

# OCN-ATM-ESS 587

## Simple and basic dynamical ideas.....

- **Newton's Laws**
- **Pressure and hydrostatic balance**
- **The Coriolis effect**
- **Geostrophic balance**
- **Lagrangian-Eulerian coordinate frames**

# Coupled Ocean- Atmosphere Systems

$$\sigma = z_{\text{top}} \left( \frac{z - z_{\text{tfc}}}{z_{\text{top}} - z_{\text{tfc}}} \right), \quad (1)$$

where  $z_{\text{top}}$  is the depth of the model domain and  $z_{\text{tfc}}$  is the terrain height. The complete set of the prognostic equations is as follows:

$$\begin{aligned} \frac{\partial u}{\partial t} + C_p \theta_v \left( \frac{\partial \pi'}{\partial x} + G_x \frac{\partial \pi'}{\partial \sigma} \right) + K_D \left( \frac{\partial D_3}{\partial x} + G_x \frac{\partial D_3}{\partial \sigma} \right) \\ = -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} - \sigma \frac{\partial u}{\partial \sigma} + f_v + D_u + K_H \nabla^4 u \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{\partial v}{\partial t} + C_p \theta_v \left( \frac{\partial \pi'}{\partial y} + G_y \frac{\partial \pi'}{\partial \sigma} \right) + K_D \left( \frac{\partial D_3}{\partial y} + G_y \frac{\partial D_3}{\partial \sigma} \right) \\ = -u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} - \sigma \frac{\partial v}{\partial \sigma} - f_u + D_v + K_H \nabla^4 v \end{aligned} \quad (3)$$

$$\begin{aligned} \frac{\partial w}{\partial t} + C_p \theta_v G_z \frac{\partial \pi'}{\partial \sigma} + K_D G_z \frac{\partial D_3}{\partial \sigma} \\ = g \left( \frac{\theta'}{\theta} + 0.608 q'_v - q_c - q_r - q_i - q_s \right) \\ - u \frac{\partial w}{\partial x} - v \frac{\partial w}{\partial y} - \sigma \frac{\partial w}{\partial \sigma} + D_w + K_H \nabla^4 w \end{aligned} \quad (4)$$

$$\begin{aligned} \frac{\partial \pi'}{\partial t} + \frac{c^2}{C_p \bar{\rho} \theta_v^2} (D_3) = -u \frac{\partial \pi'}{\partial x} - v \frac{\partial \pi'}{\partial y} - \sigma \frac{\partial \pi'}{\partial \sigma} \\ - \frac{R_d \bar{\pi}}{c_v} \nabla_3 \cdot \mathbf{V} + \frac{c^2}{C_p \theta_v^2} \frac{d\theta_v}{dt} \end{aligned} \quad (5)$$

$$\begin{aligned} \frac{\partial \theta}{\partial t} = -u \frac{\partial \theta}{\partial x} - v \frac{\partial \theta}{\partial y} - \sigma \frac{\partial \theta}{\partial \sigma} + \frac{Q_\theta}{\bar{\rho}} + D_\theta \\ + K_H \nabla^4 (\theta - \theta_{\text{atd}}) \end{aligned} \quad (6)$$

$$\begin{aligned} \frac{\partial e}{\partial t} = -u \frac{\partial e}{\partial x} - v \frac{\partial e}{\partial y} - \sigma \frac{\partial e}{\partial \sigma} + \text{BP} + \text{SP} \\ + D_e + K_H \nabla^4 e - \epsilon \end{aligned} \quad (7)$$

$$\begin{aligned} \frac{\partial q_v}{\partial t} = -u \frac{\partial q_v}{\partial x} - v \frac{\partial q_v}{\partial y} - \sigma \frac{\partial q_v}{\partial \sigma} + D_{q_v} \\ + K_H \nabla^4 (q_v - q_v^*) + \frac{S_v}{\rho} \end{aligned} \quad (8)$$

$$\begin{aligned} \frac{\partial q_c}{\partial t} = -u \frac{\partial q_c}{\partial x} - v \frac{\partial q_c}{\partial y} - \sigma \frac{\partial q_c}{\partial \sigma} + D_{q_c} \\ + K_H \nabla^4 q_c + \frac{S_c}{\rho} \end{aligned} \quad (9)$$

$$\begin{aligned} \frac{\partial q_r}{\partial t} = -u \frac{\partial q_r}{\partial x} - v \frac{\partial q_r}{\partial y} - \sigma \frac{\partial q_r}{\partial \sigma} + \frac{G_z}{\bar{\rho}} \frac{\partial}{\partial \sigma} (\bar{\rho} V_z q_r) \\ + D_{q_r} + K_H \nabla^4 q_r + \frac{S_r}{\rho} \end{aligned} \quad (10)$$

$$\begin{aligned} \frac{\partial q_i}{\partial t} = -u \frac{\partial q_i}{\partial x} - v \frac{\partial q_i}{\partial y} - \sigma \frac{\partial q_i}{\partial \sigma} + D_{q_i} \\ + K_H \nabla^4 q_i + \frac{S_i}{\rho} \end{aligned} \quad (11)$$

$$\begin{aligned} \frac{\partial q_s}{\partial t} = -u \frac{\partial q_s}{\partial x} - v \frac{\partial q_s}{\partial y} - \sigma \frac{\partial q_s}{\partial \sigma} + \frac{G_z}{\bar{\rho}} \frac{\partial}{\partial \sigma} (\bar{\rho} V_z q_s) \\ + D_{q_s} + K_H \nabla^4 q_s + \frac{S_s}{\rho} \end{aligned} \quad (12)$$

