Lecture 17: Analytic Functions and Integrals (See Chapter 14 in Boas)

This is a good point to take a brief detour and expand on our previous discussions of complex variables and complex functions of complex variables. In particular, we want to consider the properties of so-called analytic functions, especially those properties that allow us to perform integrals relevant to evaluating an inverse Fourier transform. More generally we want to motivate the process of thinking about the functions that appear in physics in terms of their properties in the complex plane (where “real life” is the real axis). In the complex plane we can more usefully define a well “behaved function”; we want both the function and its derivative to be well defined.

Consider a complex valued function of a complex variable defined by

$$F(z) = U(x, y) + iV(x, y),$$

(17.1)

where $U, V$ are real functions of the real variables $x, y$. The function $F(z)$ is said to be analytic (or regular or holomorphic) in a region of the complex plane if, at every (complex) point $z$ in the region, it possesses a unique (finite) derivative that is independent of the direction in the complex plane along which we take the derivative. (This implies that there is always a, perhaps small, region around every point of analyticity within which the derivatives exist.) This property of the function is summarized by the Cauchy-Riemann conditions, i.e., a function is analytic/regular at the point $z = x + iy$ if and only if (iff)

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \quad \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}$$

(17.2)

at that point. To see this result start with the fact that $F$ is a function of $z$ and then apply the chain rule in the $x$ direction and in the $y$ direction to find

$$\frac{\partial F}{\partial x} = \frac{dF}{dz} \frac{\partial z}{\partial x} = \frac{dF}{dz} \frac{dx}{\partial x} = \frac{dF}{dz} \frac{\partial U}{\partial x} + \frac{\partial V}{\partial x},$$

$$\frac{\partial F}{\partial y} = \frac{dF}{dz} \frac{\partial z}{\partial y} = \frac{dF}{dz} \frac{dy}{\partial y} = -i \frac{dF}{dz} \frac{\partial U}{\partial y} + \frac{\partial V}{\partial y}.$$
By equating the real and imaginary parts of the 2 expressions for $dF/dz$ we are led to Eq. (17.2).

For example, the function $F(z) = z^2 = x^2 - y^2 + i2xy$, $U = x^2 - y^2$, $V = 2xy$, satisfies the C-R equations at all $z$, while $F(z) = z^* = x - iy$ satisfies the C-R equations nowhere. Next consider the function

$$F(z) = \frac{1}{z} = \frac{z^*}{|z|^2} = \frac{x}{(x^2 + y^2)} = -\frac{iy}{(x^2 + y^2)},$$

(17.4)

for which

$$U = \frac{x}{x^2 + y^2}, \quad \frac{\partial U}{\partial x} = \frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

$$\frac{\partial U}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2},$$

$$V = -\frac{y}{x^2 + y^2}, \quad \frac{\partial V}{\partial x} = \frac{2xy}{(x^2 + y^2)^2}, \quad \frac{\partial V}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$ 

(17.5)

We see that this function is analytic everywhere except the point $z = 0$ where it is said to be singular (both the function and the derivatives diverge). Unlike points of analyticity, singular points can be isolated points.

In the (simply connected) region $R$ of the complex plane where the function $F(z)$ is analytic/regular it has many truly important properties that we summarize here.

- For an analytic/regular function (that satisfies the C-R equations) the real functions $U, V$ are harmonic functions, i.e., they satisfy Laplace’s equation

$$\nabla^2 U = \nabla^2 V = 0.$$ 

(17.6)

- The integral of an analytic function is independent of the path of integration in the complex plane as long as the function is analytic/regular at all points along all paths considered and between all pairs of paths, i.e. an analytic function is
holomorphic in the region $R$.

- The integral of an analytic/regular function along a closed path vanishes as long as the function is analytic/regular at every point along and inside the path. Let $F(z)$ be analytic/regular everywhere in a region $R$ in the complex plane and let $C$ be a simple (i.e., does not cross itself), smooth (with, at most, a finite number of corners) closed path entirely inside $R$. Then we have

$$\oint_{C \subset R} F(z)\,dz = 0, \quad (17.7)$$

which is called Cauchy’s theorem. (This closed line integral is called a contour integral in the complex plane.)

