VI.
Why do samples allow inference? How sure do we have to be? How many do I need to be that sure?

Sampling Distributions, Confidence Intervals, & Sample Size

PBAF 527 Winter 2005

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Today

Theory

Sampling Distributions
- Describe the Properties of Estimators
- Explain Sampling Distribution
- Describe the Relationship between Populations & Sampling Distributions
- State the Central Limit Theorem
- Solve Probability Problems Involving Sampling Distributions

Practice

Tools for Samples
- Making Inference
- Confidence Intervals
  - State What is Estimated
  - Distinguish Point & Interval Estimates
  - Explain Interval Estimates
  - Compute Confidence Interval Estimates for Population Mean & Proportion
- Compute Sample Size

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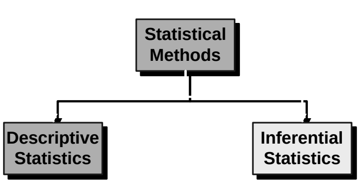
Statistical Methods



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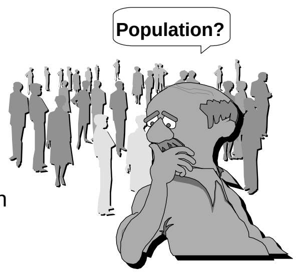
Inferential Statistics

1. Involves:
   - Estimation
   - Hypothesis Testing
2. Purpose
   - Make Decisions about Population Characteristics



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Inference Process

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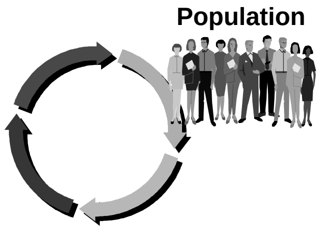
Inference Process



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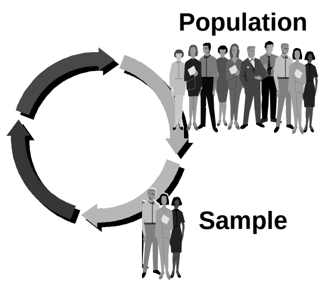
Inference Process



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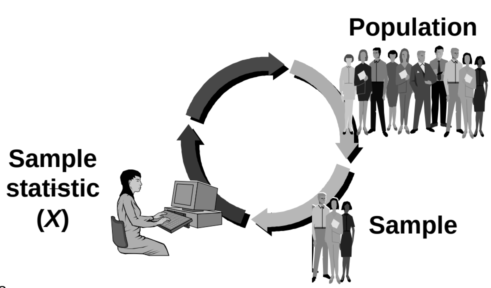
Inference Process



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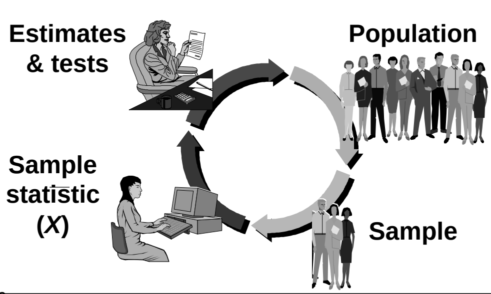
Inference Process



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Non-Probability Sampling

- Cannot tell the probability of choosing each member of the population.
  - Quota Sampling
  - Volunteer Sampling
  - Snowball Sampling

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Probability Sampling

Each population member sampled with known probability.
- ESSENTIAL for a representative sample
- Simple Random Sample (SRS)
- Stratified Random Sample
- Cluster Sample
- Systematic Sample

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Estimators

- Random Variables Used to Estimate a Population Parameter
  - Sample Proportion $Var(\hat{P}) = \frac{P(1-P)}{n}$
    - The mean of the sample p= $\hat{P}$
  - Sample Mean

$$E(\overline{X}) = m \qquad Var(\overline{X}) = \frac{s^2}{n}$$

- Theoretical Basis Is Sampling Distribution

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Sampling Distribution

1. **Theoretical** Probability Distribution
2. Random Variable is **Sample Statistic**
   - Sample Mean, Sample Proportion etc.
3. Results from Drawing **All** Possible Samples of a **Fixed** Size
4. List of All Possible [$\bar{X}$, P($\bar{X}$)] Pairs
   - Sampling Distribution of Mean

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Standard Error of Mean

1. Standard Deviation of All Possible Sample Means, $\bar{x}$
   - How far a typical x-bar is from mu.
2. Less Than Pop. Standard Deviation
3. Formula (Sampling With Replacement)

For a sample proportion

$$SE \equiv SD(\hat{P}) = \sqrt{\frac{p(1-p)}{n}}$$

For a sample mean

$$SE \equiv SD(\bar{X}) = \frac{s}{\sqrt{n}}$$

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Creating a Sampling Distribution

Remember, from a class of 27, there are over 33,000 possible samples of 5.

