1. The self-energy $\tilde\Sigma(k) \equiv \tilde G(k)^{-1} - \tilde G^0(k)^{-1}$. Solving for $\tilde G(k)$ and inserting the explicit form of the relativistic free scalar propagator $\tilde G^0(k)$ gives $\tilde G(k) = 1/\big[\tilde G^0(k)^{-1} + \tilde\Sigma(k)\big] = 1/\big[k^2 + m^2 + \tilde\Sigma(k) - i\epsilon\big]$.

   (a) If the interacting theory describes a scalar particle of physical mass $m_{\rm ph}$, then there must exist single particle states $|\underset{\sim}{p}\rangle$ whose exact energy is $E_{\rm ph}(\underset{\sim}{p}) = (\underset{\sim}{p}^2 + m_{\rm ph}^2)^{1/2}$. The interacting propagator will have a contribution from these single particle intermediate states which varies with the phase e^{ipx} where $p^0 = E_{\rm ph}(\underset{\sim}{p})$:

$$
\begin{aligned}
G(x) &= i\langle 0|\mathscr{T}\big(\phi(x)\phi(0)\big)|0\rangle = i\Theta(x^0)\sum_n \langle 0|\phi(x)|n\rangle\langle n|\phi(0)|0\rangle + (x\to -x)\\
&= i\Theta(x^0)\int \frac{d^3p}{(2\pi)^3}\,\langle 0|\phi(x)|\underset{\sim}{p}\rangle\langle \underset{\sim}{p}|\phi(0)|0\rangle + \text{multi-particle stuff} + (x\to -x)\\
&= i\Theta(x^0)\int \frac{d^3p}{(2\pi)^3}\,e^{-iE_{\rm ph}(\underset{\sim}{p})x^0 + i\underset{\sim}{p}\cdot\underset{\sim}{x}}\big|\langle \underset{\sim}{p}|\phi(0)|0\rangle\big|^2 + \text{multi-particle stuff} + (x\to -x)\,.
\end{aligned}
$$

   So the Fourier transform $\tilde G(p)$ receives contributions from single particle states proportional to $i\int_0^\infty dx^0\,\Big(e^{i(p^0 - E_{\rm ph}(\underset{\sim}{p}))x^0} + e^{-i(p^0 + E_{\rm ph}(\underset{\sim}{p}))x^0}\Big) = [-p^0 + E_{\rm ph}(\underset{\sim}{p}) - i\epsilon]^{-1} + [p^0 + E_{\rm ph}(\underset{\sim}{p}) - i\epsilon]^{-1}$ $\propto [-(p^0)^2 + E_{\rm ph}(\underset{\sim}{p})^2 - i\epsilon]^{-1}$, and necessarily diverges when $p^0 = \pm E_{\rm ph}(\underset{\sim}{p})$. Therefore, a pole in the interacting propagator at $p^0 = \pm E$ is a signal that there exists, in the full theory, single particle states with energy E and momentum $\underset{\sim}{p}$. From the above representation of $\tilde G(p)$ in terms of the self-energy, such a pole will occur when the denominator has a zero, which occurs whenever there exists a four-momentum $\bar p$ for which $\bar p^2 + m^2 + \tilde\Sigma(\bar p) = 0$. If $|\tilde\Sigma(\bar p)| \ll m^2$ then the self-energy is a small perturbation, so the root $\bar p$ will be close to an unperturbed on-shell momentum p satisfying $p^2 + m^2 = 0$. Therefore, one may expand $\tilde\Sigma(\bar p)$ about the unperturbed on-shell momentum p and find $-\bar p^2 = m^2 + \tilde\Sigma(p) + \tilde\Sigma'(p)\,(\bar p - p) + \cdots = m^2 + \tilde\Sigma(p) + O(\tilde\Sigma^2/m^2)$ [since $\bar p^2 - p^2 = O(\tilde\Sigma)$]. Therefore, $m_{\rm ph}^2 = m^2 + \tilde\Sigma(p) + O(\tilde\Sigma^2/m^2)$. If $\operatorname{Im}\tilde\Sigma(p) \neq 0$, then the physical mass is not real, and the "on-shell" energy of the excitation is complex,

$$
E_{\rm ph} = \sqrt{\underset{\sim}{p}^2 + m^2 + \tilde\Sigma(p)} = \sqrt{\underset{\sim}{p}^2 + \operatorname{Re} m_{\rm ph}^2} + \frac{\frac{i}{2}\operatorname{Im}\tilde\Sigma(p)}{\sqrt{\underset{\sim}{p}^2 + \operatorname{Re} m_{\rm ph}^2}} + O(\operatorname{Im}\tilde\Sigma)^2\,.
$$

   Consequently, on-shell "plane-waves" will decay exponentially in time, $|e^{i\bar p\cdot x}| = \exp(-\frac{1}{2}\Gamma x^0)$ with a decay rate $\Gamma = -\operatorname{Im}\tilde\Sigma(p)/\operatorname{Re}E_{\rm ph}$, indicating that the interacting "particle" is not stable. To be physically sensible, the decay rate had better be positive, which means $\operatorname{Im}\tilde\Sigma(p)$ must be negative. Note that this is the same sign as the $-i\epsilon$ imaginary part of the free propagator.