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t} = \left( -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} - w \frac{\partial u}{\partial z} + f_v + D_u + K_H \nabla^4 u \right) \\ - \frac{1}{\rho_0} \frac{\partial \bar{p}_r}{\partial x} \end{aligned} \quad (33)$$

$$\begin{aligned} \frac{\partial \bar{v}}{\partial t} = \left( -u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} - w \frac{\partial v}{\partial z} - f_u + D_v + K_H \nabla^4 v \right) \\ - \frac{1}{\rho_0} \frac{\partial \bar{p}_r}{\partial y} \end{aligned} \quad (35)$$

$$\frac{\partial p}{\partial z} = -g \rho$$

$$\begin{aligned} \frac{\partial v'}{\partial t} = \left( -u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} - w \frac{\partial v}{\partial z} - f_u + D_v + K_H \nabla^4 v \right) \\ - \frac{1}{\rho_0} \frac{\partial p'_r}{\partial y} \end{aligned} \quad (36)$$

$$\frac{\partial T}{\partial t} = -u \frac{\partial T}{\partial x} - v \frac{\partial T}{\partial y} - w \frac{\partial T}{\partial z} + D_T + K_H \nabla^4 T \quad (37)$$

$$\frac{\partial S}{\partial t} = -u \frac{\partial S}{\partial x} - v \frac{\partial S}{\partial y} - w \frac{\partial S}{\partial z} + D_S + K_H \nabla^4 S \quad (38)$$

$$\frac{\partial h}{\partial t} = -\frac{\partial(hu_x)}{\partial x} - \frac{\partial(hv_x)}{\partial y} - H \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \quad (39)$$

$$\rho = f_\rho(T, S, p), \quad (40)$$

$$D_u = \overline{w'u'} = -l S_m q \frac{\partial u}{\partial z} = -K_{om} \frac{\partial u}{\partial z} \quad (45)$$

$$D_v = \overline{w'v'} = -l S_m q \frac{\partial v}{\partial z} = -K_{om} \frac{\partial v}{\partial z} \quad (46)$$

$$D_T = \overline{w'T'} = -l S_m q \frac{\partial T}{\partial z} = -K_{om} \frac{\partial T}{\partial z} \quad (47)$$

$$D_S = \overline{w'S'} = -l S_m q \frac{\partial S}{\partial z} = -K_{om} \frac{\partial S}{\partial z}, \quad (48)$$

**Coupled Ocean-  
Atmosphere  
Systems  
(simplified)**

$$\frac{\partial u}{\partial t} + C_p \theta_v \left( \frac{\partial \pi'}{\partial x} + G_x \frac{\partial \pi'}{\partial \sigma} \right) + K_v \left( \frac{\partial D_3}{\partial x} + G_x \frac{\partial D_3}{\partial \sigma} \right) = -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} - \sigma \frac{\partial u}{\partial \sigma} + f_v + D_u + K_H \nabla^4 u \quad (2)$$

$$\frac{\partial v}{\partial t} + C_p \theta_v \left( \frac{\partial \pi'}{\partial y} + G_y \frac{\partial \pi'}{\partial \sigma} \right) + K_v \left( \frac{\partial D_3}{\partial y} + G_y \frac{\partial D_3}{\partial \sigma} \right) = -u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} - \sigma \frac{\partial v}{\partial \sigma} - f_u + D_v + K_H \nabla^4 v \quad (3)$$

$$\frac{\partial w}{\partial t} + C_p \theta_v G_z \frac{\partial \pi'}{\partial \sigma} + K_D G_z \frac{\partial D_3}{\partial \sigma} = g \left( \frac{\theta'}{\theta} + 0.608 q'_v - q_c - q_r - q_i - q_s \right)$$

$$\frac{\partial \bar{u}}{\partial t} = \left( -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} - w \frac{\partial u}{\partial z} + f_v + D_u + K_H \nabla^4 u \right) - \frac{1}{\rho_0} \frac{\partial p_r}{\partial x} \quad (33)$$

$$\frac{\partial \bar{v}}{\partial t} = \left( -u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} - w \frac{\partial v}{\partial z} - f_u + D_v + K_H \nabla^4 v \right) - \frac{1}{\rho_0} \frac{\partial p_r}{\partial y} \quad (35)$$

$$\frac{\partial p}{\partial z} = -g \rho$$

**Hydrostatic balance (vertical motion)**  
**Coriolis effects (horizontal motion)**

## Newton's Laws....

$$\sum \underline{\underline{F}} = m \underline{\underline{a}} = m \underline{\underline{\ddot{x}}}$$

[a 2nd order vector ODE or PDE;  
actually 3 scalar ODE/PDEs ]

where  $\underline{\underline{x}} = (x, y, z)$  ;  $\underline{\underline{F}} = (F_x, F_y, F_z)$

the 3 ODEs are:

$$\sum F_x = m \ddot{x}$$
$$\sum F_y = m \ddot{y}$$
$$\sum F_z = m \ddot{z}$$

note notation variants:  $\underline{\underline{F}} = \vec{F}$

## Properties of an atmosphere and ocean at rest:

Note that gases and liquids are *fluids*. Fluids differ from solids in that fluids at rest cannot support a shearing stress....the distinctive property of fluids.

A fluid at rest cannot sustain a tangential force...the fluid would simply glide over itself in layers in such a case.

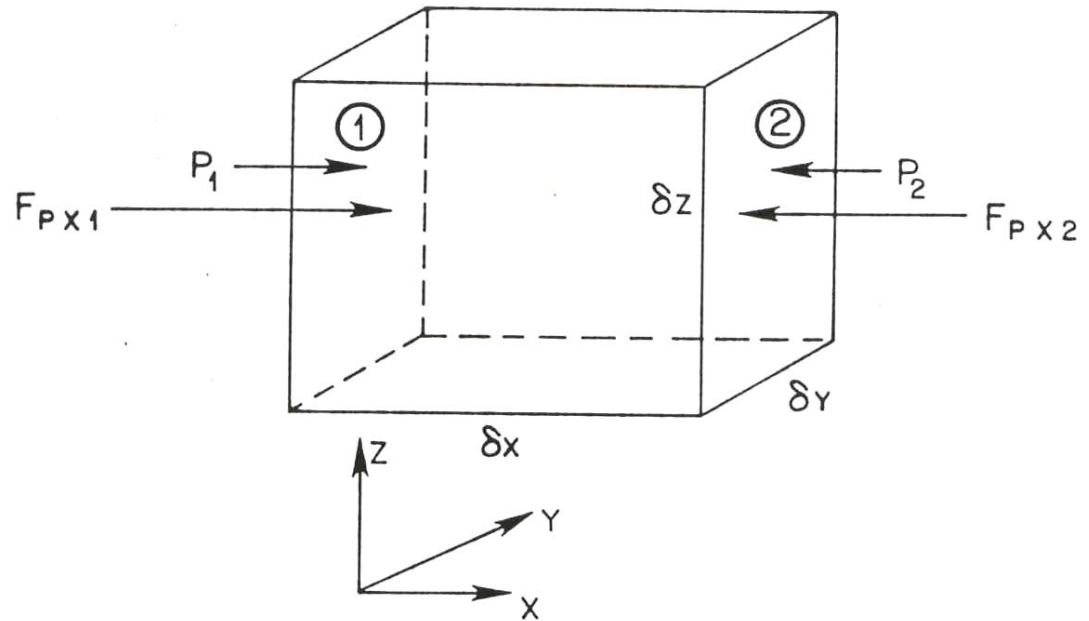
At rest, fluids can only sustain forces normal to their shape (a container, for example). *Pressure* is the magnitude of such a normal force, normalized by the area of the fluid normal to the force.

