Using *Mathematica* to solve oscillator differential equations

**Unforced, damped oscillator**

General solution to forced harmonic oscillator equation (which fails when $b^2=4k$, i.e. at perfect damping). $C[1]$ and $C[2]$ are integration constants.

```
In[8]:= DSolve[x''[t] + b x'[t] + k x[t] == 0, x[t], t]
```

```
Out[8]= \\
\{x[t] \to e^{\frac{1}{2} \left( -b - \sqrt{b^2 - 4k} \right) t} C[1] + e^{\frac{1}{2} \left( -b + \sqrt{b^2 - 4k} \right) t} C[2]\}
```

Now let’s solve with boundary conditions: $x=1$ at $t=0$ and it is at rest. Then get a fully defined solution.

```
In[9]:= res1 = DSolve[{x''[t] + b x'[t] + k x[t] == 0, x[0] == 1, x'[0] == 0}, x[t], t]
```

```
Out[9]= \\
\{x[t] \to \frac{1}{2 \sqrt{b^2 - 4k}} b e^{\frac{1}{2} \left( -b - \sqrt{b^2 - 4k} \right) t} + \\
\frac{1}{2 \sqrt{b^2 - 4k}} \left( b e^{\frac{1}{2} \left( -b + \sqrt{b^2 - 4k} \right) t} e^{\frac{1}{2} \left( -b - \sqrt{b^2 - 4k} \right) t} \sqrt{b^2 - 4k} + e^{\frac{1}{2} \left( -b + \sqrt{b^2 - 4k} \right) t} e^{\frac{1}{2} \left( b + \sqrt{b^2 - 4k} \right) t} \sqrt{b^2 - 4k}\right)\}
```

Set up a function to plot the result, here setting $k=1$ so the undamped frequency is 1, and keeping the damping constant as a variable. Note the construction with two "/." in turn ("use the assignment res1, and, within that, make the assignments to b and k").

```
In[10]:= res1plot[b0_] := Plot[x[t] /. res1 /. {b -> b0, k -> 1}, {t, 0, 20}, \\
PlotRange -> {{0, 20}, {-1, 1}}, PlotStyle -> Thick, PlotLabel -> b0]
```

Here’s what the oscillations look like with no damping:

```
In[11]:= res1plot[0]
```

![Plot of oscillations with no damping](image)

Underdamping ($b^2 < 4k$). First, very weak damping ($b=0.1$), which is the value used below for the driven oscillator.
In[12]:= resiplot[.1]

Out[12]=

Next, a much more damped case (b=1).

In[13]:= resiplot[1]

Out[13]=

Essentially perfect damping (b^2=4k). The function is singular at b exactly equal to 2 because the solution is not correct there.

In[14]:= resiplot[2.01]

Out[14]=
Overdamping:

\textbf{In}[15]:= res1plot[3]

\textbf{Out}[15]=

Using \textit{Animate} to show how the behavior varies with damping:

\textbf{In}[16]:= \textbf{Animate}[\text{res1plot}[b0],\{b0, 0, 4\}]

\textbf{Out}[16]=

Perfect damping

\textit{Mathematica} gets the perfectly damped case correct if it is asked for specifically:

\textbf{In}[17]:= \text{res2} = \text{DSolve}[\{x''[t] + 2 x'[t] + x[t] = 0, x[0] = 1, x'[0] = 0\}, x[t], t]

\textbf{Out}[17]= \{\{x[t] \rightarrow e^{-t} (1 + t)\}\}
Out[18]=

Comparing to almost perfect damping case shown above—identical to the eye.

Mathematica can correctly find the perfect damping solution from the general solution by taking a limit.

Which agrees with "res2" for k=1

Forced, damped oscillator

General solution in forced case (except forcing amplitude is set to unity). Note the presence of the two constants in the first two terms as above. Here there is also the long-time "particular solution".
\textbf{In[21] := } \text{DSolve\{}\left\{x''[t] + b x'[t] + k x[t] = \sin(\omega_0 t)\right\}, x[t], t\}\text{\}}

\text{Out[21] = } \left\{\left\{x[t] \rightarrow e^{\frac{t}{2} \left(-\sqrt{b^2 - 4 k} \right)} C[1] + e^{\frac{t}{2} \left(\sqrt{b^2 - 4 k} \right)} C[2] + \frac{4 \left(b \omega_0 \cos(\omega_0 t) - k \sin(\omega_0 t) + \omega_0^2 \sin(\omega_0 t)\right)}{\left(\left(-b^2 + b \sqrt{b^2 - 4 k} + 2 k - 2 \omega_0^2\right) \left(b^2 + b \sqrt{b^2 - 4 k} - 2 k + 2 \omega_0^2\right)\right)}\right\}\right\}

Now include boundary conditions to completely fix the solution:

\textbf{In[22] := } \text{res3 = DSolve\{}\left\{x''[t] + b x'[t] + k x[t] = \sin(\omega_0 t), x[0] = 1, x'[0] = 0\right\}, x[t], t\}\text{\};}

Now specify that \(b=0.1, k=1\) (very underdamped) for definiteness, and make plots of the time dependence as the driving frequency is varied.

\textbf{In[23] := } \text{res3f[t0\_, \omega__\_] = x[t] /. res3 /. \{b \rightarrow 0.1, k \rightarrow 1, \omega_0 \rightarrow \omega\} /. t \rightarrow t0;}

\textbf{In[24] := } \text{res3plot[omega\_] := } \text{Plot\[\text{res3f[t, omega], \{t, 0, 200\},}
\text{PlotRange \rightarrow \{\{0, 200\}, \{-5, 5\}\},} 
\text{PlotStyle \rightarrow Thick,} \text{PlotLabel \rightarrow omega\];}

The following plots show a quick scan as the driving frequency varies from smaller than the natural frequency (which is unity) to above the natural frequency. We see the expected resonant behavior: The smaller "wiggles" are at a frequency very close to the natural frequency, and correspond to the damped solution to the homogeneous equation. Eventually, the particular solution takes over.

\textbf{In[25] := } \text{res3plot[.2]}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{res3plot.png}
\end{figure}

Just below the resonant frequency: notice the increase in amplitude.
Just above the resonant frequency:

Well above the resonant frequency: the amplitude has greatly decreased.

Scanning through the frequencies:
In[29]= Animate[res3plot[omega], {omega, 0, 2}]

Out[29]=

1.72104