   (b) Let the self-energy have a perturbative expansion $\tilde\Sigma(k) \sim \sum_{n=1}^\infty \tilde\Sigma^{(n)}(k)$ where $\tilde\Sigma^{(n)}(k) = O(\lambda^n)$. Note that $\tilde\Sigma^{(0)}(k) = 0$ since, by assumption, $G^0(k)$ is the correct propagator in the unperturbed theory. Start with the defining relation for $\tilde G(k)$ and expand in powers of $\tilde\Sigma$,

$$
\tilde G(k) = \Big[\tilde G^0(k)^{-1} + \tilde\Sigma(k)\Big]^{-1} = \tilde G^0(k)\,\Big[1 + \tilde\Sigma(k)\,\tilde G^0(k)\Big]^{-1} \sim \tilde G^0(k)\sum_{n=0}^\infty \Big(-\tilde\Sigma(k)\,\tilde G^0(k)\Big)^n\,.
$$

   Plug in the perturbative expansion of $\tilde\Sigma$ and collect terms at the same order to find

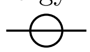
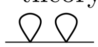
$$
\tilde G^{(n)} = \sum_{m=1}^n (-)^m \sum_{j_1=1}^n \cdots \sum_{j_m=1}^n \delta^n_{j_1+\cdots+j_m}\,\tilde G^0\,\tilde\Sigma^{(j_1)}\tilde G^0\,\tilde\Sigma^{(j_2)}\tilde G^0\cdots\tilde G^0\,\tilde\Sigma^{(j_m)}\,\tilde G^0\,,
$$

or

$$
\begin{aligned}
\tilde{G}^{(1)} &= -\tilde{G}^0\,\tilde{\Sigma}^{(1)}\,\tilde{G}^0\,, \\
\tilde{G}^{(2)} &= -\tilde{G}^0\,\tilde{\Sigma}^{(2)}\,\tilde{G}^0 + (-)^2\tilde{G}^0\,\tilde{\Sigma}^{(1)}\,\tilde{G}^0\,\tilde{\Sigma}^{(1)}\,\tilde{G}^0\,, \\
\tilde{G}^{(3)} &= -\tilde{G}^0\,\tilde{\Sigma}^{(3)}\,\tilde{G}^0 + (-)^2\tilde{G}^0\,\tilde{\Sigma}^{(1)}\,\tilde{G}^0\,\tilde{\Sigma}^{(2)}\,\tilde{G}^0 \\
&\qquad + (-)^2\tilde{G}^0\,\tilde{\Sigma}^{(2)}\,\tilde{G}^0\,\tilde{\Sigma}^{(1)}\,\tilde{G}^0 + (-)^3\tilde{G}^0\,\tilde{\Sigma}^{(1)}\,\tilde{G}^0\,\tilde{\Sigma}^{(1)}\,\tilde{G}^0\,\tilde{\Sigma}^{(1)}\,\tilde{G}^0\,.
\end{aligned}
$$

(c) To get some idea about what's going on, first look at the inverses of the above explicit expressions. Solving for $\tilde{\Sigma}^{(1)}$, $\tilde{\Sigma}^{(2)}$, etc., gives

$$
\begin{aligned}
-\tilde{\Sigma}^{(1)} &= (\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}\,, \\
-\tilde{\Sigma}^{(2)} &= (\tilde{G}^0)^{-1}\,\tilde{G}^{(2)}\,(\tilde{G}^0)^{-1} - (\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}\,, \\
-\tilde{\Sigma}^{(3)} &= (\tilde{G}^0)^{-1}\,\tilde{G}^{(3)}\,(\tilde{G}^0)^{-1} - (\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}\,\tilde{G}^{(2)}\,(\tilde{G}^0)^{-1} \\
&\qquad - (\tilde{G}^0)^{-1}\,\tilde{G}^{(2)}\,(\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1} \\
&\qquad + (\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}\,.
\end{aligned}
$$

Every diagram (beyond 0'th order) for the propagator $\tilde{G}(p)$ has two external lines which together contribute two powers of the free propagator, $\tilde{G}^0(p)^2$. The above result shows that if one simply omits these factors for the external lines when calculating diagrams which contribute to the first-order correction to the propagator, then the result is precisely (minus) the first-order self-energy $\tilde{\Sigma}^{(1)}$. Now consider second-order diagrams for the propagator. Some diagrams, such as  (in a ϕ^4 theory) consist of a single "blob" in between the two external lines. In contrast, the diagram  has two "blobs" connected by a single line. Momentum conservation forces the line connecting the two blobs to carry exactly the same momentum as the external lines, and therefore it simply contributes a factor $\tilde{G}^0(k)$ (where k is the external momentum). Each of the individual blobs represents exactly the same contribution which appeared in the first-order self-energy. Consequently, the entire contribution from this diagram, omitting the external lines, is $(-\tilde{\Sigma}^{(1)}(k))\,\tilde{G}^0(k)(-\tilde{\Sigma}^{(1)}(k)) = (\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}$. Therefore, the contribution of this diagram to the $(\tilde{G}^0)^{-1}\,\tilde{G}^{(2)}\,(\tilde{G}^0)^{-1}$ term in the expression for $\tilde{\Sigma}^{(2)}$ is exactly cancelled by the second term subtracting $(\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}$. So if one calculates the second-order propagator diagrams which do *not* contain two "blobs" connected by a single line, and omits the free propagator factors corresponding to the external lines, then the result is precisely (minus) the second-order self-energy $\tilde{\Sigma}^{(2)}(k)$. Similarly, the terms appearing in the expression for the third-order self-energy involving products of lower-order propagator corrections serve to exactly cancel the contribution from any third-order diagram which consists of multiple "blobs" connected by single lines. Therefore, (minus) the third order self-energy is obtained by omitting the propagators representing external lines in third-order propagator diagrams which contain only one "blob".

This is a general result. To see why, it is easiest to work backwards. Assume that the perturbative expansion of $-\tilde{\Sigma}(k)$ is given by the sum of all propagator diagrams with only one "blob" in which the free propagators for external lines are omitted. Then the n-th term in the expansion of the propagator in powers of the self-energy, $\tilde{G}(k) = \tilde{G}^0(k)\sum_{n=0}^{\infty}\left[-\tilde{\Sigma}(k)\,\tilde{G}^0(k)\right]^n$, is precisely the sum of all propagator diagrams consisting of n "blobs" connected by single lines. Consequently, every propagator diagram appears once (and only once) in this expansion, and makes the correct contribution to the full propagator. So the suggested diagrammatic interpretation of the self-energy must be correct.

(d) "Amputating" a diagram is defined to mean omitting the factors of free propagators for *external* lines, or equivalently multiplying by the inverse free propagator for each external line, so $\tilde{G}_{\rm amp}(p_1 \cdots p_n) \equiv \left[\prod_{j=1}^n i\,(p_j^2+m^2)\right] \tilde{G}(p_1 \cdots p_n)$. In coordinate space, this is a derivative operation, $G_{\rm amp}(x_1 \cdots x_n) \equiv \left[\prod_{j=1}^n i\,(-\partial_j^2+m^2)\right] G(x_1 \cdots x_n)$, where ∂_j is the gradient with respect to coordinate x_j.

