\text{pair}, L, S, I\rangle \,, \tag{8.10.4}$$

i.e., the isospin rotation (by π) generates an extra factor $(-1)^I$ as the analog of the more familiar $(-1)^L$ from an ordinary rotation. For example, the pion, $L = 0$, $S = 0$, $I = 1$, has *odd* ($(-1)^1$) G-parity, as does the ω, $L = 0$, $S = 1$, $I = 0$. Hence the strong decays of the ω must involve odd numbers of pions (3 in this case). On the other hand, the ρ, $L = 0$, $S = 1$, $I = 1$, has *even* ($(-1)^2$) G-parity and decays into 2 pions.

8.11 CP

As we have discussed the charged current part of the weak interactions (*i.e.*, the exchange of the charged W's) explicitly and separately violate C and P, coupling only to left-handed particles and right-handed antiparticles. On the other hand this very structure is connected by the (simultaneous) operation of CP, which turns a left-handed particle into a right-handed antiparticle. Until the 1960's it was believed that the weak interactions conserve the eigenvalue of the combined operator CP. As already mentioned the interesting structure seen in the decays of the $|K^0\rangle - |\bar{K}^0\rangle$ system explicitly confirms the importance of CP conservation. Combining the results of the previous two Sections tells us that for a fermion - anti-fermion system, like the π^0, the CP eigenvalue is defined by

$$CP|f\bar{f}\text{pair}, L, S\rangle = (-1)^L(-1)^{S+1}(-1) * (-1)(-1)^L|f\bar{f}\text{pair}, L, S\rangle = (-1)^{S+1}|f\bar{f}\text{pair}, L, S\rangle \,. \tag{8.11.1}$$

Thus the π^0 is CP odd and the decay of a state of definite CP into π^0's will be determined by the initial CP, even $CP \rightarrow$ an even number of π^0's (typically 2) while odd $CP \rightarrow$ an odd number π^0's (typically 3). Further, in this argument, we can replace a pair of π^0's by a $\pi^+\pi^-$ pair with $L = 0$, which dominates at low energies. As we consider in some detail in the HW, this means that the CP even combination of $|K^0\rangle$ and $|\bar{K}^0\rangle$ can decay (rather quickly) to 2 pions, while the CP odd state decays (more slowly) to 3 pions. So these definite CP states are labeled K_{short}, or K_S, and K_{long}, or K_L. Note also that the mixing of $|K^0\rangle$ and $|\bar{K}^0\rangle$ can happen via the weak interactions because the weak interactions do not conserve the strange quark number. Finally note that with our conventions $C|K^0\rangle = (+1)|\bar{K}^0\rangle$ (although this is not uniformly true in the literature) and thus, similarly to the neutral pion, $CP|K^0\rangle = (-1)|\bar{K}^0\rangle$, where, unlike the π^0 and due to the nonzero strangeness, the K^0 is not a CP eigenstate. However, CP eigenstates can be constructed from K^0 *and* $\bar{K}^0$ and are proportional to $|K^0\rangle \mp |\bar{K}^0\rangle$, with the minus sign corresponding to *even* CP.

ASIDE: A relevant way to think about the K^0 system is in terms of 2 sets of basis vectors. The kaons are typically *produced* via the strong interactions where the strangeness conserving basis is the appropriate one, while the decay process via the weak interaction is more simply viewed in the CP eigenstate basis,

$$\text{Strangeness:} \begin{pmatrix} |K^0\rangle \\ |\bar{K}^0\rangle \end{pmatrix} \quad \text{vs} \quad \text{CP:} \begin{pmatrix} |K_S\rangle \\ |K_L\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} |K_0\rangle - |\bar{K}^0\rangle \\ |K_0\rangle + |\bar{K}^0\rangle \end{pmatrix} \,. \tag{8.11.2}$$

These two choices of basis are clearly related by a simple rotation.

Finally we must note that in the 1960's it was learned that the weak interactions, in fact, do *not* precisely conserve CP, but rather exhibit a tiny (1 part in a thousand) violation. The experimental observation of CP violation in the neutral kaon system led to the 1980 Nobel Prize in Physics. This is an extremely interesting story, now being repeated at the LHC in the bottom quark sector, but we do not have time to discuss it more fully here (except in the HW).

8.12 Time reversal and CPT

Time reversal, denoted T, is a transformation which has the effect of flipping the sign of time, $t \to -t$.[16] So time reversal interchanges the past and the future. If some state $|\Psi_1\rangle$ evolves into state $|\Psi_2\rangle$ after a time interval Δt, then the time-reversed final state $T|\Psi_2\rangle$ will evolve into the time-reversed initial state $T|\Psi_1\rangle$ (after the same time interval Δt) — if time reversal is a symmetry of the dynamics.