pressure = normal force / unit area  
[pressure is a scalar]

$$p = \vec{F} \cdot \vec{n} / A$$

## The pressure cube...

Let  $F_p$  be the vector force per unit volume, in order to take the size of the cube out of the problem:



$$\sum F_{px} = F_{px1} - F_{px2} = \{p_1 \delta y \delta z - p_2 \delta y \delta z\} / (\delta x \delta y \delta z)$$

$$\sum F_{px} = (p_1 - p_2) / \delta x = -\frac{\delta p}{\delta x}$$

**Note:**

$$\lim_{\delta x \rightarrow 0} \frac{\delta p}{\delta x} \rightarrow \frac{\partial p}{\partial x}$$

**The force/cube equations for all of the faces of the cube can be combined to yield the vector equation**

$$\vec{F}_p = \left\{ -\frac{\partial p}{\partial x}, -\frac{\partial p}{\partial y}, -\frac{\partial p}{\partial z} \right\} = -\nabla p$$

**Suppose the force is redefined on a per unit mass basis; then**

$$\vec{F}_m = \left\{ -\frac{1}{\rho} \frac{\partial p}{\partial x}, -\frac{1}{\rho} \frac{\partial p}{\partial y}, -\frac{1}{\rho} \frac{\partial p}{\partial z} \right\} = -\frac{1}{\rho} \nabla p$$

**This equation says, simply, that the net pressure force acting on a fluid element is given by the pressure gradient.**

## Pressure, continued.....

If only pressure forces are acting on an element of fluid (air or seawater), then due to Newton's Laws the element must be accelerating. If the pressure forces are balanced by some other force, then the fluid element can be in equilibrium (ie, at rest).

What other forces are there?

Consider gravity...

$$\vec{F}_{mg} = -mg \hat{k} \quad ; \quad \vec{F}_g = -g \hat{k}$$



If the pressure force is balanced by the force of gravity, then

$$\sum \vec{F} = 0 \quad \Rightarrow \quad \vec{F}_m + \vec{F}_g = 0 \quad \text{or,}$$

[horizontal equations]

$$-\frac{1}{\rho} \nabla p - g \hat{k} = 0 \quad \Rightarrow \quad -\frac{1}{\rho} \frac{\partial p}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial y} = 0$$

[vertical equation]

$$-\frac{1}{\rho} \frac{\partial p}{\partial z} = g \quad \Rightarrow \quad \frac{\partial p}{\partial z} = -\rho g$$

*the hydrostatic relation*

The hydrostatic relation....

$$\frac{\partial p}{\partial z} = -g\rho \quad , \quad \text{which can be easily integrated.}$$

Oceanic case:

$$p(z) = -g\rho z + p(0) \quad \text{assuming } \rho = \text{constant}$$

If we define the pressure at the sea surface to be zero, then

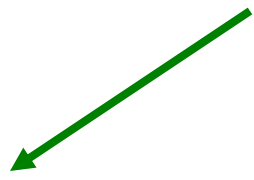
$$p(z) = -\rho g z \quad \text{pressure at any depth can be calculated}$$

**Note:** SI units of pressure are *pascales*, but oceanographers typically use *decibars*; numerically the depth in meters is the same as the pressure in decibars.

Check the units (oceanic case)....

$$p(z) = -\rho g z$$

weight per unit area



$$\frac{Mg}{L^2} \sim g \frac{M}{L^3} L = \frac{Mg}{L^2} \quad OK$$

Notes...

- Depth ( $z$ ) is negative – downwards.
- The hydrostatic pressure is just the weight of the water column above (per unit area).

**Atmospheric case:**

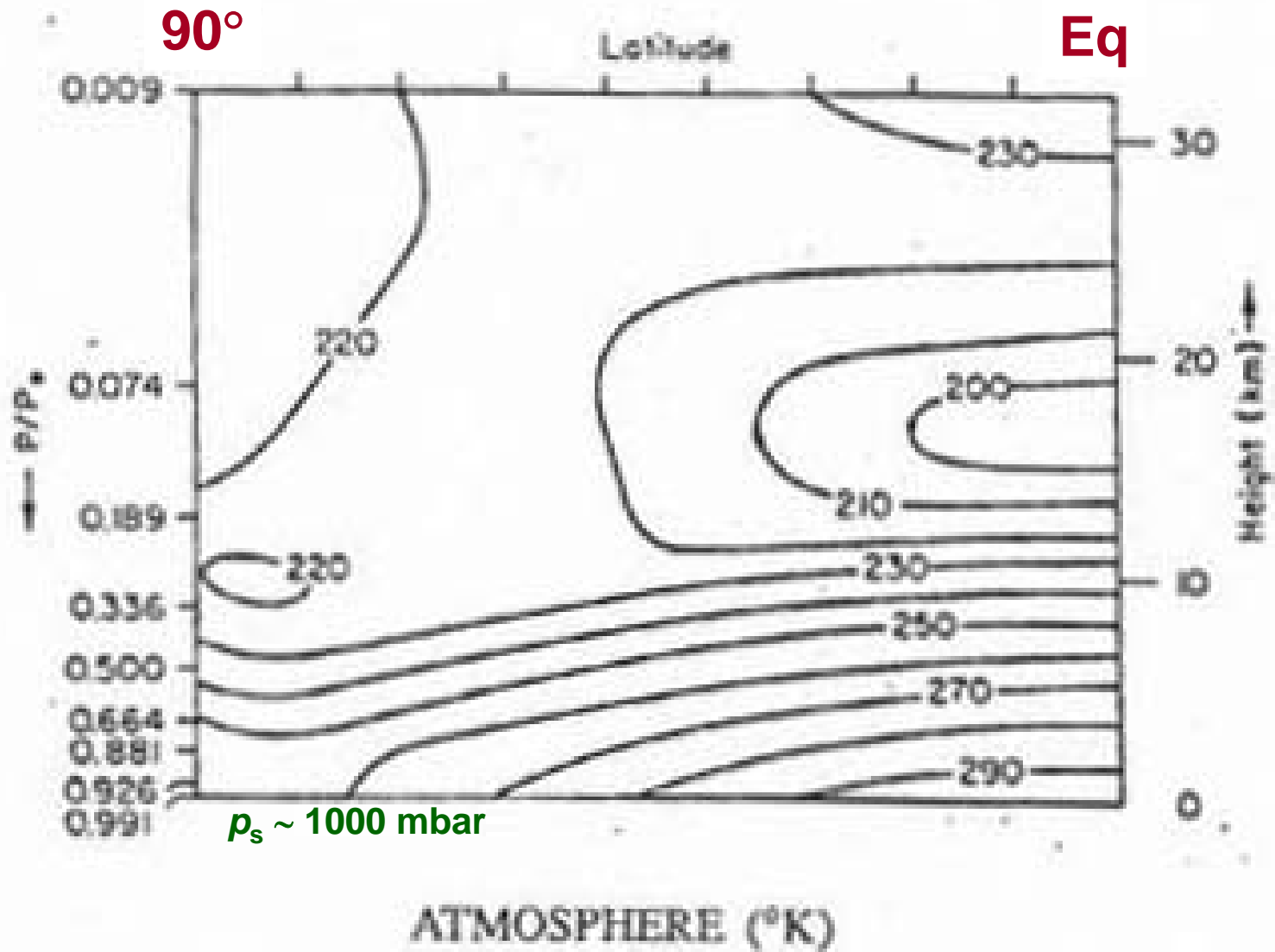
$$\frac{\partial p}{\partial z} = -g\rho \quad , \quad p = \rho RT \quad \text{[ideal gas law]}$$