So, we can take multiple samples from the class

For each sample, we can calculate a mean and a standard deviation.

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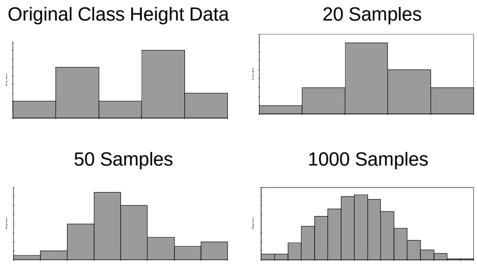
Sampling Distributions



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Sampling Distributions

	Mean	SD($\bar{x}$)
10 Samples	66.6	0.55
20 Samples	67.3	0.41
50 Samples	67.2	0.21
250 Samples	67.0	0.10
1000 Samples	67.0	0.05

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Properties of Sampling Distribution of Mean

1. Unbiasedness
   - Mean of Sampling Distribution Equals Population Mean
2. Efficiency
   - Sample Mean Comes Closer to Population Mean Than Any Other Unbiased Estimator
3. Consistency
   - As Sample Size Increases, Variation of Sample Mean from Population Mean Decreases

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Central Limit Theorem

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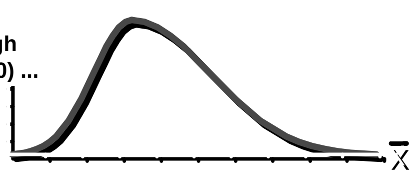
Central Limit Theorem

As sample size gets large enough (n ^{3} 30) ...



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Central Limit Theorem

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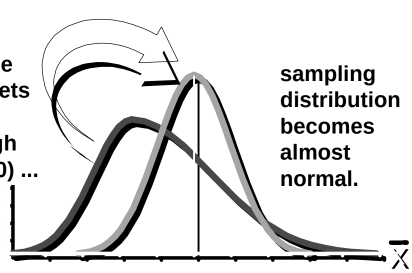

sampling distribution becomes almost normal.

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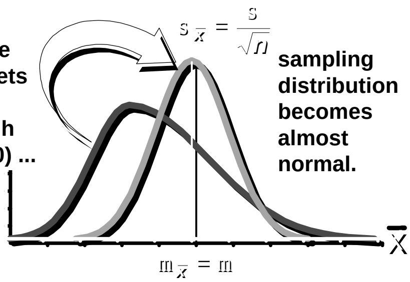
Central Limit Theorem

As sample size gets large enough ($n \geq 30$) ...



sampling distribution becomes almost normal.

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Problem 1

I know from the U.S. Current Population Survey that 1999 household income had a mean of $44,000 and a standard deviation of $62,000. If I were to pick a random sample of 100 U.S. households, how likely am I to get a sample mean of household income more than $1000 above the population mean?

1. Write down the target and draw a picture.
2. Find the standard error.
3. Find the standard score for our sample mean target value.

   SE … 6200

4. Look up the score in the z-table.

   there is a 44% probability of seeing a value of over $45K

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Problem 1 (Part 2)

Suppose instead that we had a random sample of 10,000 U.S. household. What is the chance of getting a sample mean for income more than 1000 above the population mean of $44,000?

1. Find the target and draw picture.

   We want $P(\bar{X} > \$45K)$

2. Find the standard error.

   $\sqrt{n}$ $\sqrt{10,000}$

3. Find the stand…

   $P(\bar{X} > \$45K) = P\left(Z > \frac{\ldots}{SE}\right) = P\left(Z > \frac{\ldots}{620}\right) = P(Z > 1.61)$

4. Look up score in Z tab…

   percent of the time.