"One-particle irreducible" diagrams (commonly abbreviated as 1PI) are diagrams which remain connected when any single line is cut. Any propagator diagram consisting of two (or more) "blobs" connected by a single line is "one-particle reducible" since it may be separated into two disconnected pieces by cutting the single line between blobs. Therefore, the result of part (c) may be stated more formally as "the perturbative expansion of (minus) the self-energy is given by the sum of all two-point amputated one-particle irreducible diagrams". Nothing in this discussion depends on the particular form of the interaction.

2. We have $-S = \int d^4x\, \big(\partial^\mu\phi^\dagger\partial_\mu\phi + m^2\phi^\dagger\phi - \Omega_0\big)$, with ϕ a complex scalar field.

(a) When varying the action, one may separate the complex field into real and imaginary parts, $\phi = ({\rm Re}\phi) + i({\rm Im}\phi)$, and then vary ${\rm Re}\phi$ and ${\rm Im}\phi$ independently. Or, more conveniently, one may choose to regard ϕ and $\phi^\dagger$ as independent, so that $\delta S = \int d^4x\, \frac{\delta\mathcal{L}}{\delta\phi^\dagger}\,(\delta\phi^\dagger) + \frac{\delta\mathcal{L}}{\delta\phi}\,(\delta\phi)$. Performing the variation gives $\delta S = \int d^4x\, \big[(\partial^\mu\delta\phi^\dagger)\partial_\mu\phi + m^2(\delta\phi^\dagger)\phi + \text{h.c.}\big]$, (with $+$ h.c. meaning "plus hermitian conjugate"). Integrating by parts to move the spacetime gradient off the variation produces $\delta S = \int d^4x\, (\delta\phi^\dagger)\left[-\partial^\mu\partial_\mu\phi + m^2\phi\right] + \text{h.c.}$. Demanding that this vanish for all $\delta\phi^\dagger$ and $\delta\phi$ implies that ϕ (as well as $\phi^\dagger$) satisfies the Klein-Gordon equation.

(b) The conjugate momenta are $\Pi \equiv \frac{\delta\mathcal{L}}{\delta\partial_0\phi^\dagger} = \partial_0\phi$ and $\Pi^\dagger \equiv \frac{\delta\mathcal{L}}{\delta\partial_0\phi} = \partial_0\phi^\dagger$. [Defining Π, and not $\Pi^\dagger$, as the variation with respect to $\partial_0\phi^\dagger$ is just a convention — but it is convenient.] The Hamiltonian density is $\mathcal{H} = \Pi(\partial_0\phi^\dagger) + \Pi^\dagger(\partial_0\phi) - \mathcal{L} = \Pi^\dagger\Pi + (\underset{\sim}{\nabla}\phi)^\dagger\cdot(\underset{\sim}{\nabla}\phi) + m^2\phi^\dagger\phi - \Omega_0$.

(c) Let $\widetilde{dk} \equiv \frac{d^3k}{(2\pi)^3 2\epsilon(\underset{\sim}{k})}$, with $\epsilon(\underset{\sim}{k}) = (\underset{\sim}{k}^2+m^2)^{1/2}$. Given $\phi(x) = \int \widetilde{dk}\,\big[a(\underset{\sim}{k})\,e^{ik\cdot x} + b(\underset{\sim}{k})^\dagger\,e^{-ik\cdot x}\big]$, the time derivative is $\Pi(x) = \partial_0\phi(x) = \int \widetilde{dk}\;\epsilon(\underset{\sim}{k})\,\big[-ia(\underset{\sim}{k})\,e^{ik\cdot x} + ib(\underset{\sim}{k})^\dagger\,e^{-ik\cdot x}\big]$. To solve for $a(\underset{\sim}{k})$ and $b(\underset{\sim}{k})$, one must do an inverse spatial Fourier transform of a suitable linear combination of $\phi(x)$ and $\Pi(x)$. Namely, $a(\underset{\sim}{k}) = \int d^3x\; e^{-ik\cdot x}\,[\epsilon(\underset{\sim}{k})\,\phi(x) + i\Pi(x)]$ and $b(\underset{\sim}{k})^\dagger = \int d^3x\; e^{ik\cdot x}\,[\epsilon(\underset{\sim}{k})\,\phi(x) - i\Pi(x)]$, or $b(\underset{\sim}{k}) = \int d^3x\; e^{-ik\cdot x}\,\big[\epsilon(\underset{\sim}{k})\,\phi(x)^\dagger + i\Pi(x)^\dagger\big]$.

(d) The canonical commutation relations for ϕ and Π are $i[\Pi(\underset{\sim}{x})^\dagger, \phi(\underset{\sim}{y})] = i[\Pi(\underset{\sim}{x}), \phi(\underset{\sim}{y})^\dagger] = \delta^3(\underset{\sim}{x}-\underset{\sim}{y})$ together with $[\Pi(\underset{\sim}{x}), \Pi(\underset{\sim}{y})] = [\Pi(\underset{\sim}{x})^\dagger, \Pi(\underset{\sim}{y})] = [\phi(\underset{\sim}{x}), \phi(\underset{\sim}{y})] = [\phi(\underset{\sim}{x})^\dagger, \phi(\underset{\sim}{y})] = [\Pi(\underset{\sim}{x}), \phi(\underset{\sim}{y})] = 0$. (It is $\Pi^\dagger$, and not Π, which has a non-zero commutator with ϕ because of our convention that the variation of the Lagrangian with respect to $\dot{\phi}$ is $\Pi^\dagger$, not Π.) Since the only non-zero commutators involve a $\Pi^\dagger$ with ϕ, or Π with $\phi^\dagger$, the above expressions for $a(\underset{\sim}{k})$ and $b(\underset{\sim}{k})$ immediately imply that $[a(\underset{\sim}{k}), a(\underset{\sim}{k}')] = [b(\underset{\sim}{k}), b(\underset{\sim}{k}')] = [a(\underset{\sim}{k}), b(\underset{\sim}{k}')^\dagger] = [b(\underset{\sim}{k}), a(\underset{\sim}{k}')^\dagger] = 0$. The remaining possibilities take a bit more work. First,