**For an *isothermal* atmosphere ( $T$  constant) the hydrostatic relation can be integrated to find that**

$$p = p_s e^{-z/H} \quad H = RT / g \quad \text{[the scale height]}$$

**Here  $p_s$  is the sea level pressure and  $H \sim 7.6$  km. Pressure decreases exponentially with altitude.**

# Variation of pressure with altitude in the atmosphere....



**When can the hydrostatic relation be used? (ie, what has been assumed here?)**

**We assumed that pressure forces balance gravity.....when will this be true?**

$$\sum \vec{F} = 0 \quad \Rightarrow \quad \vec{F}_m + \vec{F}_g = 0$$

**But suppose instead that**

$$\sum \vec{F} = \frac{\partial^2 z}{\partial t^2} = \ddot{z} \quad \Rightarrow \quad \vec{F}_m + \vec{F}_g = \ddot{z}$$

$$\sum \vec{F} = \frac{\partial^2 z}{\partial t^2} = \ddot{z} \quad \Rightarrow \quad \vec{F}_m + \vec{F}_g = \ddot{z}$$

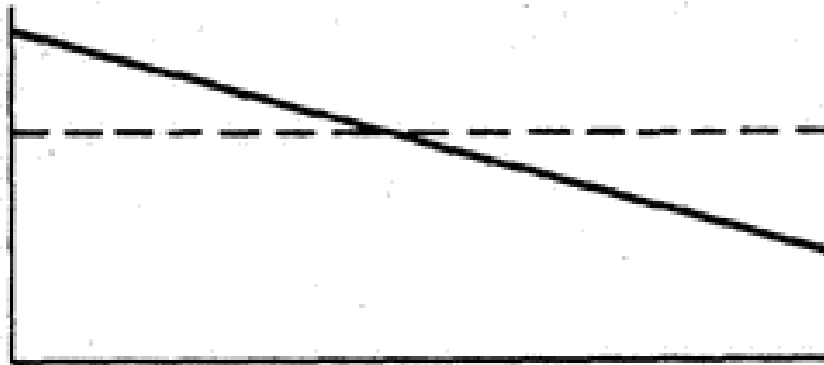
As long as  $\frac{\ddot{z}}{F_g} \ll 1$  the flow will still be hydrostatic.

$$\frac{L/T^2}{g} \ll 1 \Rightarrow T \gg \sqrt{\frac{L}{g}}$$

for  $L = 1$  meter,  
this will be true as long as  
 $T \geq 1$  sec (ocean)

for  $L = 1$  km, this will be true as long as  $T \geq 30$  sec (atmosphere)

# pressure (continued).....



**$p$  high**

**$p$  low**

$$\leftarrow \nabla p$$

$$-\nabla p \rightarrow$$

$$F_m \rightarrow$$

[thus, a force equal to  $-F_m$  is needed to balance the pressure force]



## The Coriolis effect....

### (i) Motion on a nonrotating Earth

→ The motion is determined by Newton's Laws with gravity,

$$\sum \underline{\tilde{F}} = m \underline{\tilde{\ddot{x}}} = -g \hat{r}$$

### Example: projectile motion

**Note:** Newton's Laws as normally used are true only in an inertial coordinate frame (one fixed with respect to distant, fixed stars. An Earth-based coordinate system (longitude, latitude, altitude; east, north, up) is not an inertial coordinate system.

## Coriolis effect, continued

### (ii) Stationary motion on a rotating Earth

→The motion is determined by Newton's Laws with gravity; the motion is only stationary in an Earth-based coordinate system [ east, north, up; longitude, latitude, altitude].