- For our purposes the related but more relevant result is Cauchy’s integral formula,

$$F(z) = \frac{1}{2\pi i} \oint_{C \subset R} \frac{F(z')}{z' - z} \,dz', \quad (17.8)$$

where the contour encloses the point $z$ in the counter-clockwise direction and $F(z)$ is analytic/regular everywhere in the region $R$. Note that there is a sign change if the point is encircled in the clockwise direction. We can understand the factors if we consider the special case $F(z) = 1$ and define $C$ to be the unit circle about $z$. For $z'$ on this path we have $z' - z = e^{i\theta}$, $dz' = id\theta e^{i\theta}$. Thus we have

$$F(z) = 1 = \frac{1}{2\pi i} \oint_{C \subset R} \frac{id\theta e^{i\theta}}{e^{i\theta}} = \int_{0}^{2\pi} \frac{d\theta}{2\pi} = 1. \quad (17.9)$$

The fundamental point is that, although $F(z')$ is analytic/regular at $z' = z$, the integrand $F(z')/(z' - z)$ is not. It has a simple pole at $z' = z$ and the contour integral around the point $z' = z$ serves to select out the coefficient of this pole, $F(z)$, also called the residue of the integrand at the pole at $z' = z$. 
In a region $R$ where $F(z)$ is analytic/regular, all of its derivatives exist and we can write a Taylor series expansion about any point $z_0 \subset R$,

$$F(z) = \sum_{n=0}^{\infty} F^{(n)}(z_0) \frac{(z - z_0)^n}{n!},$$

where $z_0$ is inside the contour $C$. If we substitute the first expression into the integral in the second, we see that only the term proportional to $1/(z' - z_0)$, i.e., the simple pole singularity, contributes to the integral. This is just like the previous result. The circle of convergence of this series expansion about the point $z = z_0$ is given by the distance from the point $z = z_0$ to the closest point on the boundary of $R$, i.e., the closest point to $z = z_0$, where the function $F(z)$ is not analytic/regular, i.e., where it is singular.

**ASIDE:** Thus any function for which we know that the power series expansion is convergent everywhere, e.g., $e^{z - z_0}$, $\sin(z - z_0)$, $\cos(z - z_0)$, $\sinh(z - z_0)$, $\cosh(z - z_0)$, must be analytic/regular everywhere in the (finite) complex plane. These functions are often labeled as entire functions.

Next we want to consider the series expansion of the function about a point where it is not analytic/regular, a Laurent series. Call the singular point $z_s$ and assume for the moment that it is isolated, i.e., that there is a region $R$ around $z_s$ (presumably in the form of an annulus) where the function is analytic/regular. We can still express $F(z)$ as a series expansion about this singular point, but now we must include negative powers (the $a_n, n < 0$ here are the same as the $b_{-n}$ in Boas),

$$F(z) = \sum_{n=0}^{\infty} \frac{F^{(n)}(z_0) (z - z_0)^n}{n!} + \sum_{n=1}^{\infty} \frac{F^{(n)}(z_0)}{n!} \frac{1}{(z - z_0)^n}.$$
\[ F(z) = \sum_{n=\infty}^{\infty} a_n (z - z_s)^n, \]

\[ a_n = \frac{1}{2\pi i} \oint_{C \subset R} \frac{F(z')}{(z' - z_s)^{n+1}} dz', \]

where the integral defining the coefficients in the expansion is essentially identical to that for the previous Taylor series. The contour \( C \) is again entirely in the analytic/regular region \( R \), but now encircles the singular point \( z_s \) in the counter-clockwise direction. Note the special case \( n = -1 \),

\[ a_{-1} = \frac{1}{2\pi i} \oint_{C \subset R} F(z') dz'. \]