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Proportions

Proportions are means of binary variables, so the sampling distribution is approximately normal.

Remember:

- Population Mean≡ P
- Population Variance≡ P(1-P)
- s =√P(1-P)

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Problem 2

In 1999, about 12 % of persons in the U.S. lived in households with income under the poverty level. What's the probability that the poverty rate in a sample of 1000 people will be within 1 percentage point of the "true" proportion?

To find the probability for a sample proportion:

1. Find target and draw picture.
2. Find SE (Need pop SD first).
3. Find Z scores.

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Tools for Samples

- Point Estimate
  - Best guess of a population parameter based upon a sample
- Confidence Interval
  - Range estimate around point estimate
- Hypothesis Test
  - Decision rule for rejecting hypothesized population values (null hypotheses)
- p-value
  - Continuous measure of support for null hypothesis (a probability, α)

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Making Inference

How can we make assertions about the unknown?

Scientific Method:

Theory ⇒	Hypothesis ⇒	Empirical Evidence
(model of how the world works)	(specific statement about the model)	(test of the hypothesis)

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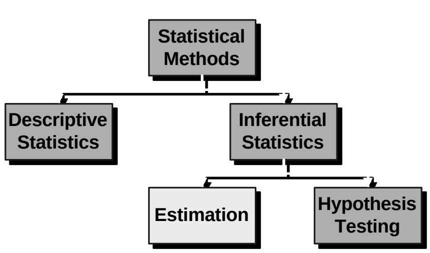
Statistical Methods



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Example: the Polls

In the 1992 Presidential Election, George Bush was expected to receive 38% of the vote, and Bill Clinton 40%. Yet, Bush was hopeful of winning. Why?

THE MARGIN OF ERROR

38% ± 3% = [35%, 41%]

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Example: the Polls

In the 1992 Presidential Election, George Bush was expected to receive 38% of the vote, and Bill Clinton 40%. Yet, Bush was hopeful of winning. Why?

THE MARGIN OF ERROR

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Example: the Polls (2)

In the 2000 Presidential Election, we had a very different situation. By Nov 1, 2000, the polls indicated that George W. Bush could expect to receive 47% and Al Gore 43%. The pundits commented on Bush's slim lead. Why?

THE MARGIN OF ERROR

47% ± 2% = [45%, 49%]

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Estimation Process

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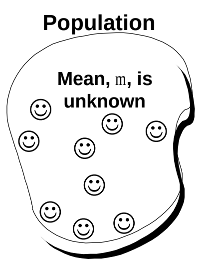
Estimation Process



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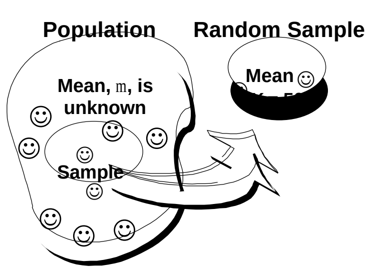
Estimation Process



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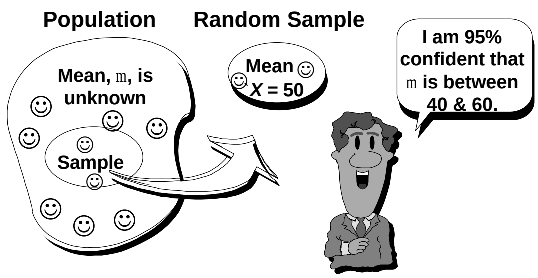
Estimation Process



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Unknown Population Parameters Are Estimated

Estimate Population Parameter...		with Sample Statistic
Mean	m	$\bar{x}$
Proportion	p	$\hat{p}$
Differences	$m_1 - m_2$	$\bar{x}_1 - \bar{x}_2$

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Estimation Methods

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Estimation Methods

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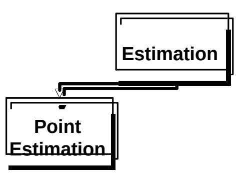
Estimation Methods



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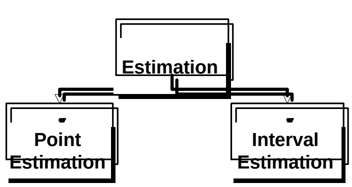
Estimation Methods



