Mathematica examples relevant to Bessel functions

The “original” Bessel function---that discussed extensively in Boas
Also called Bessel functions of the first kind, or cylindrical Bessel functions

It is a built-in function in Mathematica.

\[ \text{BesselJ}[0, x] \]
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Note that half-integer Bessels (spherical Bessels) can be given in terms of elementary functions.

\[ \text{BesselJ}[1/2, x] \]
\[ \sqrt{\frac{2}{\pi}} \sin(x) \]
\[ \sqrt{x} \]

\[ \text{BesselJ}[3/2, x] \]
\[ \sqrt{\frac{2}{\pi}} \left( -\cos(x) + \frac{\sin(x)}{x} \right) \]
\[ \sqrt{x} \]

It can be evaluated numerically

\{\text{BesselJ}[0, 3.2], \text{BesselJ}[5, 3.2], \text{BesselJ}[0.5, 3.2]\}
\{-0.320188, 0.056238, -0.0260367\}

\text{Plot}[\{\text{BesselJ}[0, x], \text{BesselJ}[1, x], \text{BesselJ}[2, x], \text{BesselJ}[3, x]\}, \{x, 0, 10\},
\text{PlotStyle} \rightarrow \{\{\text{Thick, Red}\}, \{\text{Thick, Green}\}, \{\text{Thick, Blue}\}, \{\text{Thick, Black}\}\},
\text{LabelStyle} \rightarrow \"Medium\",
\text{AxesLabel} \rightarrow \{x, J\},
\text{PlotLabel} \rightarrow \"First 4 cylindrical Bessel functions\",
\text{PlotLegends} \rightarrow \{\text{Subscript}[J, 0], \text{Subscript}[J, 1], \text{Subscript}[J, 2], \text{Subscript}[J, 3]\}]
Using “BesselJZero” to obtain the zeros of \( J_p(x) \)

Note that the results agree with the zeroes in the plot

\[
\text{Table} \{ \{ n, \text{BesselJZero}[0, n], \text{BesselJZero}[1, n], \text{BesselJZero}[2, n], \text{BesselJZero}[3, n], \} \} /\!\!/ \text{N} /\!\!/ \text{TableForm}
\]

\[
1. 2.40483 \quad 3.83171 \quad 5.13562 \quad 6.38016
2. 5.52008 \quad 7.01559 \quad 8.41724 \quad 9.76102
3. 8.65373 \quad 10.1735 \quad 11.6198 \quad 13.0152
4. 11.7915 \quad 13.3237 \quad 14.796 \quad 16.2235
\]

Orthogonality

\[
\{ \alpha = \text{BesselJZero}[0, 1], \beta = \text{BesselJZero}[0, 2], \gamma = \text{BesselJZero}[0, 3] \} /\!\!/ \text{N}
\]

\[
\text{Plot} \{ \{ \text{BesselJ}[0, \alpha x], \text{BesselJ}[0, \beta x], \text{BesselJ}[0, \gamma x] \}, \{ x, 0, 1 \}, \text{PlotStyle} \rightarrow \text{Thick}, \text{ PlotLegends} \rightarrow \{ \text{Subscript}[\text{J}, 0][\alpha x], \text{Subscript}[\text{J}, 0][\beta x], \text{Subscript}[\text{J}, 0][\gamma x] \} \}
\]

Mathematic can work out symbolically (using recursion relations, etc.) that these functions are exactly orthogonal, e.g.

\[
\text{Integrate} \left[ x \text{BesselJ}[0, \alpha x] \text{BesselJ}[0, \beta x], \{ x, 0, 1 \} \right] = 0
\]

Asymptotic forms (i.e. behavior at large \( x \))

\text{Mathematica} can determine these using \text{Series} and exapanding about infinity:

\[
\text{Series} \left[ \text{BesselJ}[p, x], \{ x, \text{Infinity}, 0 \} \right]
\]

\[
\cos \left[ \frac{\pi}{4} + \frac{p \pi}{2} - x \right] \left( \frac{2}{\sqrt{\pi}} \sqrt{\frac{1}{x} - 0.1^{-1}} \right)
\]

\[
\text{Jasy}[_, x_] := \text{Sqrt}\left[ \frac{2}{\pi x} \right] \cos\left[ x - (2 p + 1) \frac{\pi}{4} \right]
\]