This term is just the residue mentioned earlier and is the path integral of the function around the contour (divided by \( 2\pi i \)). Consider now some terminology for special cases:

- If \( a_n = 0, n < 0 \), the function \( F(z) \) is (in fact) analytic/regular at \( z_s \);
- If \( a_n = 0, n < -1, a_{-1} \neq 0 \), the function \( F(z) \) has a simple pole at \( z_s \);
- If \( a_n = 0, n < -2, a_{-2} \neq 0 \), the function \( F(z) \) has a double pole at \( z_s \);
- If \( a_n \neq 0 \), all \( n < 0 \) the function \( F(z) \) has an essential singularity at \( z_s \) (e.g., think of the function \( e^{1/(z-z_s)} \)).

In any of these cases, if the point \( z_s \) is the only singular (non-analytic/non-regular) point enclosed by \( C \), the integral of \( F(z) \) about the closed contour \( C \) is given as above by \( 2\pi i a_{-1} \), i.e., \( 2\pi i \) times the residue at the singular point. If the singularity is just a simple pole we can always find the singularity via \( a_{-1} = \lim_{z \to z_s} (z - z_s) F(z) \). For higher order poles more care must be taken.

- Next consider what happens if more than one singular point is enclosed by the closed contour \( C \). Noting that we can always move the closed contour around by adding and subtracting closed contours with no singularities inside, which
have zero integral from above, we are led to the Theorem of Residues -

\[ \oint_{C \subset R} dzF(z) = 2\pi i \sum \text{Residues}, \tag{17.13} \]

where we must be careful about the signs. We add the residues at singular points encircled in a counter-clockwise sense and subtract those encircled in a clockwise sense. For example, if \( F(z) \) is singular at \( z_1, z_2, z_3 \) inside \( C \) with residues \( a_{-1}(z_1), a_{-1}(z_2), a_{-1}(z_3) \) and \( C \) encircles \( z_1, z_2 \) in a counter-clockwise sense and \( z_3 \) in a clockwise sense (somewhat hard to imagine, but still), we have

\[ \oint_{C \subset R} dzF(z) = 2\pi i (a_{-1}(z_1) + a_{-1}(z_2) - a_{-1}(z_3)). \tag{17.14} \]

Now let’s consider some examples of using the Residue Theorem, Eq. (17.13), to perform integrals. There are 3 basic steps:

1. If the integral is not yet a contour (closed path) integral (in the complex plane), complete the contour by adding another path integral.

2. Find all singular points inside the contour and evaluate the residues at these points.

3. Sum up the residues with a +1 coefficient for CCW enclosure and -1 for CW enclosure. Multiply by \( 2\pi i \).

Comments: The tricky part is step 1., since there is no universal recipe for how to proceed. It is clear that the path integral along the added paths must either vanish (the usual case) or be easy to evaluate. The latter situation will typically arise due to the symmetry properties of the integrand (see the first example below). A typical added path with vanishing contribution is an arc at “infinity” in the complex plane where the integrand vanishes when the magnitude of the complex integration variable becomes large. Of particular interest are complex exponential factors in the integrand, which will tell us the sort of arc that is called for (see below). It is often necessary to break the integrand into a sum of terms and close the contour for the different terms in different ways. Care must be taken when branch cuts are present as described in Example 5 in Boas and exercise 14.7:33 in
Appendix B.