$$\begin{aligned}
[a(\underset{\sim}{k}), b(\underset{\sim}{k}')] &= \int d^3x\, d^3y\; e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\underset{\sim}{k}'\cdot\underset{\sim}{y}}\; \Big[\epsilon(\underset{\sim}{k})\,\phi(\underset{\sim}{x}) + i\Pi(\underset{\sim}{x}), \epsilon(\underset{\sim}{k}')\,\phi(\underset{\sim}{y})^\dagger + i\Pi(\underset{\sim}{y})^\dagger\Big] \\
&= \int d^3x\, d^3y\; e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\underset{\sim}{k}'\cdot\underset{\sim}{y}}\; \delta^3(\underset{\sim}{x}-\underset{\sim}{y})\;(\epsilon(\underset{\sim}{k}') - \epsilon(\underset{\sim}{k})) \\
&= \int d^3x\; e^{-i(\underset{\sim}{k}+\underset{\sim}{k}')\cdot\underset{\sim}{x}}\,(\epsilon(\underset{\sim}{k}') - \epsilon(\underset{\sim}{k})) = (\epsilon(\underset{\sim}{k}') - \epsilon(\underset{\sim}{k}))\,(2\pi)^3\delta^3(\underset{\sim}{k}+\underset{\sim}{k}') = 0\,.
\end{aligned}$$

(In the last step, $\epsilon(\underset{\sim}{k}) - \epsilon(-\underset{\sim}{k}) = 0$ since $\epsilon(\underset{\sim}{k})$ is even.) Conjugating gives $[a(\underset{\sim}{k})^\dagger, b(\underset{\sim}{k}')^\dagger] = 0$.

Next,

$$
\begin{aligned}
[a(\utilde{k}), a(\utilde{k}')^\dagger] &= \int d^3x\, d^3y\; e^{-i\utilde{k}\cdot\utilde{x}+i\utilde{k}'\cdot\utilde{y}} \left[\epsilon(\utilde{k})\, \phi(\utilde{x}) + i\Pi(\utilde{x}), \epsilon(\utilde{k}')\, \phi(\utilde{y})^\dagger - i\Pi(\utilde{y})^\dagger\right] \\
&= \int d^3x\, d^3y\; e^{-i\utilde{k}\cdot\utilde{x}+i\utilde{k}'\cdot\utilde{y}}\, \delta^3(\utilde{x}-\utilde{y})\, (\epsilon(\utilde{k}') + \epsilon(\utilde{k})) \\
&= \int d^3x\; e^{-i(\utilde{k}-\utilde{k}')\cdot\utilde{x}}\, (\epsilon(\utilde{k}') + \epsilon(\utilde{k})) = 2\epsilon(\utilde{k})\, (2\pi)^3\delta^3(\utilde{k}-\utilde{k}')\,,
\end{aligned}
$$

and exactly the same steps give $[b(\utilde{k}), b(\utilde{k}')^\dagger] = 2\epsilon(\utilde{k})\, (2\pi)^3\delta^3(\utilde{k}-\utilde{k}')\,,$.

(e) It is convenient to rewrite the mode expansion of ϕ and Π as $\phi(\utilde{x}) = \int \widetilde{dk}\, e^{i\utilde{k}\cdot\utilde{x}}\, \big(a(\utilde{k}) + b(-\utilde{k})^\dagger\big)$ and $\Pi(\utilde{x}) = -i\int \widetilde{dk}\; e^{i\utilde{k}\cdot\utilde{x}}\, \epsilon(\utilde{k})\, \big(a(\utilde{k}) - b(-\utilde{k})^\dagger\big)$. Plugging these into the Hamiltonian gives

$$
\begin{aligned}
H = \int d^3x\; \widetilde{dk}\; \widetilde{dk}'\; e^{-i(\utilde{k}-\utilde{k}')\cdot\utilde{x}} \Big\{ &\epsilon(\utilde{k})\epsilon(\utilde{k}')\, \Big(a(\utilde{k})^\dagger - b(-\utilde{k})\Big) \Big(a(\utilde{k}') - b(-\utilde{k}')^\dagger\Big) \\
&+ (\utilde{k}\cdot\utilde{k}' + m^2)\, \Big(a(\utilde{k})^\dagger + b(-\utilde{k})\Big) \Big(a(\utilde{k}') + b(-\utilde{k}')^\dagger\Big)\Big\} - \int d^3x\; \Omega_0\,.
\end{aligned}
$$

The $\utilde{x}$ integral yields $(2\pi)^3\delta^3(\utilde{k}-\utilde{k}')$, which makes the $\utilde{k}'$ integral trivial (but don't forget the $2\epsilon(\utilde{k})$ denominator hidden in $\widetilde{dk}'$), giving

$$
\begin{aligned}
H &= \int \widetilde{dk}\; (2\epsilon(\utilde{k}))^{-1}\, \Big\{\epsilon(\utilde{k})^2\, \Big(a(\utilde{k})^\dagger - b(-\utilde{k})\Big) \Big(a(\utilde{k}) - b(-\utilde{k})^\dagger\Big) \\
&\qquad\qquad + (\utilde{k}^2 + m^2)\, \Big(a(\utilde{k})^\dagger + b(-\utilde{k})\Big) \Big(a(\utilde{k}) + b(-\utilde{k})^\dagger\Big)\Big\} - \int d^3x\; \Omega_0 \\
&= \int \widetilde{dk}\; \epsilon(\utilde{k}) \Big\{a(\utilde{k})^\dagger a(\utilde{k}) + b(\utilde{k}) b(\utilde{k})^\dagger\Big\} - \int d^3x\; \Omega_0 \\
&= \int \widetilde{dk}\; \epsilon(\utilde{k}) \Big\{a(\utilde{k})^\dagger a(\utilde{k}) + b(\utilde{k})^\dagger b(\utilde{k})\Big\} - \int d^3x \Big[\Omega_0 - \int d^3k\; \epsilon(\utilde{k})\Big].
\end{aligned}
$$