The asymptotic for is exact for \( p=1/2 \)
\[ Jasy[1/2, x] \]
\[ \sqrt{\frac{2}{\pi}} \sqrt{\frac{1}{x}} \sin[x] \]

\[ \text{BesselJ}[1/2, x] \]
\[ \sqrt{\frac{2}{\pi}} \sin[x] \]
\[ \sqrt{x} \]

Asymptotic form only breaks down for \( x < 1 \) for \( J_0 \)

\[ \text{Plot}[\{\text{BesselJ}[0, x], Jasy[0, x]\}, \{x, 0, 30\}, \]
\[ \text{PlotRange} \rightarrow \{\{0, 30\}, \{-1, 1.4\}\}, \text{PlotStyle} \rightarrow \text{Thick}, \text{PlotLegends} \rightarrow \text{"Expressions"}] \]

Another way of testing the asymptotic forms: take out the \( 1/\sqrt{x} \) and shift the argument so the cosines are in phase

\[ \text{Plot} \left[ \{\text{BesselJ}[1, x], Jasy[1, x]\}, \{x, 0, 30\}, \right. \]
\[ \text{PlotRange} \rightarrow \{\{0, 30\}, \{-0.6, 0.6\}\}, \text{PlotStyle} \rightarrow \text{Thick} \]

\[ \text{testasy}[p_\_, x_\_] := \text{Sqrt}[\pi x/2] \text{BesselJ}[p, x + (2p + 1) \pi/4] \]
\textbf{Small }x\textbf{ behavior}

Similarly, we can expand around small \(x\)

\begin{equation}
\text{Series}[\text{BesselJ}[p, x], \{x, 0, 2\}]
\end{equation}

\begin{equation}
x^p \left(\frac{2^{-p}}{\Gamma[1 + p]} - \frac{2^{-2p} x^2}{(1 + p) \Gamma[1 + p]} + O[x]^3\right)
\end{equation}

\begin{equation}
\text{Jsmall}[p, x_] := \frac{x^p}{2^p p!}
\end{equation}

\text{Plot}[[\text{BesselJ}[0, x], \text{Jsmall}[0, x]], \{x, 0, 2\},
\text{PlotRange} \rightarrow \{\text{Automatic}, \{-1, 1.4\}\}, \text{PlotStyle} \rightarrow \text{Thick}, \text{PlotLegends} \rightarrow "\text{Expressions}\text{"}]

This is not a very good estimate for \(J_0\) as it is just a constant.
Plot[{BesselJ[1, x], Jsmall[1, x]}, {x, 0, 2},
PlotRange -> {Automatic, {-1, 1.4}}, PlotStyle -> Thick, PlotLegends -> "Expressions"]

This estimate is better for a larger range for larger p

Plot[{BesselJ[2, x], Jsmall[2, x]}, {x, 0, 2},
PlotRange -> {Automatic, {-1, 1.4}}, PlotStyle -> Thick, PlotLegends -> "Expressions"]

We can see there is a small region where neither the asymptotic form nor the small x form is a good estimate, this region grows larger with p. In this region more terms in the series expansions can be used.
Manipulate[Plot[{BesselJ[p, x], Jsmall[p, x], Jasy[p, x]}, {x, 0, 20}, PlotRange → {Automatic, {-1, 1.4}}, PlotStyle → Thick, PlotLegends → {"J_p(x)", "Jsmall[p,x]", "Jasy[p,x]"}, {p, 0, 5, .5}]

Bessel function of the second kind: BesselY

These functions diverge at x=0, as can be seen by using “Series”.
Here we see that Y_0 diverges as log(x):

Series[BesselY[0, x], {x, 0, 2}]
\[\frac{2}{\pi} \left(\text{EulerGamma} - \log(2) + \log(x)\right) + \frac{1}{2} \left(-\text{EulerGamma} + \log(2) - \log(x)\right) x^2 + O[x]^3\]

For half integer order have exact analytic expressions, e.g.