Step 2. is straightforward (well defined) mathematically, e.g., find the Laurent expansion, but may involve some arithmetic. When only simple poles are present, as is typically the case in physics problems, we can use the relation
\[ a_{-1}(z) = \lim_{z \to z_n} (z - z_n) F(z). \]

Step 3. is just more arithmetic except that care must be taken to correlate the sign of the coefficient with the direction of the path around the singularity (see the examples)

As a first example consider
\[
I = \int_{0}^{\infty} \frac{dx}{1 + x^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{(x + i)(x - i)} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{2i} \left[ \frac{1}{x - i} - \frac{1}{x + i} \right],
\]

where we use the symmetry of the integrand to extend the integral along the entire real axis (with a factor of \( \frac{1}{2} \)) and then factor the denominator to “expose” the poles. Now comes the essential “trick”. We choose to close the contour by adding a line integral along a semi-circle of radius \( R \), connected to the real axis at both ends, and then take the limit \( R \to \infty \) where this added path integral vanishes (\( z = Re^{i\theta} \) along the arc),

\[
I_{arc} = \text{Lim}_{R \to \infty} \frac{1}{2} \int_{\text{arc}} \frac{dz}{1 + z^2} = \text{Lim}_{R \to \infty} \frac{1}{2} \int_{0}^{\pm \pi} \frac{id\theta Re^{i\theta}}{1 + R^2 e^{2i\theta}}
\]
\[
= \text{Lim}_{R \to \infty} \frac{1}{2R} \int_{0}^{\pm \pi} id\theta e^{-i\theta} \to 0.
\]

Here the \( \pm \) in the limit of the integral (\( \pm \pi \)) reminds us we could put the semicircle in either the upper half-plane (\( \text{Im} \ z \geq 0, 0 \leq \theta \leq \pi \)) or in the lower half-plane (\( \text{Im} \ z \leq 0, 0 \geq \theta \geq -\pi \)). The only difference is which poles we encircle and the sense of the encirclement. If we close in the upper half-plane, we have
Here we noted that, since there is only a simple pole at $z = i$, we can use the expression above, $a_{-1}(z_s) = \lim_{z \to z_s} \left( z - z_s \right) F(z)$. An alternative, and often simpler approach, is to use the “partial fractionated” expression in the last version of Eq. (17.15), which tells us by inspection that $a_{-1}(i) = 1/2i$ and $a_{-1}(-i) = -1/2i$.

If we close the contour the other way in the lower half-plane ($\Im z \leq 0$, $0 \geq \theta \geq -\pi$), we pick up the other pole and encircle it in the clockwise sense,

$$
I = I + I_{\text{arc}} = \frac{1}{2} \oint_{C \subset \text{lower half-plane}, \text{CW}} \frac{dz}{(z+i)(z-i)} = -\frac{2\pi i}{2} a_{-1}(z = -i)
$$

$$
= -\pi i \lim_{z \to -i} \left( \frac{1}{z-i} \right) = -\frac{\pi i}{2i} = \frac{\pi}{2}.
$$

Note that with either choice we obtain the same result as required. As long as we close the contour correctly, we must obtain the same result.

An example closer to our goals in this course is to evaluate the inverse Fourier transform of the particular solution of the 2nd order differential equation, $a\ddot{x} + b\dot{x} + cx = F(t)$, where the right-hand-side is an impulse, $F(t) = C\delta(t)$. (Nothing happens until we hit the system with a hammer at $t = 0$.) The Fourier transform of the driving function is trivially found to be $F(t) = C\delta(t) \Rightarrow G(\omega) = C/\sqrt{2\pi}$. Thus we can write (see the end of Lecture 16) the particular solution as

$$
F(t) = C\delta(t) \Rightarrow G(\omega) = C/\sqrt{2\pi}.
$$
\[ x_p(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \dot{x}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \frac{G(\omega)e^{i\omega t}}{a(i\omega - \alpha_1)(i\omega - \alpha_2)} \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{C e^{i\omega t}}{a(i\omega - \alpha_1)(i\omega - \alpha_2)} \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{C e^{i\omega t}}{-a(\omega - (-i\alpha_1))(\omega - (-i\alpha_2))}. \] (17.19)

\[ -i\alpha_{1,2} = \frac{1}{2a} \pm \sqrt{\frac{c}{a} - \frac{b^2}{4a^2}} = i\gamma \pm \omega_0. \]