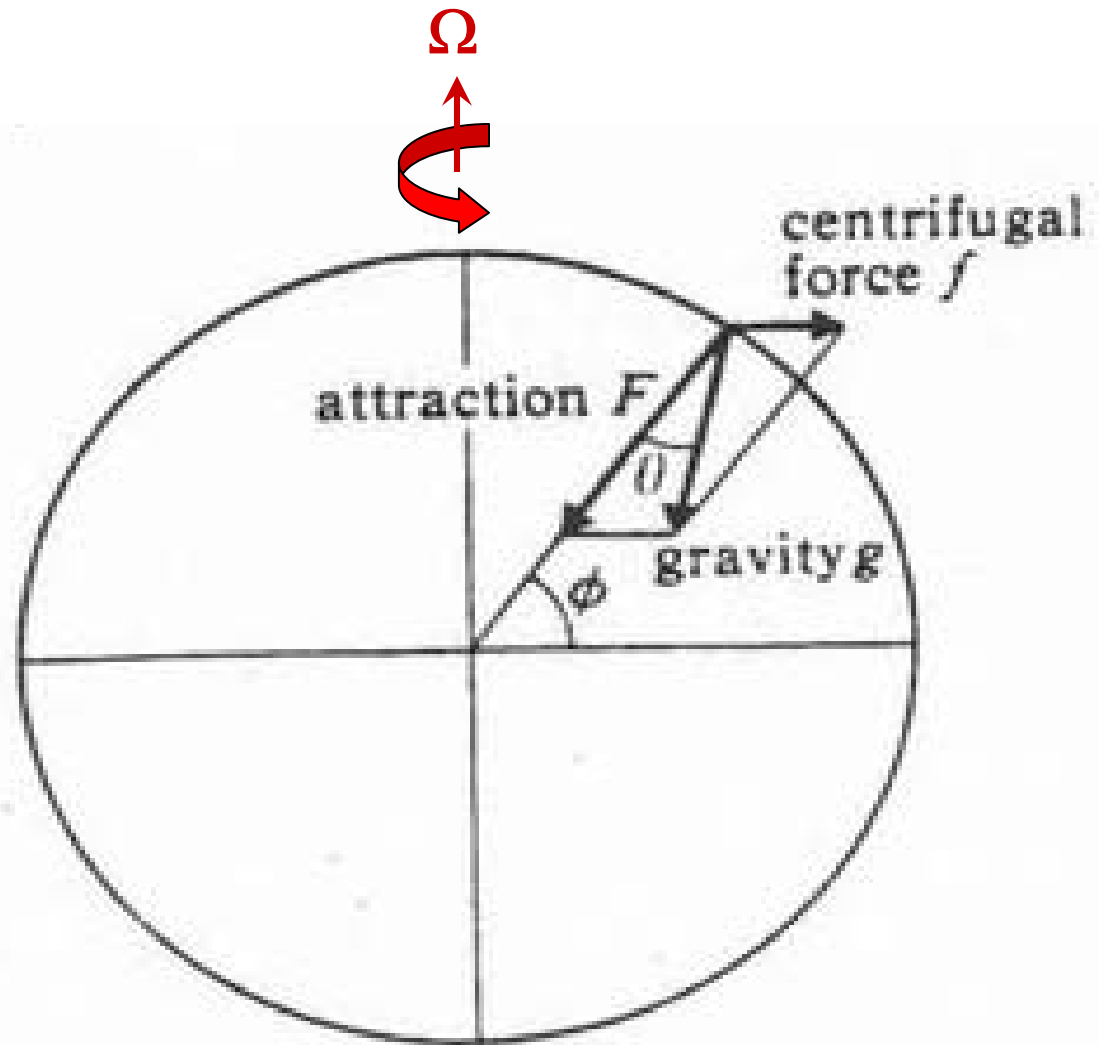
Motion in an Earth-based coordinate system leads to a new effect: *the centrifugal force*.

[inertial frame: centripetal acceleration]

[rotating frame: centrifugal force]

## Coriolis effects....continued

On a rotating Earth the effective gravity is the sum of the force of attraction and the centrifugal force.

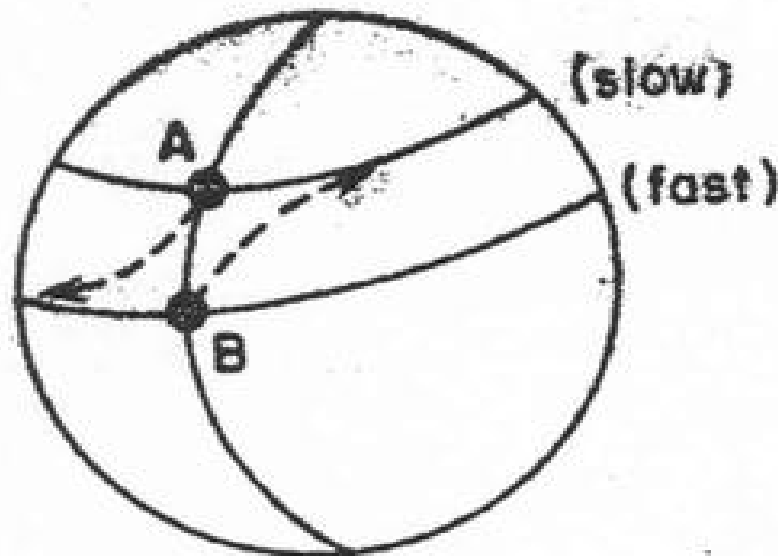


## Coriolis effect, continued

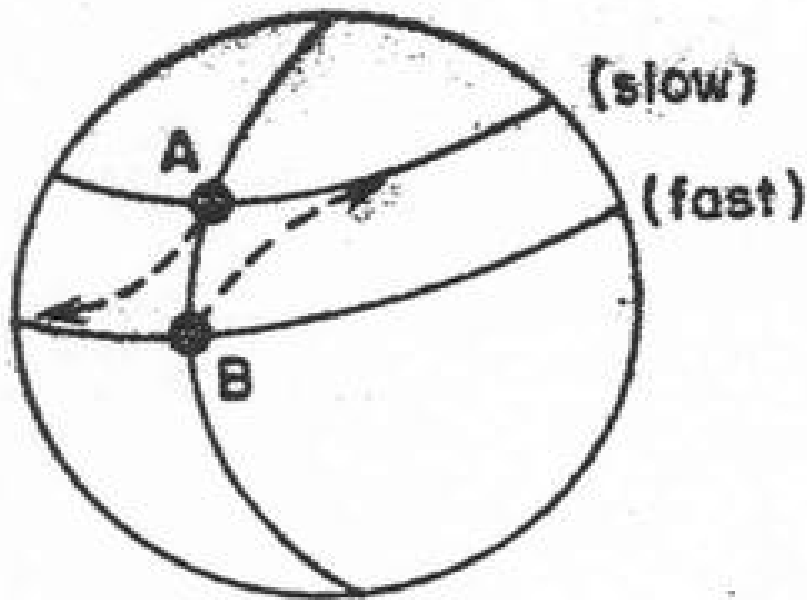
### (iii) Moving particles on a rotating Earth

→ The motion is determined by Newton's Laws with gravity; the motion is not stationary in an inertial frame or the rotating frame.