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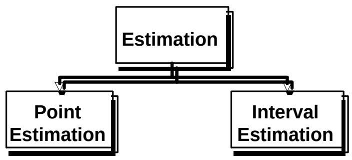
Estimation Methods



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Point Estimation

1. Provides Single Value
   - Based on Observations from 1 Sample
2. Gives No Information about How Close Value Is to the Unknown Population Parameter
3. Example: Sample Mean $\bar{X}$ = 3 Is Point Estimate of Unknown Population Mean

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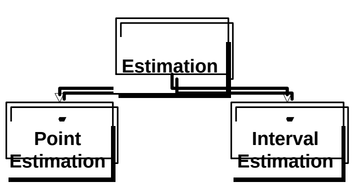
Estimation Methods



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Interval Estimation

1. Provides Range of Values
   - Based on Observations from 1 Sample
2. Gives Information about Closeness to Unknown Population Parameter
   - Stated in terms of Probability
     - Knowing Exact Closeness Requires Knowing Unknown Population Parameter
3. Example: Unknown Population Mean Lies Between 50 & 70 with 95% Confidence

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Key Elements of Interval Estimation

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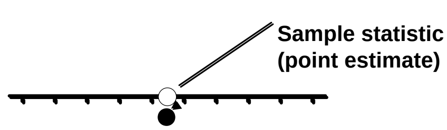
Key Elements of Interval Estimation



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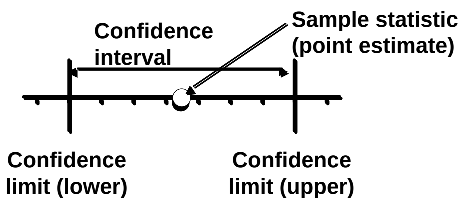
Key Elements of Interval Estimation



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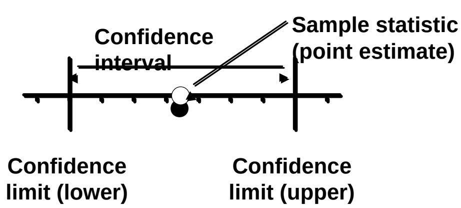
Key Elements of Interval Estimation

A probability that the population parameter falls somewhere within the interval.



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Confidence Limits for Population Mean

(1) $m = \bar{X} \pm Error$

(2) $Error = \bar{X} - m$ or $\bar{X} + m$

(3) $Z = \frac{\bar{X} - m}{s_{\bar{x}}} = \frac{Error}{s_{\bar{x}}}$

(4) $Error = Zs_{\bar{x}}$

(5) $m = \bar{X} \pm Zs_{\bar{x}}$

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Many Samples Have Same Interval

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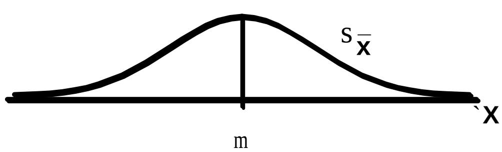
Many Samples Have Same Interval



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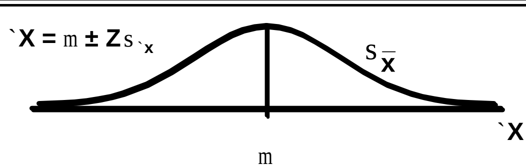

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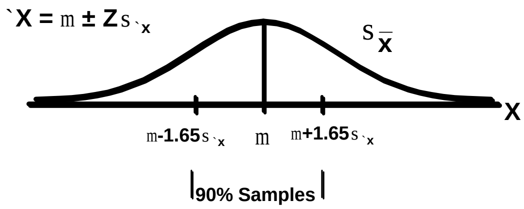

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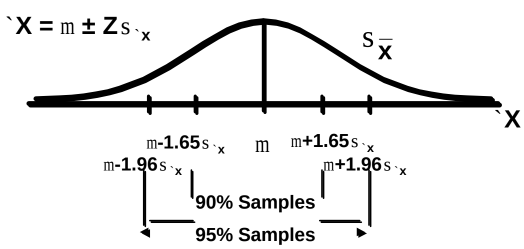

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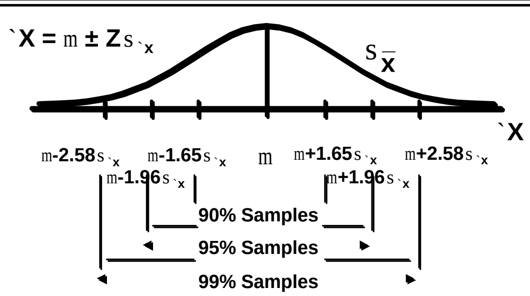

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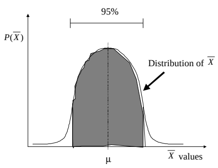
Sampling Distribution Gives the Probability of Sample Mean Close to μ.