To perform this integral using the Residue Theorem we must close the contour using the ideas discussed above. In particular, we must deal with the exponential factor \( e^{i\omega t} = e^{i(\text{Re}\omega) t} e^{-t(\text{Im}\omega)} \). In order to close the contour with a semicircle “at infinity” we must ensure that the absolute magnitude of the integrand vanishes as \( R \to \infty \) (formally this rule is called Jordan’s Lemma). The second factor in the complex exponential is a real exponential, which we must ensure is a damping exponential, and not a divergent one. So in this case we are not free to choose to close in either half-plane, but rather this restriction produces the following correlation: requiring the product \(-t\text{Im}[\omega]<0 \) means that A) if \( t > 0 \), we must require that \( \text{Im}\omega \geq 0 \) and thus we must close in the upper half-plane; B) if \( t < 0 \), we must require that \( \text{Im}\omega \leq 0 \) and thus we must close in the lower half-plane.

Noting that \(-i\alpha_{1,2} = \alpha_{1,2} = \frac{ib}{2a} \pm \sqrt{\frac{c}{a} - \frac{b^2}{4a^2}} \) or \( \omega_{1,2} \equiv i\gamma \pm \omega_0 \), we see that the two poles in the complex \( \omega \) plane are both in the upper half-plane (this remains true even if the system is over-damped) and we have...
\[ x_p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{C e^{i\omega t}}{-a(\omega + i\alpha_1)(\omega + i\alpha_2)} \]

\[
= \begin{cases} 
\frac{1}{2\pi} \int_{C_{\text{upper} \text{ half-plane}}} d\omega Ce^{i\omega t} \frac{d\omega}{-a(\omega - \omega_1)(\omega - \omega_2)} & \text{[} t \geq 0 \text{]} \\
\frac{1}{2\pi} \int_{C_{\text{lower} \text{ half-plane}}} d\omega Ce^{i\omega t} \frac{d\omega}{-a(\omega - \omega_1)(\omega - \omega_2)} & \text{[} t < 0 \text{]} 
\end{cases}
\]

\[
= \begin{cases} 
\frac{2\pi iCe^{-\gamma t}}{a2\pi} \left( \frac{e^{+i\omega_0 t} - e^{-i\omega_0 t}}{-2\omega_0} \right) & \text{[} t \geq 0 \text{]} \\
= \frac{Ce^{-bt/2a}}{a\omega_0} \sin \omega_0 t & \text{[} t < 0 \text{]} 
\end{cases}
\]

Note that the “turn-on” of motion at \( t = 0 \) (i.e., the presence of the explicit theta or step function) is automatically handled by this method of evaluation! The particular solution automatically knows that it exhibits no motion before the “hammer hits” at \( t = 0 \). Thus the combined mathematical magic of the Fourier integral transform and the Theorem of Residues (analytic function magic) ensures the correct time behavior (usually treated with initial conditions).

**ASIDE:** To practice our facility with the functions of mathematical physics, let us explicitly verify that the result in Eq. (17.20) is a solution of the initial equation \( a\ddot{x} + b\dot{x} + cx = C\delta(t) \). By explicit differentiation we find
\[ x_p(t) = \frac{Ce^{-bt/2a}}{a\omega_0} \sin \omega_0 t \Theta(t) = \frac{C(e^{\alpha_1 t} - e^{\alpha_2 t})}{2i\omega_0} \Theta(t) \]

\[ \Rightarrow \dot{x}_p(t) = \frac{C(\alpha_1 e^{\alpha_1 t} - \alpha_2 e^{\alpha_2 t})}{2i\omega_0} \Theta(t) + \frac{C(e^{\alpha_1 t} - e^{\alpha_2 t})}{2i\omega_0} \dot{\Theta}(t) \]

\[ = \frac{C(\alpha_1 e^{\alpha_1 t} - \alpha_2 e^{\alpha_2 t})}{2i\omega_0} \Theta(t) + \frac{C(e^{\alpha_1 t} - e^{\alpha_2 t})}{2i\omega_0} \delta(t) \]