The last step used the fact that $b(\utilde{k})b(\utilde{k})^\dagger = b(\utilde{k})^\dagger b(\utilde{k}) + 2\epsilon(\utilde{k})\,(2\pi)^3\delta^3(\utilde{k}-\utilde{k}) = b(\utilde{k})^\dagger b(\utilde{k}) + 2\epsilon(\utilde{k})\, \int d^3x$. The Hamiltonian will equal $\int \widetilde{dk}\; \epsilon(\utilde{k})\big\{a(\utilde{k})^\dagger a(\utilde{k}) + b(\utilde{k})^\dagger b(\utilde{k})\big\}$, and have vanishing ground state energy, if $\Omega_0 \equiv \int d^3k\; \epsilon(\utilde{k})$. This is exactly the expected vacuum energy subtraction: $\frac{1}{2}\epsilon(\utilde{k})$ for each mode of the a-particles, and an equal amount for the b-particles.

3. We are given the Lagrangian density $-\mathcal{L} = |\partial\phi_e|^2 + m_e^2|\phi_e|^2 + |\partial\phi_\mu|^2 + m_\mu^2|\phi_\mu|^2 + \frac{1}{2}(\partial\chi)^2 + g\,\chi\,(|\phi_e|^2 + |\phi_\mu|^2)$ with m_e positive and much smaller than m_μ.

   (a) In the free theory with $g = 0$, the complex scalar field ϕ_μ will destroy a spinless "muon" particle, which will be denoted symbolically as μ^-, or create its antiparticle, an "antimuon" (μ^+). Similarly, ϕ_e will destroy a scalar "electron" (e^-) or create a "positron" (e^+), and the real field χ will create or destroy a scalar "photon" (γ). When the coupling g is non-zero, the $g\,\chi\,|\phi_\mu|^2$ interaction generates non-zero matrix elements between states with a single (bare) photon and states with a (bare) muon and antimuon, as well as non-zero matrix elements between single muon and muon plus photon states. Similarly, the $g\,\chi\,|\phi_e|^2$ interaction generates non-zero matrix elements between single photon and electron-positron pair states, or between electron and electron plus photon states. Note that the interactions do not change the net electron number (*i.e.*, number of electrons minus positrons) or the net muon number.

   Despite these non-zero matrix elements, a single electron (or other charged particle) cannot spontaneously emit a photon because 4-momentum cannot be conserved in such a process.

Similarly, a single photon cannot decay to an electron-positron (or muon-antimuon) pair because this would also violate 4-momentum conservation. [The easiest way to prove this is to note that the 4-momentum of a massless photon is null, $k^2 = 0$, while the total 4-momentum of an electron-positron pair satisfies $(p_1 + p_2) \leq (2m_e)^2$, and hence cannot equal a null vector.] More generally, as long as the photon mass is less then twice the electron or muon rest masses, 4-momentum conservation guarantees that the photon must be stable. And 4-momentum conservation combined with the conservation of net electron and muon numbers gurantees that electrons and muons must also be stable. Therefore, the interacting theory will have the same spectrum of stable particles as the $g = 0$ theory: the massless photon and a massive electron and muon, plus their antiparticles.

(b) The (Minkowski space) Feynman rules are:

i. Draw diagrams of the appropriate topology (*i.e.*, number of external lines) composed of directed solid lines (representing the bare electron propagator), directed dashed lines (reprsenting the muon propagator), and undirected wavy lines (representing the photon propagator), and cubic vertices at which a photon line connects to either an incoming and outgoing electron line, or an incoming and outgoing muon line.[1]

ii. Label each line with a four-momentum, and conserve momentum at every vertex.

iii. Each vertex corresponds to a factor of $-ig$.

iv. Each electron line labeled by momentum p corresponds to a factor of $-i/(p^2 + m_e^2 - i\epsilon)$,

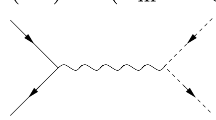
v. Each muon line labeled by momentum p corresponds to a factor of $-i/(p^2 + m_\mu^2 - i\epsilon)$.

vi. Each photon line labeled by momentum k corresponds to a factor of $-i/(k^2 - i\epsilon)$.

vii. Integrate [with measure $d^4l/(2\pi)^4$] over each undetermined loop momentom l.

viii. Divide by the symmetry factor of the diagram, which is number of permutations exchanging vertices, or lines connecting vertices, which leave the diagram (before labeling momenta) unchanged.

The result is a contribution to the Fourier transformed vacuum expectation value of the time-ordered product of fields corresponding to the given number and types of external lines. To obtain a contribution to the associated covariant scattering amplitude (times $-i$), amputate (*i.e.*, ignore) the external lines, divide by appropriate wavefunction renormalization factors, ignore the overall momentum conserving $(2\pi)^4\,\delta^4(P_{\rm in} - P_{\rm out})$, and evaluate on-shell.

(c) There is a single lowest order diagram:  . This gives a covariant scattering amplitude $-i\mathcal{M} = (-ig)^2/[(p_1+p_2)^2 - i\epsilon]$, where p_1 and p_2 are the momenta of the incoming electron and positron, corresponding to the external propagators attached to (say) the left vertex in the diagram.