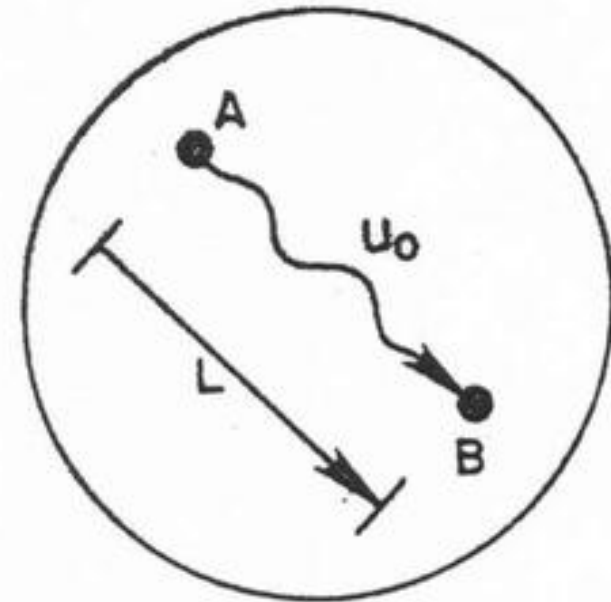
Motion in the rotating frame leads to a new effect: *the Coriolis force*.



## Coriolis effect (continued)....



A point on latitude  $\lambda$  is rotating at a tangential velocity  $V_T$  given by  $V_T = V_0 \cos \lambda$ , where  $V_0$  is the equatorial value.



Moving a distance  $L$  between A and B at speed  $U_0$  requires a time  $L/U_0$ .

Earth rotation rate =  $\Omega$

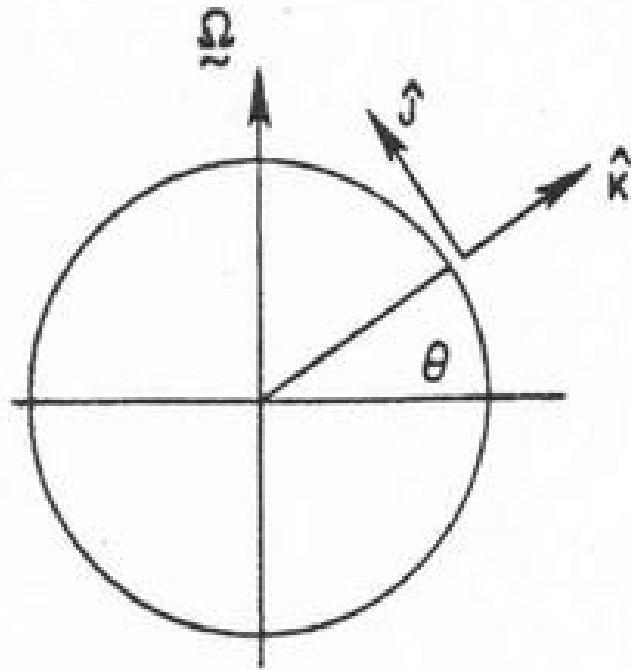
Earth rotation time =  $1/\Omega$

$$\varepsilon = (\text{Earth time})/(\text{AB time}) = U_0/(L \Omega)$$

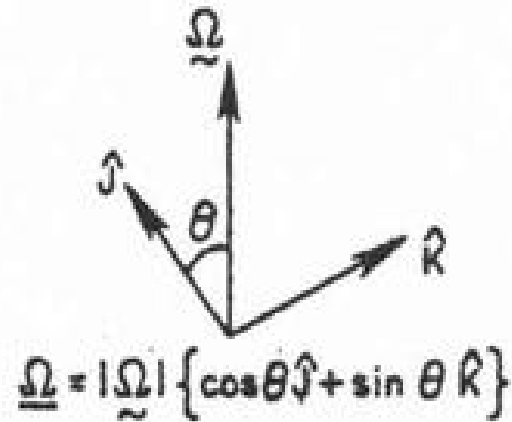
$\varepsilon \ll 1$  (slow AB motion, strong rotation)

$\varepsilon \gg 1$  (fast AB motion, weak rotation)

## Coriolis effect (continued)....



$\hat{i}, \hat{j}, \hat{k}$  = east, north, up unit vectors



**Coriolis acceleration =  $2\Omega \times u$      $u = (u, v, w)$**

**Coriolis force =  $-2\Omega \times u$**

## Coriolis effect (continued)....

$$2\Omega \times \mathbf{u} = 2\Omega \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & \cos \lambda & \sin \lambda \\ u & v & w \end{bmatrix}$$
$$= 2\Omega ( (w \cos \lambda - v \sin \lambda) \hat{\mathbf{i}} + u \sin \lambda \hat{\mathbf{j}} - u \cos \lambda \hat{\mathbf{k}} )$$

## Coriolis effect, continued....

$$2\Omega \times \mathbf{u} = 2\Omega \left( (w \cos \lambda - v \sin \lambda) \hat{\mathbf{i}} + u \sin \lambda \hat{\mathbf{j}} - u \cos \lambda \hat{\mathbf{k}} \right)$$

### Initial velocity

north  $(0, v, 0)$ ,  $v > 0$   
south  $(0, v, 0)$ ,  $v < 0$

east  $(u, 0, 0)$ ,  $u > 0$   
west  $(u, 0, 0)$ ,  $u < 0$

up  $(0, 0, w)$ ,  $w > 0$   
down  $(0, 0, w)$ ,  $w < 0$

### Coriolis force

east  
west

south, up  
north, down

west  
east

**The Coriolis effect is due to rotation + spherical geometry**



## Coriolis effect (continued)....

**Newton's Laws with rotation:**

$$\sum \underline{\tilde{F}} = m \underline{\tilde{\ddot{x}}} + 2\Omega \times \underline{\tilde{u}} = m \underline{\tilde{\dot{u}}} + 2\Omega \times \underline{\tilde{u}}$$

or

$$\sum \underline{\tilde{F}} - 2\Omega \times \underline{\tilde{u}} = m \underline{\tilde{\ddot{x}}} = m \underline{\tilde{\dot{u}}}$$

## Coriolis effect, continued....

$$\dot{\tilde{\mathbf{u}}} + 2\boldsymbol{\Omega} \times \tilde{\mathbf{u}} = -\frac{1}{\rho} \nabla p$$

