Quasinormal modes of Schwarzschild – AdS_5 black branes

Quasinormal modes are eigenfunctions of linearized perturbations around equilibrium or stationary solutions. Their (generally complex) frequecies characterize the relaxation of perturbations away from a thermal equilibrium state.

The metric $ds^2 = 2\,dt(dr - A(r)\,dt) + r^2(dx^2 + dy^2 + dz^2)$, with $A(r)=1/2\,(r^2 - m/r^2)$, describes a Schwarzschild – AdS_5 black brane which, via gauge/string duality, represents a thermal equilibrium state of a 3+1 dimensional quantum field theory at a temperature $T = m^{1/4}/\pi$. If an infinitesimal metric perturbation $\delta g_{xy} = r^2\,e^{i(qz-\omega t)}\,f(r)$ is added, then Einstein's equations require that f(r) be a solution to the equation:

$$(m - r^4)\,f''[r] + (m - 5\,r^4 + 2\,i\,r^3\,\omega)\,f'[r]/r + (q^2 + 3\,i\,r\,\omega)\,f[r] = 0$$

The physical domain is $r \in [r_h, \infty]$, where $r_h = m^{1/4}$ is the black brane horizon, and r=∞ is the AdS boundary. Plysical solutions must be regular at the horizon and vanish at the boundary. For a given value of the wavevector *q,* such solutions only exist for a discrete set of frequencies $\omega = \omega_i$ known as quasinormal mode freqencies.

Your task: accurately compute the first few (smallest) quasinormal mode frequencies at *q*=0.

Warm-up :

Is the horizon a singular point of the equation? Is the boundary?

What is the local behavior of the two linearly independent solutions near $r = \infty$?

What is the local behavior of the two linearly independent solutions near $r = r_h$?

Can you make a redefinition of f(r) which will map the semi-infinite domain into a finite interval (say [0,1]) and at the same time make the only needed boundary conditions regularity at either endpoint?

Based on answers to these warm-up questions, appropriately rewrite the equation in a more computationally amenable form, choose a spectral grid, and convert the equation into a finite dimensional generalized eigenvalue problem of the form $\alpha\,v = \omega\,\beta\,v$, where α and β are *M*x*M* matrices, *v* is the eigenvector, and ω is one of the desired eigenvalues. (Mathematica, Matlab, and most any linear algebra library has routines to solve such generalized eigenvalue problems.) Examine convergence with increasing truncation size *M*.