(d) The differential cross section $d\sigma = \frac{d^3p_1'}{(2\pi)^3(2E_1')}\,\frac{d^3p_2'}{(2\pi)^3(2E_2')}\,\frac{|\mathcal{M}|^2}{(2E_1)(2E_2)v_{\rm rel}}\,(2\pi)^4\delta^4(p_1+p_2-p_1'-p_2')$, where p_1' and p_2' are the momenta of the outgoing muon and antimuon, respectively, and $E_1 = (p_1)^0$, etc. Evaluating this in the center-of-mass frame, we have $\tilde{p}_1 = -\tilde{p}_2$, $E_1 = E_2 = \frac{1}{2}E_{\rm c.m.}$, and $(p_1+p_2)^2 = -(E_{\rm c.m.})^2$. Since this cannot vanish, the $i\epsilon$ in the denominator of $\mathcal{M}$ is

[1] Using solid, dashed, and wavy lines to distinguish electron, muon, and photon propagators, respectively, is merely a convention — but one needs to adopt some means of distinguishing the different types of propagators.

irrelevant. The relative velocity $v_{\rm rel} = 2|\underset{\sim}{p}_1|/E_1$ with $|\underset{\sim}{p}_1| = \sqrt{E_1^2 - m_e^2} = \frac{1}{2}\sqrt{E_{\rm c.m.}^2 - 4m_e^2}$. So

$$\begin{aligned}
\sigma_{\rm c.m.} &= \frac{g^4}{(E_{\rm c.m.})^6}\,\frac{E_1}{2|\underset{\sim}{p}_1|}\int \frac{d^3p_1'}{(2\pi)^3(2E_1')}\,\frac{d^3p_2'}{(2\pi)^3(2E_2')}\,(2\pi)^4\delta^4(p_1+p_2-p_1'-p_2') \\
&= \frac{g^4}{(E_{\rm c.m.})^8}\,\frac{E_1}{2|\underset{\sim}{p}_1|}\int \frac{d\Omega}{4\pi^2}\int_0^\infty dp'\,p'^2\;\delta(E_{\rm c.m.}{-}2E') \\
&= \frac{g^4}{(E_{\rm c.m.})^8}\,\frac{E_1}{2|\underset{\sim}{p}_1|}\,\frac{1}{\pi}\int_{m_\mu}^\infty dE'\,E'p'\;\tfrac{1}{2}\delta(E'{-}\tfrac{1}{2}E_{\rm c.m.}) \\
&= \Theta(E_{\rm c.m.}{-}2m_\mu)\,\frac{g^4}{(E_{\rm c.m.})^8}\,\frac{E_1}{2|\underset{\sim}{p}_1|}\,\frac{E'|\underset{\sim}{p}'|}{2\pi}\bigg|_{E'=E_1=\frac{1}{2}E_{\rm c.m.}} \\
&= \Theta(E_{\rm c.m.}{-}2m_\mu)\,\frac{1}{16\pi}\,\frac{g^4}{(E_{\rm c.m.})^6}\left[\frac{E_{\rm c.m.}^2 - 4m_\mu^2}{E_{\rm c.m.}^2 - 4m_e^2}\right]^{1/2}.
\end{aligned}$$

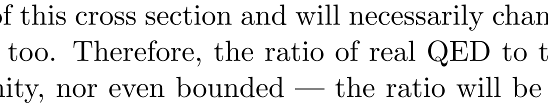
Note that $|\underset{\sim}{p}'| \neq |\underset{\sim}{p}_1|$, even though $E' = E_1$, because the initial and final particles have different masses. The step function $\Theta(E_{\rm c.m.}{-}2m_\mu)$ reflects the fact that the delta-function in the last integral cannot be satisfied if $E_{\rm c.m.} < 2m_\mu$.

(e) The threshold energy $E_{\rm min} = 2m_\mu$. For $E_{\rm c.m.} - E_{\rm min} \ll E_{\rm min}$, the cross section $\sigma_{\rm c.m.} \sim \frac{g^4}{16\pi}\,(2m_\mu)^{-7}\sqrt{E_{\rm c.m.}^2 {-} (2m_\mu)^2}$, where $m_\mu \gg m_e$ has been used to simplify the form. Consequently, the cross-section rises from threshold proportional to $\sqrt{\Delta E}$ where $\Delta E = E_{\rm c.m.} - 2m_\mu$ is the energy above threshold.

For $E_{\rm c.m.} \gg E_{\rm min}$, the muon (and electron) rest mass may be neglected relative to $E_{\rm c.m.}$ so the cross section simplifies to $\sigma_{\rm c.m.} \sim \frac{g^4}{16\pi}\,(E_{\rm c.m.})^{-6}$.

(f) The fact that real fermions have spin will change the angular dependence of the differential cross section — but this is to be expected. A much bigger difference comes from the fact that the coupling constant g in our toy model has dimensions of energy. This is why the asymptotic high energy cross section falls as $E_{\rm c.m.}^{-6}$ [since $g^4/E_{\rm c.m.}^6$ is the only quantity with dimensions of area which is $O(g^4)$ and depends only on g and $E_{\rm c.m.}$]. In real QED, the coupling constant e is dimensionless (in natural units). Hence the cross section $\sigma_{e^+e^-\to\mu^+\mu^-} \propto e^4/E_{\rm c.m.}^2 \sim \alpha^2/E_{\rm c.m.}^2$ at high energies. So replacing real (spin one) photons by spinless toy photons. changes the high energy behavior of this cross section and will necessarily change the high energy behavior of every other process too. Therefore, the ratio of real QED to toy QED results will not, in general, be close to unity, nor even bounded — the ratio will be arbitrarily large (or small). So I would call this toy model a rather poor approximation to real QED.

4. We have a real scalar field with Lagrange density $\mathcal{L} \equiv \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m_0^2\phi^2 + \frac{1}{4!}\lambda_0\phi^4 - (\text{const.})$. To organize perturbation theory, it is convenient to rewrite this as $\mathcal{L} = \mathcal{L}_{\rm free} + \mathcal{L}_{\rm int}$ with $\mathcal{L}_{\rm free} \equiv \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2$, and $\mathcal{L}_{\rm int} \equiv \frac{1}{4!}\lambda\phi^4 + \frac{1}{2}\delta m^2\phi^2 + \frac{1}{4!}\delta\lambda\,\phi^4$. All we have done is defined $m_0^2 = m^2 + \delta m^2$ and $\lambda_0 = \lambda + \delta\lambda$, where m is the physical particle mass and λ is a renormalized coupling (to be defined below). Choosing to regard the δm^2 and $\delta\lambda$ terms as new interactions, we will have three interaction vertices:



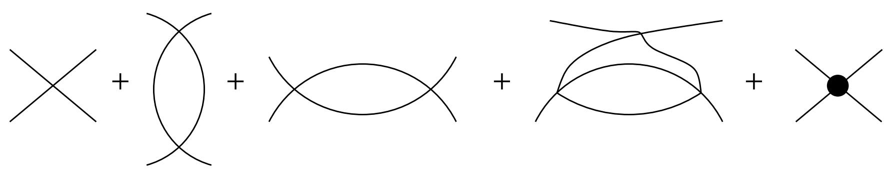
which, in Euclidean Feynman rules, represent factors of $-\lambda$, $-\delta m^2$, and $-\delta\lambda$, respectively. The mass counterterm δm^2 will be $O(\lambda)$, while the coupling counterterm $\delta\lambda \sim O(\lambda^2)$.