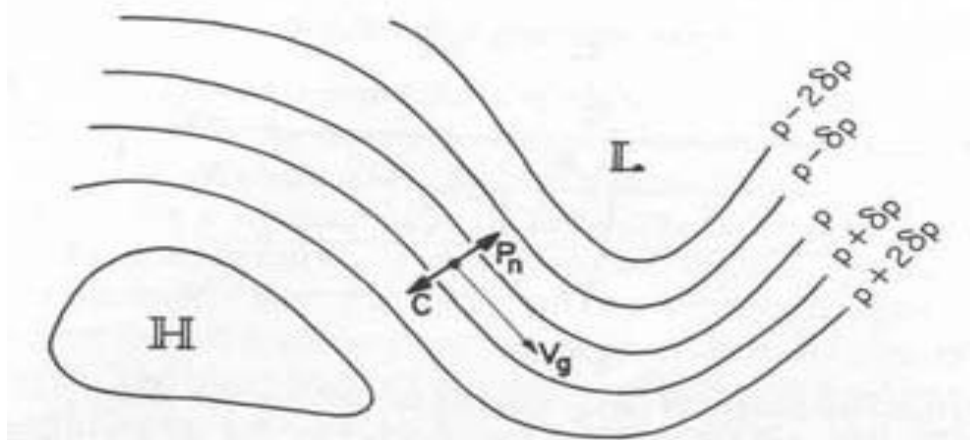
add the pressure gradient as a force

$$2\boldsymbol{\Omega} \times \tilde{\mathbf{u}} = -\frac{1}{\rho} \nabla p$$

steady or nearly steady motion

geostrophic balance

geostrophic flow: pressure gradient balances the Coriolis force



flow is *along* lines of constant pressure

## The Coriolis effect and geostrophic balance....

Recall the equation for the Coriolis acceleration:

$$2\Omega \times \mathbf{u} = 2\Omega \left( (w \cos \lambda - v \sin \lambda) \hat{\mathbf{i}} + u \sin \lambda \hat{\mathbf{j}} - u \cos \lambda \hat{\mathbf{k}} \right)$$

Note in the  $i$  term that  $(w \cos \lambda)/(v \sin \lambda) = (w/v) \cot \lambda \ll 1$ , except near the Equator. So, neglect the  $w$  term with respect to the  $v$  term.

Also, note that we have already examined hydrostatic balance and found that vertical accelerations are small compared to horizontal. Thus, ignore the  $k$  term here.

## Coriolis effect and geostrophic motion....

With these simplifications the geostrophic equations become

$$2|\Omega| \sin \lambda v = \frac{1}{\rho} \frac{\partial p}{\partial x}$$

+x pressure gradient: northward flow in the N. hemisphere

$$2|\Omega| \sin \lambda u = -\frac{1}{\rho} \frac{\partial p}{\partial y}$$

+y pressure gradient: westward flow in the N. hemisphere

These equations provide a simple diagnostic tool for examining the circulation of the atmosphere or the ocean.

## Coriolis effect (continued)....

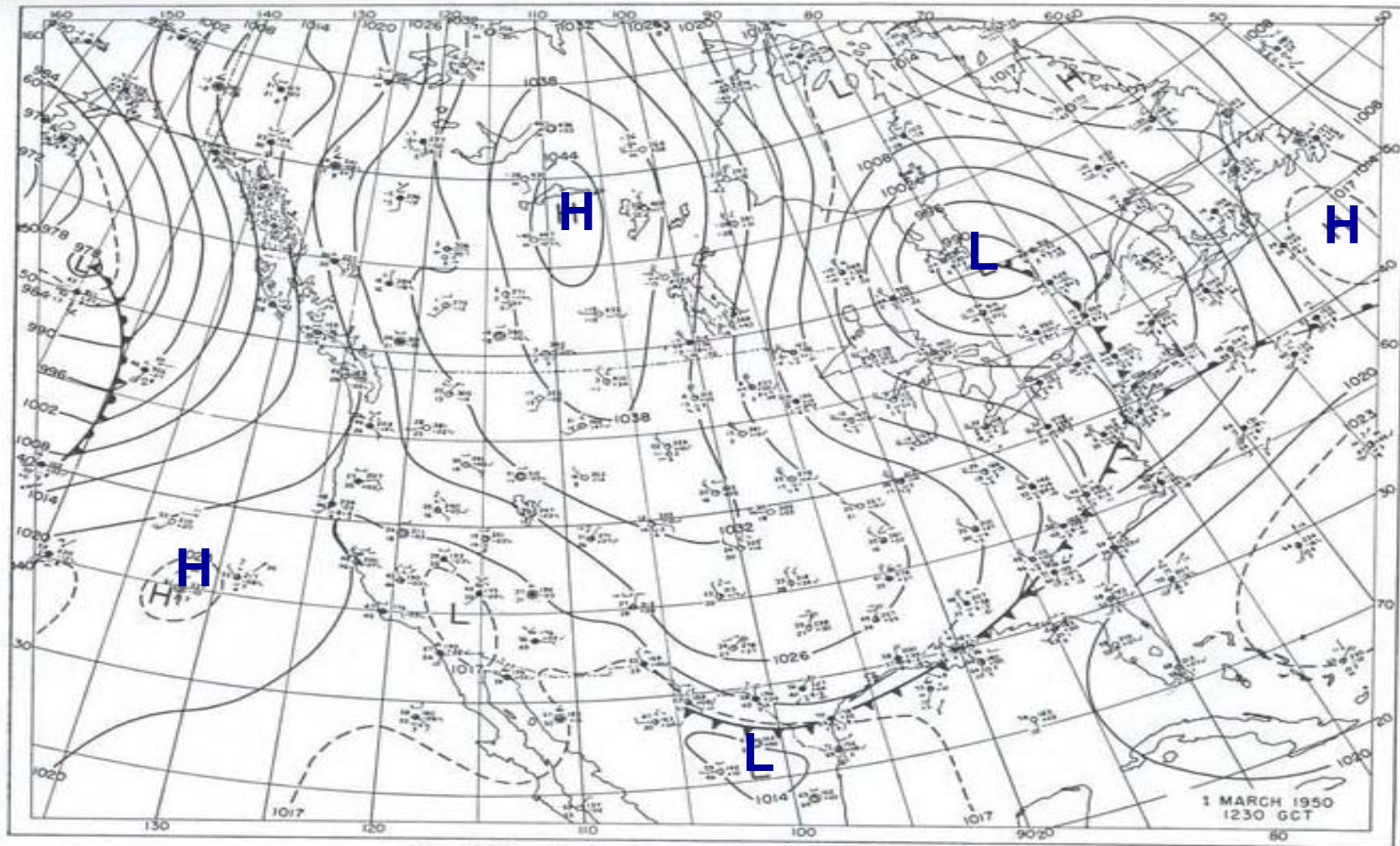
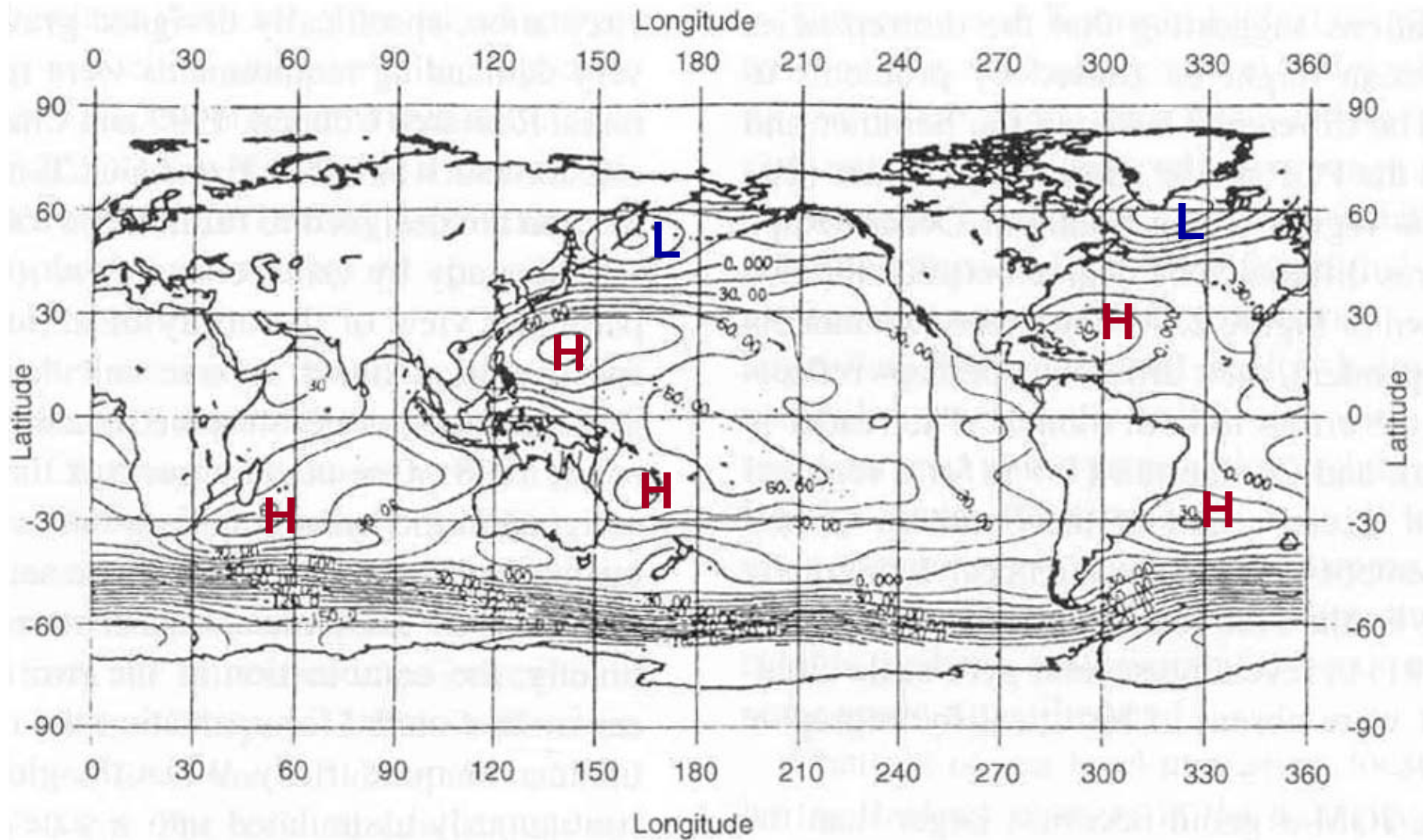