What's the probability of getting a sample mean within 1.96 standard errors of the population mean?



$P(\mu-1.96SE) < \overline{X} < P(\mu + 1.96SE)$

$$= P\left[\frac{(m-1.96SE)-m}{SE} < Z < \frac{(m+1.96SE)-m}{SE}\right]$$

=1-.025-.025=.4750+.4750=.95

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Confidence Intervals Explained: 95% CI

The interval from xbar-1.96*SE* to xbar+1.96*SE* should contain the population mean, μ, 95% of the time. Xbar is the mean of one sample.

In other words, 95% of the time when we take a sample of size n and take its mean, the population mean, μ, will fall within ±1.96*SE*'s around xbar. That is, in 95 of 100 samples.

We call the interval xbar ± 1.96*SE* a 95% confidence interval for the unknown population mean, μ.

μ either lies inside the confidence interval or it lies outside the confidence interval.

What we know is that 95% of all possible intervals constructed around all of the xbar (that we calculate from random samples of size n) will contain μ.

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Confidence Interval Example

U.S. household income has a standard deviation of $62,000. Suppose you took a random sample of 100 households and got a sample mean of $40,000. What range estimate could you construct that will have a 95% chance of including the unknown population mean?

Want 95% confidence interval (with known s):

P(xbar- 1.96SE < μ < xbar+ 1.96SE)= .95

1.Get standard error $SE = \frac{s}{\sqrt{n}} = \frac{\$62,000}{\sqrt{100}} = 6200$

2. Plug in SE and sample mean:

P(xbar- 1.96SE < μ < xbar + 1.96SE)= .95

=P[40,000 -(1.96)(6200) < μ < 40,000 + (1.96)(6200)]

=P[27,848 < μ < 52,152]=.95 $\overline{X} \pm 1.96\frac{s}{\sqrt{n}}$=[27848,52152]

▪So, 95 percent of the time, the household mean for U.S. households is within the range $27,848 and $52,152.

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Confidence Level

1. Probability that the Unknown Population Parameter Falls Within Interval
2. Denoted (1 - α) %
   - a Is Probability That Parameter Is **Not** Within Interval (the error)
3. Typical Values Are 99%, 95%, 90%

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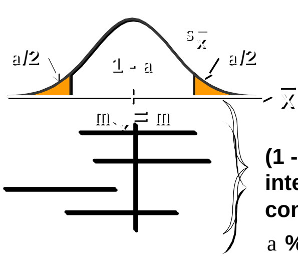
Intervals & Confidence Level

Sampling Distribution of Mean



Intervals extend from `X - Zs`x to `X + Zs`x

(1 - a) % of intervals contain m.

a % do not.

Large number of intervals

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Steps to Confidence Intervals

1. Decide what percentage of the probability you want inside the interval.
2. Take the percentage outside the interval and label it α
3. Calculate $\alpha/2$, the percentage of the probability in one of the tails outside the interval.
4. Look up the value of $z_{\alpha/2}$, the z value that cuts off a right tail area of $\alpha/2$, under the standard normal curve.