(a) Let $G(p)$ be the Fourier transformed Euclidean two-point function $\langle 0|\mathcal{T}\phi(x)\phi(y)|0\rangle$. The inverse of the full momentum space propagator has the form $G(p)^{-1} = p^2 + m^2 + \Sigma(p^2)$, where $-\Sigma$ is the sum of all amputated 1PI diagrams that correct the free propagator, . To one-loop order, $-\Sigma(p^2) = -\lambda\, K(m^2) - \delta m^2 + O(\lambda^2)$, where $K(m^2) \equiv \frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2+m^2}$. This integral diverges. Regulate it by introducing a UV momentum cutoff, $|q| < \Lambda$. This choice of regulator preserves the $O(4)$ rotational symmetry of the integral, so one may do the angular integration immediately. This gives a factor of the volume of a three-sphere, $2\pi^2$. The remaining radial integral is straightforward:

$$K(m^2) = \tfrac{1}{2}\int^{\Lambda} \frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2+m^2} \equiv \frac{2\pi^2}{32\pi^4}\int_0^{\Lambda} \frac{dq\,q^3}{q^2+m^2} = \frac{1}{32\pi^2}\int_0^{\Lambda^2} \frac{dq^2\,q^2}{q^2+m^2}$$
$$= \frac{m^2}{32\pi^2}\left[\frac{\Lambda^2}{m^2} - \ln\left(1+\frac{\Lambda^2}{m^2}\right)\right] = \frac{m^2}{32\pi^2}\left[\frac{\Lambda^2}{m^2} - \ln\left(\frac{\Lambda^2}{m^2}\right) + O\left(\frac{m^2}{\Lambda^2}\right)\right].$$

We will be sending $\Lambda \to \infty$, so $O(m^2/\Lambda^2)$ terms may be ignored. To one-loop order, the inverse propagator $G(p)^{-1} = p^2 + m^2 + \delta m^2 + \lambda\, K(m^2)$. Since, by assumption, m is the physical particle mass, $G(p)$ must have a single-particle pole at $p^2 = -m^2$. Equivalently, $G(p)^{-1}$ must vanish at $p^2 = -m^2$. Therefore, $\delta m^2 = -\lambda\, K(m^2) + O(\lambda^2) = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{m^2}\right)\right] + O(\lambda^2)$. Hence, bare mass varies with the cutoff as $m_0^2(\Lambda) \equiv m^2 + \delta m^2(\Lambda) = m^2 - \frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{m^2}\right)\right] + O(\lambda^2)$. Note that the quartic interactions act to increase the mass, so the bare mass must decrease with increasing λ to keep the physical mass fixed.

(b) The Fourier transformed connected four-point correlator can be expressed as $G^{(4)}_{\text{conn.}}(\{p_i\}) = -\Gamma^{(4)}(\{p_i\}) \prod_{i=1}^4 G(p_i)$ with the vertex function $-\Gamma^{(4)}$ given by the sum of all amputated connected diagrams with four external lines. It is convenient to regard all momenta as incoming. The one-loop correction to the vertex function is a sum of s, t, and u channel contributions, plus the one-loop coupling counterterm:



or $-\Gamma^{(4)}(\{p_i\}) = -\lambda + \lambda^2\left[J(p_1{+}p_2) + J(p_1{+}p_3) + J(p_1{+}p_4)\right] - \delta\lambda + O(\lambda^3)$, where we have defined the loop integral $J(p) \equiv \frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\,\frac{1}{q^2+m^2}\,\frac{1}{(q-p)^2+m^2}$. Since the full propagators at external momenta are factored out of the amputated vertex function, the one-loop self-energy corrections for external lines, and corresponding mass counterterms, are not part of $\Gamma^{(4)}$.

By Lorentz invariance, the integral J can only depend on p^2. For vanishing external momenta, $\Gamma^{(4)}(\{p_i{=}0\}) = \lambda - 3\lambda^2 J(0) + \delta\lambda + O(\lambda^3)$. To keep the renormalized coupling λ, defined to equal the value of the vertex function at vanishing external momenta, fixed as the cutoff $\Lambda \to \infty$, we need to choose the coupling counter-term $\delta\lambda = 3\lambda^2 J(0) + O(\lambda^3)$. Regulate the integral defining $J(0)$ by imposing a momentum cutoff, $J(0) = \frac{1}{2}\int^{\Lambda} \frac{d^4q}{(2\pi)^4}\,\frac{1}{(q^2+m^2)^2} \equiv \frac{1}{32\pi^2}\int_0^{\Lambda^2} dq^2\,\frac{q^2}{(q^2+m^2)^2}$. One may evaluate this directly, or notice that $J(0) = -\frac{\partial}{\partial m^2} K(m^2)$ and use the previous result for $K(m^2)$. Either way, one finds $J(0) = \frac{1}{32\pi^2}\left[\ln\left(\Lambda^2/m^2\right) - 1\right] + O(m^2/\Lambda^2)$. Thus, $\lambda_0(\Lambda) \equiv \lambda + \delta\lambda(\Lambda) = \lambda + \frac{3}{32\pi^2}\lambda^2 \ln\left(\Lambda^2/m^2\right) + O(\lambda^3)$. Note that the bare coupling λ_0 is larger than the renormalized coupling by an amount which grows logarithmically as the UV cutoff increases.