FIG. 9.02a.—Sea-level chart. 1230 GCT. 1 March 1950

**Weather map shows geostrophic flow**

# Coriolis effects and geostrophic motion....



**Global mean sea level from 10 years of Topex/Poseidon**

## **Lagrangian and Eulerian descriptions of motion....**

**Newton's Laws are formulated in terms of the motion of a particle in an inertial reference frame. The Coriolis effect provides a modification to Newton's Laws for a rotating frame.**

**What happens if we don't want to follow a particle, but instead prefer to examine the flow on a grid? This is the type of observation made by ships on a regular survey pattern or at fixed weather stations. This situation is not included in Newton's Laws, but we can easily modify the Laws to allow this description.**

**Lagrangian description: following a particle**  
**Eulerian description: on a grid (fixed points)**

## Eulerian and Lagrangian descriptions....

Consider some property  $\phi$  that we want to measure, where  $\phi = \phi(x, y, z, t)$ , and  $(x_0, y_0, z_0)$  is a fixed point.

In the vicinity of  $(x_0, y_0, z_0)$ , we can write that

$$\phi(\underline{x}, t) = \phi(\underline{x}_0, t) + \frac{\partial \phi}{\partial x} \delta x + \frac{\partial \phi}{\partial y} \delta y + \frac{\partial \phi}{\partial z} \delta z + \frac{\partial \phi}{\partial t} \delta t$$

$$\underline{x} = (x, y, z); \quad \underline{x}_0 = (x_0, y_0, z_0); \quad \delta \underline{x} = (\delta x, \delta y, \delta z)$$

$$\underline{x} = \underline{x}_0 + \delta \underline{x}$$



## Lagrangian-Eulerian descriptions....

This can be rearranged to yield

$$\frac{\phi(\underline{x}, t) - \phi(\underline{x}_0, t_0)}{\delta t} = \frac{\partial \phi}{\partial x} \left( \frac{\partial x}{\partial t} \right) + \frac{\partial \phi}{\partial y} \left( \frac{\partial y}{\partial t} \right) + \frac{\partial \phi}{\partial z} \left( \frac{\partial z}{\partial t} \right) + \frac{\partial \phi}{\partial t}$$

Taking the limit as the  $\delta$  terms go to zero, it is found that

$$\frac{d\phi}{dt} = \frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} + w \frac{\partial \phi}{\partial z}$$

**Lagrangian**

**Eulerian**

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \underline{u} \cdot \nabla \phi$$

**(total)**

**(local)**

**(motion)**

## Newton's Laws....

The Eulerian version of Newton's Laws with rotation are

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} + 2\Omega \times \underline{u} = -\frac{1}{\rho} \nabla p - g \hat{k} + \sum \underline{F}$$

[note: this vector PDE has no general solution]