As with C and P, time reversal is a symmetry of strong and electromagnetic interactions, but not of weak interactions. However, the product of charge conjugation, parity, and time reversal, or CPT, is a symmetry of all known interactions. In fact, one can prove that any Lorentz invariant theory (which satisfies causality) must be CPT invariant. This is one of the deepest results which follows from combining special relativity and Lorentz invariance, and essentially follows from analytic continuation applied to Lorentz transformations.[17] Note finally that CPT invariance implies that the violation of time reversal invariance is intimately related to CP violation. Considerable effort, including locally at the UW, is now going into directly detecting T violating observables.[18]

8.13 Sample calculations

Let us illustrate the use the technology of the ladder operators to determine the various 3-quark spin wavefunctions for the baryons we discussed in Chapter 7. We start by combining the notation of the raising/lowering operators of Chapter 5 with the arrow notation (for the individual quarks) of Chapter 7 and applying them to the *symmetric* spin $\frac{3}{2}$ state,

$$|\frac{3}{2},\frac{3}{2}\rangle = |\uparrow\uparrow\uparrow\rangle\,. \tag{8.13.1}$$

For a single quark state operated on by the lowering ladder operator we have

$$J_{-,1}|\frac{1}{2},\frac{1}{2}\rangle = \hbar\sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)}|\frac{1}{2},-\frac{1}{2}\rangle = \hbar|\frac{1}{2},-\frac{1}{2}\rangle\,, \tag{8.13.2}$$

[16] Because this transformation changes the meaning of time, it is not represented by a unitary operator which commutes with the Hamiltonian. In fact, unlike all other symmetries discussed so far, time reversal, in quantum mechanics, is not represented by a linear operator, but rather by an "anti-linear" operator. Such operators do not satisfy the defining relation of linear operators, $\mathcal{O}(c_1|\Psi_1\rangle + c_2|\Psi_2\rangle) = c_1\,(\mathcal{O}|\Psi_1\rangle) + c_2\,(\mathcal{O}|\Psi_2\rangle)$. Instead, for anti-linear operators, $\mathcal{O}(c_1|\Psi_1\rangle + c_2|\Psi_2\rangle) = c_1^*\,(\mathcal{O}|\Psi_1\rangle) + c_2^*\,(\mathcal{O}|\Psi_2\rangle)$.

[17] The Wikipedia entry on CPT symmetry has a nice sketch of the proof of the CPT theorem, together with a summary of its history.

[18] Results from the *BaBar* experiment reported in autumn 2012, *Phys. Rev. Lett.* **109**, 211801 (2012), and describing the $B^0\bar{B}^0$ system exhibit the expected correlation between CP and T violation with very good precision.

or

$$J_{-,1}|\uparrow\rangle = \hbar|\downarrow\rangle\,. \tag{8.13.3}$$

To apply to the 3-quark state we just need to note that, although for the total angular momentum we need to understand how to add *representations*, for the components it is just the ordinary (linear, algebraic) sum,

$$J_{3,\text{tot}} = J_{3,1} + J_{3,2} + J_{3,3} \quad \text{and} \quad \mathrm{J}_{\pm,\text{tot}} = \mathrm{J}_{\pm,1} + \mathrm{J}_{\pm,2} + \mathrm{J}_{\pm,3}\,. \tag{8.13.4}$$

The overall normalization is given by Eq. (5.1.33) via

$$J_{-,\text{tot}}|\frac{3}{2},\frac{3}{2}\rangle = \hbar\sqrt{\left(\frac{3}{2}+\frac{3}{2}\right)\left(\frac{3}{2}-\frac{3}{2}+1\right)}|\frac{3}{2},\frac{1}{2}\rangle = \hbar\sqrt{3}|\frac{3}{2},\frac{1}{2}\rangle\,. \tag{8.13.5}$$

Applying the total lowering ladder to the explicit spin state we have

$$J_{-,\text{tot}}|\uparrow\uparrow\uparrow\rangle = J_{-,1}|\uparrow\uparrow\uparrow\rangle + J_{-,2}|\uparrow\uparrow\uparrow\rangle + J_{-,3}|\uparrow\uparrow\uparrow\rangle = \hbar\left(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle\right)\,. \tag{8.13.6}$$

Combining we have the desired result for the normalized spin wavefunction (note that the $\hbar$ factors cancel)

$$|\frac{3}{2},\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}\left(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle\right)\,. \tag{8.13.7}$$