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Common Values of $z_{?/2}$

Confidence Level	α=error probability	$\alpha/2$=one tail area	$z_{\alpha/2}$=critical value
90%	0.10	0.05	1.645
95%	0.05	0.025	1.96
98%	0.02	0.01	2.326
99%	0.01	0.005	2.576

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An aside about the Empirical Rule and Common Confidence Intervals

SD on either side of the mean	Empirical Rule for Mound-Shaped Distributions	Probabilities Associated with ANY Normal Variable
1	Approximately 68%	.6826
2	Approximately 95%	.9544
3	Almost all	.9974

Estimated Probabilities

SD on either side of the mean	Probabilities Associated with ANY Normal Variable
1.645	.90
1.96	.95
2.575	.99

Exact Probabilities

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Factors Affecting Interval Width

1. Data Dispersion
   - Measured by σ
2. Sample Size
   - $\sigma_{\bar{X}} = \sigma / \sqrt{n}$
3. Level of Confidence $(1 - \alpha)$
   - Affects Z

Intervals Extend from $\bar{X} - Z\sigma_{\bar{X}}$ to $\bar{X} + Z\sigma_{\bar{X}}$

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Confidence Interval Estimates

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Confidence Interval Estimates

Confidence Intervals

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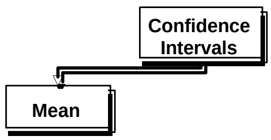
Confidence Interval Estimates



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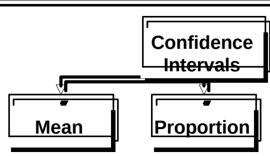

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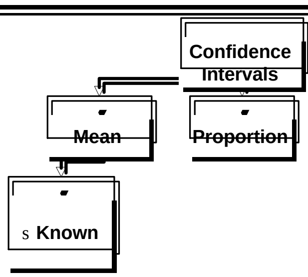
Confidence Interval Estimates



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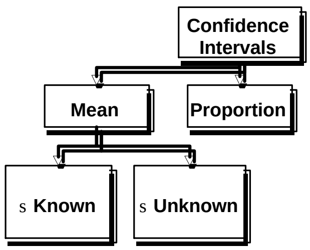
Confidence Interval Estimates



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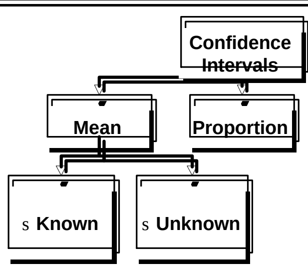
Confidence Interval Estimates



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Confidence Interval Mean (s Known)

1. Assumptions
   - Population Standard Deviation Is Known
   - Population Is Normally Distributed
   - If Not Normal, Can Be Approximated by Normal Distribution ($n \geq 30$)
2. Confidence Interval Estimate

$$\bar{X} - Z_{a/2} \cdot \frac{s}{\sqrt{n}} \leq m \leq \bar{X} + Z_{a/2} \cdot \frac{s}{\sqrt{n}}$$

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Estimation Example Mean (σ Known)

The mean of a random sample of *n* = 25 is $\bar{X}$ = 50. Set up a 95% confidence interval estimate for μ if σ = 10.

$$\bar{X} - Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

$$50 - 1.96 \cdot \frac{10}{\sqrt{25}} \le \mu \le 50 + 1.96 \cdot \frac{10}{\sqrt{25}}$$

$$46.08 \le \mu \le 53.92$$

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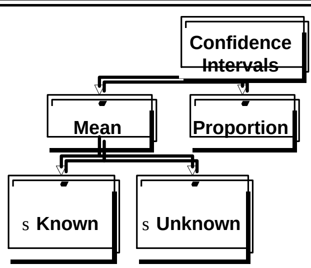
Confidence Interval Estimates



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Confidence Interval Mean (σ Unknown)

1. Assumptions
   - Population Standard Deviation Is Unknown
   - Population Must Be **Normally Distributed**
2. Estimate SE $\frac{S}{\sqrt{n}}$
3. Use Student's t Distribution
4. Confidence Interval Estimate

$$\bar{X} - t_{\alpha/2, n-1} \cdot \frac{S}{\sqrt{n}} \le \mu \le \bar{X} + t_{\alpha/2, n-1} \cdot \frac{S}{\sqrt{n}}$$

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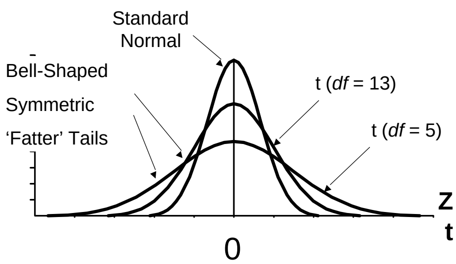
Student's t Distribution