\[ \Rightarrow \ddot{x}_p(t) = \frac{C(\alpha_1^2 e^{\alpha_1 t} - \alpha_2^2 e^{\alpha_2 t})}{2i\omega_0} \Theta(t) + \frac{C(\alpha_1 e^{\alpha_1 t} - \alpha_2 e^{\alpha_2 t})}{2i\omega_0} \dot{\Theta}(t) \]

\[ = \frac{C(\alpha_1^2 e^{\alpha_1 t} - \alpha_2^2 e^{\alpha_2 t})}{2i\omega_0} \Theta(t) + \frac{C(\alpha_1 - \alpha_2)}{2i\omega_0} \delta(t) \]

\[ = \frac{C(\alpha_1^2 e^{\alpha_1 t} - \alpha_2^2 e^{\alpha_2 t})}{2i\omega_0} \Theta(t) + \frac{C}{a} \delta(t). \]

In simplifying these expressions we have used the knowledge that the derivative of the theta function is just the delta function (see Chapter 8.11 in Boas) and that the delta function can contribute only when multiplied by a function with a nonzero value at the point defined by the argument of the delta function \((t = 0\) in the present case), and we might as well evaluate the coefficient of the delta function at that value. Thus, when we substitute these expressions into the LHS of the differential equation, we find

\[ a\ddot{x}_p + b\dot{x}_p + cx_p = \frac{C\Theta(t)}{2i\omega_0} \left\{ \left( a\alpha_1^2 + b\alpha_1 + c \right) e^{\alpha_1 t} - \left( a\alpha_2^2 + b\alpha_2 + c \right) e^{\alpha_2 t} \right\} + C\delta(t) \]

\[ = C\delta(t). \]

The two quantities in the curly brackets vanish due to the definition of the exponents \(\alpha_{1,2}, a\alpha_{1,2}^2 + b\alpha_{1,2} + c = 0\). Thus we have indeed found a particular solution of the initial equation.
As a final example of these techniques let us return to the Fourier transform in Eq. (16.13) in the previous Lecture and confirm that we can use our new contour integral techniques to perform the inverse transform. This will allow us to discuss another example of how the choice of (how to close) the contour defines the final $t$ dependence and allow us to discuss the case when the integration contour passes through a singular point. This last issue is discussed in Boas Section 14.7, Example 4 and in exercise 14.7:21 in the HW. The central point is that the contribution of the pole on the contour is $\frac{1}{2}$ what you would normally expect, but the sign still depends on how the contour is closed. Recall we had the initial function in the figure and the inverse Fourier transform

$$F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega G(\omega) e^{i\omega t} = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \frac{\sin \omega}{\omega} e^{i\omega t}$$

(17.21)

To explicitly evaluate the integral using our new techniques we first reorganize the integrand to put it explicitly in the form in Example 4 in Boas. We have

$$F(t) = \frac{1}{2i\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega t} - e^{-i\omega t}}{\omega} e^{i\omega t}$$

(17.22)

Note that in this form we are explicitly integrating through the simple pole at the origin, i.e., there is a singularity on the contour, and we need to be a bit careful. We can obtain the general result from exercise 14.7:21 in the HW. In the case that it is appropriate to close the contour in the upper half-plane we have...
where we are thinking of the contour $\Gamma$ in Figure 7.3. Note that the pole at the origin does not contribute explicitly (it is outside of the contour), but it does guarantee a nonzero contribution from the vanishingly small contour $C'$. Thus for the simple case of Example 4 in Section 14.7 of Boas, where the numerator is just the complex exponential telling us to close the contour in the upper half-plane (where $y$ gives a damping exponential and the integral along $C$ vanishes) and there are no poles except at the origin, we have ($f(0) = 1$)

