Derivation of energy balance model

GHR Oct9th 07

See in-class sketch

Definitions

Earth radius, *a.*
Latitude, φ.
Length of latitude belt, $2\pi a\cos\varphi$.
Width of latitude belt, $a\, d\varphi$.
Area of latitude belt, dA = $2\pi a^2\cos\varphi\, d\varphi$.
y is the meridional (north-south) coordinate, and is related to latitude via $y=a\varphi$.

Meridional heat flux

Let the flux per unit length of longitude = $\tilde{F} = -D\dfrac{dT}{dy} = -\dfrac{D}{a}\dfrac{dT}{dy}$. Note that the heat flux is from hot to cold (makes sense), and increases in magnitude if the meridional temperature gradient increases. Therefore the total flux past latitude φ, is given by:

$$F = \tilde{F} \times 2\pi a\cos\varphi = -2\pi D\cos\varphi\frac{dT}{d\varphi}.$$

Energy balance

Assuming an energy balance in equilibrium, then

Stuff in = stuff out

From the sketch this means that

$$sw\downarrow \cdot dA + F\big|_{\varphi} = lw\uparrow \cdot dA + F\big|_{\varphi+d\varphi}$$

Note that the radiation fluxes are per unit area and so need to be multiplies by the area of the latitude belt. Rearranging and substituting for dA:

$$(sw\downarrow - lw\uparrow)\cdot 2\pi a^2\cos\varphi\; d\varphi = F\big|_{\varphi+d\varphi} - F\big|_{\varphi}$$

Using our first order Taylor series trick $F\big|_{\varphi+d\varphi} = dF/d\varphi\big|_{d\varphi}\, d\varphi$, we get

$$(sw\downarrow - lw\uparrow)\cdot 2\pi a^2 \cos\varphi \, d\varphi = \left.\frac{dF}{d\varphi}\right|_{\varphi} d\varphi$$

and hence on rearranging and canceling the $d\varphi$s:

$$sw\downarrow - lw\uparrow = \frac{1}{2\pi a^2 \cos\varphi}\frac{dF}{d\varphi}$$

This is the energy balance equation solved by the numerical model.

Details of the radiation

Shortwave radiation
We need to account for the distribution of solar radiation as a function of latitude

$$sw\downarrow = \frac{Q_0}{4}(1-\alpha)S(\varphi),$$

$S(\varphi) = c_1 - c_2\sin^2\varphi$, with c_1=1.2 and c_2=0.7. See in-class sketch.

Longwave radiation
We linearize the longwave radiation. See in-class figure. To a fair approximation, outgoing longwave radiation can be parameterized as a linear function of temperature

$$lw\downarrow = A + BT,$$

where T is the surface temperature; A = 203 W m^{-2}; and B = 2.1 W m^{-2} ^{o}C^{-1}.

Further rearrangement of energy balance equation

We can rearrange the model with a few more steps. Substituting expressions for $sw\uparrow$, $lw\downarrow$, and F gives:

$$\frac{Q_0}{4}(1-\alpha)S(\varphi) - (A+BT) = \frac{-1}{2\pi a^2 \cos\varphi}\frac{d}{d\varphi}\left(2\pi D \cos\varphi \frac{dT}{d\varphi}\right)$$

Using the transformation of variables x=sinφ, and that

$$\frac{d}{d\varphi} = \frac{dx}{d\varphi}\frac{d}{dx} = \cos\varphi \frac{d}{dx},$$

which gives

$$\frac{Q_0}{4}(1-\alpha)S(\varphi)-(A+BT)=-\frac{D}{a^2}\frac{d}{dx}\left(\cos^2\varphi\frac{dT}{dx}\right).$$

Or finally

$$\frac{Q_0}{4}(1-\alpha)S(\varphi)-(A+BT)+\frac{D}{a^2}\frac{d}{dx}\left((1-x^2)\frac{dT}{dx}\right)=0\,.$$

This is a second order, ordinary differential equation for the annual-mean temperature as a function of latitude, given a specified distribution of forcing (i.e., the solar radiation). There are three terms in the energy balance. The net absorbed shortwave (solar) radiation, the net outgoing longwave (terrestrial) radiation, and the convergence of the poleward heat flux due to atmospheric and oceanic circulations transporting heat from hot places to cold places. At each latitude, and in the annual mean, a balance must be achieved among these three terms.

Lastly the climate model includes an albedo feedback. The albedo is specified as a function of temperature. If T < -10 ^{o}C, then α = 0.6, and if T >= -10 ^{o}C, then α = 0.3. Thus for cold annual mean temperatures snow is assumed to be on the ground and reflectivity is high. For warm temperatures the surface is ice-free, and the ground is consequently darker.

That's it. The numerical code solves the equation and spits out the annual mean temperature as a function of latitude, and also each of the three terms in the energy balance.