**Mathematica** for Dirac delta functions and Green functions

**DiracDelta function**

Mathematica has Dirac's delta function built in for use in integrals and solving differential equations. If you evaluate it directly you get 0 unless the argument is 0 in which case it gives you the function back---it is not evaluated and does not evaluate to infinity.

\[
\{\text{DiracDelta}[1], \text{DiracDelta}[0], \text{DiracDelta}[-1]\}
\]

\[
\{0, \text{DiracDelta}[0], 0\}
\]

So there is no peak in a plot of the function

\[
\text{Plot}[\text{DiracDelta}[x], \{x, -1, 1\}, \text{Exclusions} \to \text{None}, \text{PlotStyle} \to \{\text{Red}, \text{Thick}\}]
\]

![Graph of DiracDelta](image)

However, it integrates to give the theta function:

\[
\text{Integrate}[\text{DiracDelta}[x], x]
\]

\[
\text{HeavisideTheta}[x]
\]

\[
\text{Plot}[\text{HeavisideTheta}[x], \{x, -1, 1\}, \text{Exclusions} \to \text{None}, \text{PlotStyle} \to \{\text{Red}, \text{Thick}\}]
\]

![Graph of HeavisideTheta](image)
And differentiating the theta function returns the Dirac delta function

\[ D[\text{HeavisideTheta}[x], x] \]
\[ \text{DiracDelta}[x] \]

Further derivatives are just denoted symbolically

\[ D[\text{DiracDelta}[x], x] \]
\[ \text{DiracDelta}'[x] \]

Note that numerical integration over a delta function does not give the correct result because the routines cannot “find” the infinitely narrow peak. So, beware of this---use DiracDelta only in analytical integrations or solves.

\[ \text{NIntegrate}[\text{DiracDelta}, \{x, -1, 1\}] \]
\[ \text{NIntegrate::izero} : \text{Integral and error estimates are 0 on all integration subregions. Try increasing the value of the MinRecursion option. If value of integral may be 0, specify a finite value for the AccuracyGoal option.} \]
\[ 0. \]

Integrals using Dirac delta function

Defining properties

\[ \text{Integrate}[f[x] \text{DiracDelta}[x - 1], \{x, -\infty, \infty\}] \]
\[ f[1] \]

Integral vanishes if integration range does not include position of delta function

\[ \text{Integrate}[f[x] \text{DiracDelta}[x - 1], \{x, -\infty, 0.5\}] \]
\[ 0. \]

Dirac delta is an even function

\[ \text{Integrate}[f[x] \text{DiracDelta}[-x], \{x, -1, 1\}] \]
\[ f[0] \]

Correct Jacobians

\[ \text{Integrate}[f[x] \text{DiracDelta}[3 (x - 1)], \{x, -2, 2\}] \]
\[ f[1] \]
\[ \frac{1}{3} \]

Example from lecture

\[ \text{Integrate}[3 \text{Exp}[x] \text{DiracDelta}[\text{Sinh}[2 x]], \{x, -\infty, \infty\}] \]
\[ \frac{3}{2} \]

Checking explicitly:
\[ D[\text{Sinh}[2 \, x], \, x] \]
\[ 2 \, \text{Cosh}[2 \, x] \]
\[ 2 \, \text{Cosh}[0] \]
\[ 2 \]

\[ \text{Integrate}\left[\frac{3}{2} \, \text{Exp}[x] \, \text{DiracDelta}[x], \{x, -\infty, \infty\}\right] \]
\[ \frac{3}{2} \]

Correctly picks up multiple crossings of zero by argument of delta function

\[ \text{Integrate}[f[x] \, \text{DiracDelta}[\text{Sin}[2 \, x]], \{x, -1, \pi + 1\}] \]
\[ \frac{1}{2} \left( f[0] + f\left[\frac{\pi}{2}\right] + f[\pi] \right) \]

More examples from lecture

\[ \text{Integrate}[\text{Cos}[x/2] \, \text{DiracDelta}[\text{Sin}[x]], \{x, -\pi/2, 3 \, \pi/2\}] \]
\[ 1 \]

\[ \text{Integrate}[\text{Cos}[x/2] \, \text{DiracDelta}[\text{Sin}[x]], \{x, -\pi/2, -\pi/4\}] \]
\[ 0 \]

Integrals of derivatives of delta functions

\[ \text{Integrate}[f[x] \, \text{DiracDelta}'[x], \{x, -1, 1\}] \]
\[ -f'[0] \]

A few cautions about Mathematica's DiracDelta[]:

Only HeavisideTheta[x]'s derivative gives DiracDelta[], functions with similar behavior do not testfunctions =
\{ HeavisideTheta[x], UnitStep[x], (Sqrt[x^2]/x + 1)/2, (Abs[x]/x + 1)/2 \}
\{ HeavisideTheta[x], UnitStep[x], \frac{1}{2} \left( 1 + \frac{\sqrt{x^2}}{x} \right), \frac{1}{2} \left( 1 + \frac{\text{Abs}[x]}{x} \right) \}
The limits from each side have the appropriate values

\[
\text{Limit}[\text{testfunctions}, \ x \rightarrow 0, \ \text{Direction} \rightarrow 1] \rightarrow \{0, 0, 0, 0\}
\]

\[
\text{Limit}[\text{testfunctions}, \ x \rightarrow 0, \ \text{Direction} \rightarrow -1] \rightarrow \{1, 1, 1, 1\}
\]

At the discontinuous point they have different interpretations

\[
\text{testfunctions} /. \ x \rightarrow 0
\]

\[
\text{Power::infy: } \text{Infinite expression } \frac{1}{0} \text{ encountered. } \Rightarrow
\]

\[
\text{Infinity::indet: } \text{Indeterminate expression } 0 \text{ ComplexInfinity encountered. } \Rightarrow
\]

\[
\text{Power::infy: } \text{Infinite expression } \frac{1}{0} \text{ encountered. } \Rightarrow
\]

\[
\text{Infinity::indet: } \text{Indeterminate expression } 0 \text{ ComplexInfinity encountered. } \Rightarrow
\]

\[
\{\text{HeavisideTheta}[0], 1, \text{Indeterminate, Indeterminate}\}
\]

**Mathematica** will only correctly identify the derivative of HeavisideTheta[x] as DiracDelta[x]

\[
\text{D}[\text{testfunctions}, \ x]
\]

\[
\{\text{DiracDelta}[x], \begin{cases} \text{Indeterminate} & x = 0 \rightarrow 0, \\ \text{True} & 0, \frac{1}{2} \left( -\frac{\text{Abs}[x]}{x^2} + \frac{\text{Abs}'[x]}{x} \right) \end{cases}\}
\]

\[
\text{DiracDelta[} \phantom{1 + i} \text{]} \text{ is not defined for complex arguments:}
\]

\[
\text{DiracDelta}[2 + i]
\]

DiracDelta[2 + i]
Integrate[DiracDelta[x - 1], {x, -2 I, 2 I}]
\[\int_{-2}^{2} \text{DiracDelta}[-i + x] \, dx\]

Limits which can be used to define a delta function will not return \text{DeltaFunction}[ ]:
Limit[\varepsilon / (x^2 + \varepsilon^2), \varepsilon \to 0]
0

Integrals will not return a delta function:
Integrate[Exp[I k x], {x, -\infty, \infty}, \text{Assumptions} \to k \in \text{Reals}]
Integrate::idiv : Integral of \(e^{ikx}\) does not converge on \((-\infty, \infty)\).
Integrate[Exp[I k x], {x, -\infty, \infty}, \text{Assumptions} \to k \in \text{Reals}]

But \text{FourierTransform}[ ] will:
FourierTransform[Exp[I k x], x, y, \text{FourierParameters} \to \{-1, -1\}]
DiracDelta[k - y]

Sec. 8.11 #9---example worked in class

\textit{Mathematica} can deal with general \(t_0\) (which it assumes is real, as \text{DiracDelta} is only defined for reals):
Clear[t0, y]
soln9 = Flatten[DSolve[{y''[t] + 2 y'[t] + 10 y[t] == \text{DiracDelta}[t - t0], y'[0] == 0, y[0] == 0}, y[t], t] // Simplify]
\[\left\{y[t] \rightarrow \frac{1}{3} e^{-t-t0} (\text{HeavisideTheta}[3 t - 3 t0] - \text{HeavisideTheta}[-3 t0]) \sin[3 (t - t0)]\right\}\]

Make into a function:
\(\text{yy}[t_] = \text{FullSimplify}[$$\text{Flatten}[$$y[t]$$. soln9 $$]$$]
\[\frac{1}{3} e^{-t-t0} (\text{HeavisideTheta}[t - t0] - \text{HeavisideTheta}[-t0]) \sin[3 (t - t0)]\]

Next consider \(t0=1\) for definiteness.
This gives the answer obtained in class in various ways:
t0 = 1; \(\text{yy}[t]\)
\[\frac{1}{3} e^{-t} \text{HeavisideTheta}[-1 + t] \sin[3 (-1 + t)]\]