$$\int_{-\infty}^{\infty} dx \frac{e^{ix}}{x} = i\pi. \quad (17.24)$$

If the situation requires instead that we close the contour in the lower half-plane (but with the small semi-circle $C'$ the same), we now sum over the residues in the lower half-plane with the opposite sign (CW direction), include the contribution of the residue at the origin, which changes the sign of the second term, and the last integral along the contour $C$ is from 0 to $-\pi$ (also a change of sign). Thus we have for the lower half-plane
\[
\int_{-\infty}^{\infty} dx \frac{f(x)}{x} = -2\pi i \sum_{\text{Im } z < 0} \text{residues of } \frac{f(z)}{z} - 2\pi if(0) \\
+ i\pi f(0) - \lim_{R \to \infty} \int_{0}^{\pi} i d\theta f\left(Re^{i\theta}\right) \\
= -2\pi i \sum_{\text{Im } z < 0} \text{residues of } \frac{f(z)}{z} - \pi if(0) \\
- \lim_{R \to \infty} \int_{0}^{\pi} i d\theta f\left(Re^{i\theta}\right).
\] (17.25)

(Note that, if we also flipped the small semi-circle \(C'\) into the lower half-plane and did not encircle the pole at the origin, we would still obtain the change in sign due to the changed integral along \(C', 0 \geq \theta \geq -\pi\).) So in the case of the complex conjugate of the simple exponential, which is to be closed in the lower half-plane, \(y < 0\), we get the expected complex conjugate result (compare to Eq. (17.24))

\[
\int_{-\infty}^{\infty} dx \frac{e^{-ix}}{x} = \left[ \int_{-\infty}^{\infty} dx \frac{e^{ix}}{x} \right]^* = -i\pi.
\] (17.26)

So now we apply these ideas to the two similar integrals in Eq. (17.22). For the first integral with the factor \(\exp\left(i\omega(t+1)\right)\) we are led to close the contour in the upper half-plane when \(t > -1\) (\(\text{Re}\left[i\omega(t+1)\right] < 0\)) and in the lower half-plane for \(t < -1\). For the second integral with the factor \(\exp\left(i\omega(t-1)\right)\) we close in the upper half-plane for \(t > 1\) and in the lower half-plane for \(t < 1\). Thus via contour integration techniques we obtain that
\[ F(t) = \frac{1}{2i\pi} \left[ \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(t+1)}}{\omega} - \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(t-1)}}{\omega} \right] \]

\[ = \begin{cases} 
 1 < t \text{[both in upper half-plane]}, & \frac{1}{2i\pi} \left[ i\pi - (i\pi) \right] = 0 \\
 -1 < t < 1 \text{[1st in upper, 2nd in lower]}, & \frac{1}{2i\pi} \left[ i\pi - (i\pi) \right] = 1 \\
 t < -1 \text{[both in lower half-plane]}, & \frac{1}{2i\pi} \left[ -i\pi - (-i\pi) \right] = 0 
\end{cases} \quad (17.27) \]

So again the structure of the contour integral ensures the correct theta functions in the \( t \) dependence. In particular, recall that the definition of the theta function means

\[ \Theta(t+1) = \begin{cases} 
 1, & t + 1 > 0, t > -1 \\
 0, & t + 1 < 0, t < -1 
\end{cases}, \quad \Theta(1-t) = \begin{cases} 
 1, & 1 - t > 0, t < 1 \\
 0, & 1 - t < 0, t > 1 
\end{cases} \quad (17.28) \]

\[ \Rightarrow \Theta(t+1) \times \Theta(1-t) = \begin{cases} 
 0, & t < -1 \\
 1, & -1 < t < 1 \\
 0, & t > 1 
\end{cases} \]

In summary, at least in principle, we can now solve any 2\(^{nd}\) order linear inhomogeneous differential equation (with constant coefficients) whose driving function can be expressed in terms of a Fourier integral transform (or a Fourier series, if the function is periodic). The only essential limitation is that we have to assume that the function is (absolutely) integrable. We will address that limitation next using the Laplace transform.