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Student's t Table

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Student's t Table

v	$t_{.10}$	$t_{.05}$	$t_{.025}$
1	3.078	6.314	12.706
2	1.886	2.920	4.303
3	1.638	2.353	3.182

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Student's t Table

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t values

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Student's t Table

a / 2

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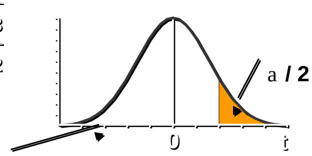

t values

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Student's t Table

a / 2

Assume:
n = 3
df = *n* - 1 = 2
a = .10
a/2 =.05

v	$t_{.10}$	$t_{.05}$	$t_{.025}$
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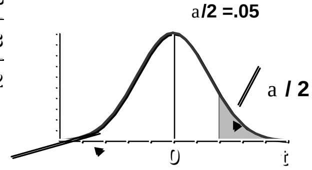

t values

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Student's t Table

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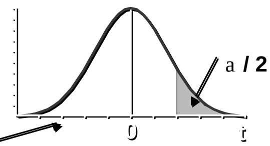

t values

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Student's t Table

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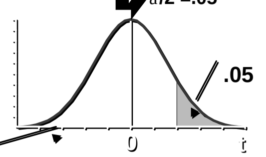

t values

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Student's t Table

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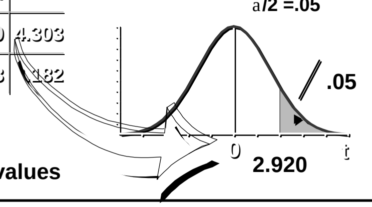

t values

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Degrees of Freedom (*df*)

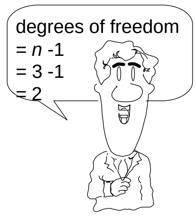
1. Number of Observations that Are Free to Vary After Sample Statistic Has Been Calculated
2. Example
   - Sum of 3 Numbers Is 6

     X_1 = 1 (or Any Number)

     X_2 = 2 (or Any Number)

     X_3 = **3** (Cannot Vary)

     Sum = 6



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Estimation Example Mean (s Unknown)

Suppose my sample of 100 households has a mean income of $40,000 and a standard deviation (s) of $59,000. What is the 95% confidence interval?

Want 95% confidence interval (with σ unknown):

$$P\left(\overline{X} - t_{.025}\hat{S}E < m < \overline{X} + t_{.025}\hat{S}E\right) = .95$$

(α=.05, so α/2=.025, or 2.5% in each tail)

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Estimation Example Mean (s Unknown) (2)

1. Find SE estimate. $\hat{S}E = \frac{s}{\sqrt{n}} = \frac{59{,}000}{\sqrt{100}} = 5900$

2. Find t-score for 95% CI with 2.5% in each tail.

degrees of freedom = n-1 = 100-1= 99

⇒$t_{.025}$= 2.00 2t rule of thumb

3. Plug in t-score, sample mean, and estimated sample SE:

$$P\left(\overline{X} - 2\hat{S}E < m < \overline{X} + 2\hat{S}E\right) = .95$$
$$= P\left[(40{,}000 - 2.00(5900)) < m < (40{,}000 + 2.00(5900))\right] = .95$$
$$= P\left[28{,}200 < m < 51{,}800\right] = .95$$

So, 95 percent of the time, the household mean for U.S. households is within the range $28,200 and $51,800.

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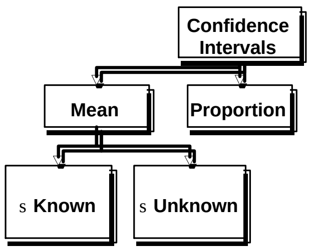
Confidence Interval Estimates



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Confidence Interval Proportion

1. Assumptions
   - Two Categorical Outcomes
   - Large sample
     - np and n(1-p) are greater than about 5 or 10.
   - Normal Approximation Can Be Used
2. Formula: a large sample (1-?)100% confidence interval for the population proportion, p:

$$\hat{p} \pm z_{a/2}\sqrt{\frac{\hat{p}\hat{q}}{n}} \qquad \hat{q} = 1 - \hat{p}$$

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Estimation Example Proportion