BesselY[1/2, x]
\[-\sqrt{\frac{2}{\pi}} \cos(\sqrt{x})\]

The asymptotic form is the same as for J_p except with Cos[ ] exchanged for a -Sin[ ]

Series[BesselY[p, x], {x, \infty, 0}]
\[-\sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{x}} + O\left[\frac{1}{x}\right]^1 \sin\left[\frac{\pi}{4} + \frac{p \pi}{2} - x\right]\]

Series[BesselJ[p, x], {x, 0, 0}] // FullSimplify
\[x^p \left(\frac{2^{-p}}{\Gamma[1 + p]} + O[x]^1\right)\]

The small x behavior has some cosmetic similarities but has a part which diverges as \(\frac{1}{x^p}\).
Series[BesselY[p, x], {x, 0, 0}] // FullSimplify

\[
x^p \left( -\frac{2^p \cos[p \pi]}{\pi} \Gamma[p] + O[x]^1 \right) + x^{-p} \left( -\frac{2^p \Gamma[p]}{\pi} + O[x]^1 \right)
\]

This is what the first few look like for integer \( p \):

Plot[{BesselY[0, x], BesselY[1, x], BesselY[2, x], BesselY[3, x]}, {x, 0, 10},
PlotStyle -> {Thick, Red}, {Thick, Green}, {Thick, Blue}, {Thick, Black}],
AxesLabel -> "\( x \), \( J \),
PlotLabel -> "First 4 cylindrical Bessel functions of 2nd kind",
PlotLegends -> {Subscript[Y, 0], Subscript[Y, 1], Subscript[Y, 2], Subscript[Y, 3]}]

First 4 cylindrical Bessel functions of 2nd kind

Spherical Bessel functions of the first kind (\( j_n \))

Spherical Bessels are obtained from cylindrical Bessels of half integer order as follows:

testspherical[p_, x_] = Sqrt[Pi/(2 x)] BesselJ[p + 1/2, x]

\[
\sqrt{\frac{\pi}{2}} \frac{1}{x} \text{BesselJ}[\frac{1}{2} + p, x]
\]

....but Mathematica has spherical Bessels already built in

Table[{testspherical[p, 0.3], SphericalBesselJ[p, 0.3]}, {p, 0, 2}]

\{\{0.985067, 0.985067\}, \{0.0991029, 0.0991029\}, \{0.00596152, 0.00596152\}\}

For any integer \( p \), we can get the explicit form of \( j_p(x) \)

FunctionExpand[Table[{p, SphericalBesselJ[p, x]}, {p, 0, 4}]] // TableForm

Spherical Bessel's behave for small \( x \) as \( x^p \). This is the same as we saw for the cylindrical Bessels as the \( \frac{1}{\sqrt{x}} \) in the definition removes the explicit \( 1/2 \) in \( J_{p+1/2}(x) \sim x^{p+1/2} \)
Spherical Bessel's behave for small $x$ as $x^{p}$. This is the same as we saw for the cylindrical Bessels as the $1/x$ in the definition removes the explicit $1/2$ in $J_{p+1/2}(x) \sim \sim x^{p+1/2}$.

Series[SphericalBesselJ[p, x], {x, 0, 0}]

\[ x^{p} \left( \frac{2^{-1-p} \sqrt{\pi}}{\Gamma\left( \frac{3}{2} + p \right)} + O(x) \right) \]

Here are the first 4:

Plot[{SphericalBesselJ[0, x], SphericalBesselJ[1, x], SphericalBesselJ[2, x], SphericalBesselJ[3, x]}, {x, 0, 20}, PlotRange -> {{0, 20}, {-0.3, 1}}, LabelStyle -> "Medium", PlotLabel -> "First 4 spherical Bessels of first kind", PlotLegends -> {Subscript[j, 0], Subscript[j, 1], Subscript[j, 2], Subscript[j, 3]}]