Physics 570 Assignment #5 December 3, 2010

Due 6pm Monday, December 13, 2010

Put completed homework in Prof. Yaffe's mailbox

1. Consider a theory of a massive real scalar field ϕ coupled to a massless real scalar field χ, with Lagrangian
$$-\mathcal{L} = \tfrac{1}{2}\left[(\partial\phi)^2 + m_0^2\phi^2 + (\partial\chi)^2\right] + \tfrac{1}{4!}\,\lambda_0\,\phi^4 + \tfrac{1}{4}\,\eta_0\,\phi^2\chi^2\,.$$
   (a) What are the Feynman rules of this theory?
   (b) Show that this theory, as it stands, is not renormalizable. In other words, show that there are contributions sensitive to the UV cutoff which cannot be absorbed by suitably adjusting bare parameters in the Lagrangian.
   (c) Show, to one-loop order, that this theory can be made renormalizable by adding a finite number of additional terms to the Lagrangian. What terms are necessary?
   (d) Find the one-loop beta-functions describing the renormalization group flow of the renormalized couplings $\lambda(\mu)$ and $\eta(\mu)$. That is, evaluate $\beta_\lambda \equiv \mu\frac{\partial}{\partial\mu}\lambda$ and $\beta_\eta \equiv \mu\frac{\partial}{\partial\mu}\eta$ to one-loop order.

2. Spontaneous symmetry breaking. Consider a real ϕ^4 scalar field theory with $m^2 < 0$ in d spacetime dimensions,
$$-\mathscr{L} = (\partial\phi)^2 + m^2\phi^2 + \tfrac{1}{4!}\lambda\phi^4\,.$$
   (a) Rewrite the potential energy in the form $(\lambda/4!)(\phi^2 - v^2)^2$ and relate v^2 to m^2 and λ. What are the global minima of the action?
   (b) Add a source term $-\int d^dx\,J(x)\phi(x)$ to the Euclidean action. Let $\bar\phi_J(x)$ be the field configuration which minimizes the action in the presence of the source. What equation does $\bar\phi_J$ satisfy? Suppose that $J(x)$ is constant. Find the explicit behavior of $\bar\phi_J$ as $J \to \pm\infty$ and $J \to \pm 0$. Sketch $\bar\phi_J$ as a function of J. Is it continuous? Sketch the the minimum action (density) as a function of J. Is it continuous?
   (c) Consider the functional integral for the Euclidean generating functional $Z[J]$. Expand about the minimum of the action (in the presence of J) by writing $\phi(x) = \bar\phi_J(x) + \chi(x)$. Reexpress the action in terms of χ. What are the Feynman rules?
   (d) Consider the limit $J \to 0^+$. What is the tree-level particle spectrum? What is the lowest order 2 to 2 particle scattering amplitude? Is the 2 to 3 particle scattering amplitude non-zero? If so, draw the relevant lowest order diagrams which describe this process.
   (e) The Lagrangian (with $J = 0$) is obviously invariant under the transformation $\phi(x) \to -\phi(x)$. Can physical particles be classified as even or odd under this symmetry? Is the ground state invariant under this symmetry? What happened to this symmetry?
   (f) How does the dimension d affect any of these results?

3. Diamagnetism. Consider the quantum theory of a charged (non-relativistic) spinless particle moving in a static external electromagnetic potential $A_\mu(\underset{\sim}{x})$.
   (a) What is the quantum Hamiltonian? What is the corresponding classical Lagrangian?
   (b) Derive a path integral representation for the partition function $Z = \mathrm{Tr}\,e^{-\beta H}$. You may wish to assume that A_μ is in Coulomb gauge.
   (c) Show that the logarithm of the resulting integrand is a discretized approximation to the integral $\int_0^\beta d\tau\,\{ip(\tau)\dot{x}(\tau) - h_{\rm cl}[p(\tau), x(\tau)]\}$, where $h_{\rm cl}[p, x]$ is the classical Hamiltonian.
   (d) Integrate (exactly) over the momenta $\{p_i\}$ and show that the logarithm of the resulting integrand may be regarded as a discretized approximation to the (Euclidean) classical action $-S_E[x(\tau)]$. How does S_E depend on the external gauge field?
   (e) Show that turning on an external magnetic field necessarily *decreases* the partition function, or increases the free energy. Explain why this provides a very simple proof of the diamagnetic behavior of spinless charged particles.