Suppose that in my random sample of 100 households I find that 58% would be willing to pay more than $100 per year in taxes to offset the national deficit. What is the 99% confidence interval for that proportion?

$$P\left(\hat{p} - z_{.005}\hat{S}E < p < \hat{p} + z_{.005}\hat{S}E\right) = .99$$

1. Find SE Estimate. For proportion $s = \sqrt{\hat{p}\hat{q}} = \sqrt{.58(1-.58)} = .49$

2. Find z-score of 99% CI. ?=.01, so ?/2=.005, or .5% in each tail
   $z_{.005}$=2.58

3. Plug in $\hat{S}E$, $\hat{p}$, z.

$$P\left(\hat{p} - 2.58\hat{S}E < p < \hat{p} + 2.58\hat{S}E\right) = .99$$
$$= P(.58 - 2.58(.049) < p < .58 + 2.58(.049)) = .99$$
$$= P(.45 < p < .71) = .99$$

93 99% of the time, the proportion will fall between .45 and .71.

A final word on confidence intervals

So, **confidence intervals** give range estimates of all values close enough for us to consider as possible population means or proportions.

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How many do I need to be that sure?

Determining Sample Size

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To determine sample size…

...You need to know something about your population!

You also need to think about the quality of the estimate:

- Close do you want to be to the unknown parameter? (B)
- Confidence level? (z)
- Estimate of Variance? (SD)

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Sample Size Formulas

Estimating a population mean $n = \frac{z_{a/2}^2 s^2}{B^2}$

Estimating a population proportion $n = \frac{z_{a/2}^2 pq}{B^2}$

Need some prior experience to estimate sample size—or conservatism. p=.05 will yield the largest sample size for a proportion

These are the minimum sample sizes for SRS (with replacement) where you only want to estimate a mean or proportion! They do not take **population size** into account.

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Finding Sample Sizes for Estimating m

(1) $Z = \frac{\overline{X} - m}{s_{\overline{x}}} = \frac{Error}{s_{\overline{x}}}$

(2) $Error = Zs_{\overline{x}} = Z\frac{s}{\sqrt{n}}$

(3) $n = \frac{Z^2 s^2}{Error^2}$

Error Is Also Called Bound, *B*

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Sample Size Example

What if I was going to draw my sample of households from the population? I know that the standard deviation of household income in the U.S. is $62,000. What sample size do I need to estimate mean income with 95% confidence to within plus or minus $500?

1. Find B: B is the half-width bound, so B=500
2. What z? At 95% confidence, z=1.96
3. Variance estimate? σ=62000, so σ^2=3,844,000,000
4. Plug in B, z, and σ^2 (or its estimate) into the equation

$$n = \frac{z_{a/2}^2 s^2}{B^2} = \frac{(1.96)^2(62000)^2}{500^2} = 59{,}068$$

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Sample Size Example (2)

What if I only wanted to come within $1000 of the mean?

$$n = \frac{z_{a/2}^2 s^2}{B^2} = \frac{(1.96)^2(62000)^2}{1000^2} = 14{,}767$$

How about $5000 of the mean?

$$n = \frac{z_{a/2}^2 s^2}{B^2} = \frac{(1.96)^2(62000)^2}{5000^2} = 591$$

At a larger distance, a smaller sample size is needed.

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Correcting for Without Replacement Sampling

- SRS_{WOR} in reality!
- Also correct for population size
  - Important for small populations
- Formula

$$n_{WOR} = \frac{n}{1 + \frac{n}{N}}$$

Where
  - n_{WOR} is the sample size needed for without replacement sampling
  - n is the sample size needed for with replacement sampling (equation in book)
  - N is the population size.

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Today

Theory	Practice

Sampling Distributions
- Describe the Properties of Estimators
- Explain Sampling Distribution
- Describe the Relationship between Populations & Sampling Distributions
- State the Central Limit Theorem
- Solve Probability Problems Involving Sampling Distributions

Tools for Samples
- Making Inference
- Confidence Intervals
  - State What is Estimated
  - Distinguish Point & Interval Estimates
  - Explain Interval Estimates
  - Compute Confidence Interval Estimates for Population Mean & Proportion
- Compute Sample Size

102 Next Time: Hypothesis Testing

End of Chapter

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