Repeating this process while being explicit about the terms generated and the normalization, we have

$$J_{-,\text{tot}}|\frac{3}{2},\frac{1}{2}\rangle = \hbar\sqrt{\left(\frac{3}{2}+\frac{1}{2}\right)\left(\frac{3}{2}-\frac{1}{2}+1\right)}|\frac{3}{2},-\frac{1}{2}\rangle = 2\hbar|\frac{3}{2},-\frac{1}{2}\rangle \tag{8.13.8}$$

and (recall that the lowering operator yields zero when applied to a spin-down state)

$$\begin{aligned} J_{-,\text{tot}}&\frac{1}{\sqrt{3}}\left(|\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle\right) \\ &= \frac{\hbar}{\sqrt{3}}\left(0 + |\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle + 0 + |\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle + 0\right) \\ &= \frac{2\hbar}{\sqrt{3}}\left(0 + |\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle\right)\,, \end{aligned} \tag{8.13.9}$$

which combine to give the expected result (*i.e.*, the factors of $\hbar$ and 2 cancel),

$$|\frac{3}{2},-\frac{1}{2}\rangle = \frac{1}{\sqrt{3}}\left(|\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle\right)\,. \tag{8.13.10}$$

Finally the last step is given by

$$J_{-,\text{tot}}|\frac{3}{2},-\frac{1}{2}\rangle = \hbar\sqrt{\left(\frac{3}{2}-\frac{1}{2}\right)\left(\frac{3}{2}+\frac{1}{2}+1\right)}|\frac{3}{2},-\frac{3}{2}\rangle = \sqrt{3}\hbar|\frac{3}{2},-\frac{3}{2}\rangle \tag{8.13.11}$$

and

$$\begin{aligned} J_{-,\text{tot}}&\frac{1}{\sqrt{3}}\left(|\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle\right) \\ &= \frac{\hbar}{\sqrt{3}}\left(0 + 0 + |\downarrow\downarrow\downarrow\rangle + 0 + |\downarrow\downarrow\downarrow\rangle + 0 + |\downarrow\downarrow\downarrow\rangle + 0 + 0\right) \\ &= \sqrt{3}\hbar|\downarrow\downarrow\downarrow\rangle\,, \end{aligned} \tag{8.13.12}$$

which combine to give the expected result (*i.e.*, now the factors of $\hbar$ and $\sqrt{3}$ cancel),

$$|\frac{3}{2},-\frac{3}{2}\rangle = |\downarrow\downarrow\downarrow\rangle\,. \tag{8.13.13}$$

Note that each of these states is symmetric under the interchange of any pair of quarks.

Recall from Eq. (7.4.2) that when we combine 3 spin $\frac{1}{2}$ quarks there are two resulting total spin $\frac{1}{2}$ representations, which we distinguished in Eq. (7.4.3) by whether the states were antisymmetric in quarks 1 and 2 (*i.e.*, quarks 1 and 2 have pair spin 0) or symmetric in quarks 1 and 2 (*i.e.*, quarks 1 and 2 have pair spin 1). (Note that the choice here to focus on quarks 1 and 2 is another *arbitrary* labeling choice, and eventually in Chapter 7 there was a sum over all choices in the overall wavefunction.) Further we know that both possibilities must be orthogonal to the corresponding spin $\frac{3}{2}$ wave function in Eq. (8.13.7). In the last Chapter we choose the former case (antisymmetric in 1 and 2, recall we made this *choice* to match the mixed symmetry of the flavor wave functions), which has the normalized wave function

$$|\frac{1}{2},\frac{1}{2}\rangle_{A12,3} = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\right)\,, \tag{8.13.14}$$

which is clearly orthogonal to the wavefunction in Eq. (8.13.7). The spin down version, which can be obtained easily with the lowering operator (and noting the cancelations), is

$$|\frac{1}{2},-\frac{1}{2}\rangle_{A12,3} = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle\right)\,, \tag{8.13.15}$$

which is orthogonal to the wavefunction in Eq. (8.13.10). Finding the other total spin $\frac{1}{2}$ wavefunction takes a bit more thought, but it must be orthogonal to the two we have already defined (and be symmetric in quarks 1 and 2). Except for the question of the overall sign it must look like

$$|\frac{1}{2},\frac{1}{2}\rangle_{S12,3} = \frac{1}{\sqrt{6}}\left(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle - 2|\uparrow\uparrow\downarrow\rangle\right)\,, \tag{8.13.16}$$

which has the correct symmetry properties and is orthogonal to the previous versions. Applying the lowering operator we quickly obtain

$$|\frac{1}{2},-\frac{1}{2}\rangle_{S12,3} = \frac{1}{\sqrt{6}}\left(2|\downarrow\downarrow\uparrow\rangle - |\downarrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\rangle\right)\,. \tag{8.13.17}$$