Recall this is a damped oscillator at rest given a unit impulse at \(t=1\), so that its slope has a discontinuity, although the function is continuous.
Finding Green function obtained in Boas using “DiracDelta”

\[
\text{soln} = \text{DSolve}\left[\{y''[t] + y[t] = \text{DiracDelta}[t - tp], y[0] = 0, y[\Pi/2] = 0\}, y[t], t\right]
\]

\[
\left\{
\{y[t] \to -\cos[tp] \text{HeavisideTheta}[\frac{\Pi}{2} - tp] \sin[t] + \cos[tp] \text{HeavisideTheta}[t - tp] \sin[t] - \cos[t] \text{HeavisideTheta}[t - tp] \sin[tp] + \cos[t] \text{HeavisideTheta}[-tp] \sin[tp]\}\right\}
\]

To simplify, need to let Mathematica know the range of “tp”. This result agrees with that we found.

\[
\text{soln2} = \text{Simplify}[\text{soln}, \text{Assumptions} \to \{0 < tp < \Pi/2\}]
\]

\[
\left\{\{y[t] \to -\cos[tp] \sin[t] + \text{HeavisideTheta}[t - tp] \sin[t - tp]\}\right\}
\]

Green function is thus

\[
GG[t\_\_, tp\_] = -\sin[t] \cos[tp] + \text{HeavisideTheta}[t - tp] \sin[t - tp];
\]

Sec. 8.12 #13---example worked in class

Forcing function.

\[
\text{ff}[x\_] = \text{Piecewise}\left[\{(x, 0 \leq x < \Pi/4), (\Pi/2 - x, \Pi/4 \leq x < \Pi/2)\}\right]
\]

\[
\begin{array}{c}
x \quad 0 \leq x < \frac{\Pi}{4} \\
\frac{\Pi}{2} - x \quad \frac{\Pi}{4} \leq x < \frac{\Pi}{2} \\
0 \quad \text{True}
\end{array}
\]
Plot[ff[x], {x, 0, Pi/2}, PlotStyle -> {Thick, Red}]

Here's the result for the solution using the Green function from above applied to the forcing function:

\[
\text{fullyy}[x_] = \text{Integrate}[GG[x, xp] ff[xp], \{xp, 0, Pi/2\}] // \text{Simplify}
\]

\[
\frac{1}{4} \left( -4 \left( -1 + \sqrt{2} \right) \sin[x] - \text{HeavisideTheta}[-\frac{\pi}{4} + x] \left( -2 \pi + 4 \pi x + 4 \cos\left[\frac{\pi}{4} + x\right] + 2 \left( \pi - 2 \pi x - 2 \cos[x] \right) \text{HeavisideTheta}\left[-\frac{\pi}{2} + x\right] + \pi \sin\left[\frac{\pi}{4} + x\right] \right) + \text{HeavisideTheta}[x] \right]
\]

\[
\left( 4 \left( x - \sin[x] \right) + \text{HeavisideTheta}\left[-\frac{\pi}{4} + x\right] \left( -4 \pi - 4 \pi x + 4 \pi \cos\left[\frac{\pi}{4} + x\right] + \pi \sin\left[\frac{\pi}{4} + x\right] \right) \right)
\]

To get simple form, need to tell \textit{Mathematica} the range of x:

\[
\text{yless}[x_] = \text{Integrate}[GG[x, xp] ff[xp], \{xp, 0, Pi/2\}, \text{Assumptions} \to \{0 < x < Pi/4\}]
\]

\[
x - \sqrt{2} \sin[x]
\]

\[
\text{ymore}[x_] = 
\quad \text{Integrate}[GG[x, xp] ff[xp], \{xp, 0, Pi/2\}, \text{Assumptions} \to \{Pi/4 < x < Pi/2\}]
\]

\[
\frac{1}{2} \left( \pi - 2 \pi x - 2 \sqrt{2} \cos[x] \right)
\]

Subsequent three plots show resulting \(y, y'\) and \(y''\), so that one can see the nature of the solution, whose third derivative is discontinuous.
Mathematica can also solve the differential equation including the piecewise function:
Simplify[DSolve[{y''[x] + y[x] == ff[x], y[0] == 0, y[Pi/2] == 0}, y[x], x]]

\[
\begin{cases}
-\left(-1 + \sqrt{2}\right) \cos[x] & 2 \, \pi > x \\
-\left(1 + \sqrt{2}\right) \sin[x] & x \leq 0 \\
\frac{1}{2} \left( x - 2 + 2 \sqrt{2} \cos[x] \right) & \frac{\pi}{4} < x \leq \frac{\pi}{2} \\
x - \sqrt{2} \sin[x] & \text{True}
\end{cases}
\]

Keep in mind when Mathematica says “True” for a piecewise function, it just means “Otherwise”