(c) Writing out the definitions: $s \equiv -(p_1+p_2)^2$, $t \equiv -(p_1-p_1')^2$, and $u \equiv -(p_1-p_2')^2$, one has

$$\begin{aligned} s+t+u &= -p_1^2 - p_2^2 - 2p_1\cdot p_2 - p_1^2 - p_1'^2 + 2p_1\cdot p_1' - p_1^2 - p_2'^2 + 2p_1\cdot p_2' \\ &= -3p_1^2 - p_2^2 - p_1'^2 - p_2'^2 + 2p_1\cdot(p_1'+p_2'-p_2) \\ &= -p_1^2 - p_2^2 - p_1'^2 - p_2'^2 = 4m^2\,, \end{aligned}$$

where the penultimate step used 4-momentum conservation, $p_1+p_2 = p_1'+p_2'$, and the last step used the condition that all four momenta are "on-shell", $p^2 = -m^2$. More generally, if the particles involved have differing masses then $s+t+u = \sum_{i=1}^4 m_i^2$. It is generally convenient to write covariant scattering amplitudes in terms of these Mandelstam variables. To relate them to quantities which can be measured in a lab, first note that all of the external momenta are timelike (future-directed) with positive energy. So p_1+p_2 is timelike, while p_1-p_1' and p_1-p_2' are spacelike. Therefore, $s>0$ and $t,u<0$. In the center-of-momentum (c.m.) frame, one may parametrize the $2\leftrightarrow 2$ scattering process by $p_1 = (E,p,0,0)$, $p_2 = (E,-p,0,0)$, $p_1' = (E, p\cos\theta, p\sin\theta, 0)$, and $p_2' = (E, -p\cos\theta, -p\sin\theta, 0)$. The outgoing momenta define a plane which we have taken to be the xy plane. Plugging in, we find

$$s = (2E)^2 = E_{\text{c.m.}}^2, \quad t = -2p^2(1-\cos\theta), \quad u = -2p^2(1+\cos\theta).$$

So s, which is bounded below by $4m^2$, is just the square of the CM-frame total energy. The other two variables characterize momentum transfer; they satisfy $-4p^2 < t,u < 0$. Observe that $t \to 0$ as $\theta \to 0$, whereas $u \to 0$ as $\theta \to \pi$. For a fixed (CM-frame) three-momentum magnitude p (which is fixed by $E_{\text{c.m.}}$), t and u are not independent, $t+u = -4p^2$.
From part (b) we found that $\delta\lambda = 3\lambda^2 J(0) + O(\lambda^3)$. Relabeling the incoming momentum p_3 as minus the outgoing p_1', and likewise $p_4 \equiv -p_2'$, and grouping terms more conveniently, we have

$$\begin{aligned} \Gamma^{(4)}(p_1,p_2;p_1',p_2') &= \lambda - \lambda^2\left[J(p_1{+}p_2){-}J(0) + J(p_1{-}p_1'){-}J(0) + J(p_1{-}p_2'){-}J(0)\right] \\ &= \lambda - \lambda^2\left[I(s)+I(t)+I(u)\right], \end{aligned}$$

where $I(-k^2) \equiv J(k) - J(0) = \frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\left[\frac{1}{q^2+m^2}\frac{1}{(q-k)^2+m^2} - \frac{1}{(q^2+m^2)^2}\right]$. According to the LSZ formula, the vertex function $\Gamma^{(4)}$ is precisely the covariant scattering amplitude when the external momenta are timelike, future-pointing, and on mass-shell.

(d) $I(s)$ is not sensitive to the UV momentum cutoff Λ, however the two integrals $J(k)$ and $J(0)$ are separately divergent. A quick-and-dirty approach is to examine the large q behavior of each integral. This procedure mangles the contribution to J from the lower limit, but gets the contribution (including any overall factors) from the upper limit right. $J(k) \sim \frac{1}{2}\int^\Lambda \frac{d^4q}{(2\pi)^4}\frac{1}{q^4} = \frac{1}{32\pi^2}\int_{q^2_{\text{min}}}^{\Lambda^2}\frac{dq^2}{q^2}$, where the lower limit of integration is roughly the momentum scale at which the approximation that q^2 is larger than any of the other physical scales in the original integral breaks down. Since $k^2 \gg m^2$, we let q^2_{min} be k^2 times some factor greater than 1. Thus, $J(k) \sim \frac{1}{32\pi^2}\ln(\Lambda^2/k^2)$. By writing the $\sim$ symbol we mean that the difference of the left and right hand sides is finite as $\Lambda \to \infty$. A similar argument can be made to extract the leading cutoff sensitive part of $J(0)$. The only difference is that now the lower limit is set by some multiple of m^2 since there is no other scale in the original integral. Thus, $J(0) \sim \frac{1}{32\pi^2}\ln(\Lambda^2/m^2)$. By taking the difference of these results, we find a Λ-independent result, $J(k) - J(0) \sim \frac{1}{32\pi^2}\ln(m^2/k^2)$. This may be analytically continued back to Minkowski space so that k is a timelike vector. Setting $k^2 = -s$, we have $I(s) \sim \kappa\ln(s/m^2)$ with

$\kappa = -1/(32\pi^2)$.[2]

(e) If s, t,and u are all much larger (in magnitude) than m^2, we may use the leading large momentum behavior of I to write the vertex function as

$$\Gamma^{(4)} \sim \lambda\big\{1 + \tfrac{\lambda}{32\pi^2}\big[\ln(s/m^2) + \ln(t/m^2) + \ln(u/m^2)\big]\,\big\} + O(\lambda^3).$$

This one-loop result should be reliable provided $\frac{\lambda}{32\pi^2}\ln(\max\{s,|t|,|u|\}/m^2) \ll 1$. For c.m. scattering angles θ which are not very close to either 0 or π, both t and u scale linearly with s. So one may simplify the reliability criterion to $\frac{3\lambda}{32\pi^2}\ln(s/m^2) \ll 1$. This inequality fails at sufficiently high energies; the one-loop correction becomes comparable to the leading order term near $E_{\text{Landau}} \sim m\, e^{16\pi^2/(3\lambda)}$. So one-loop perturbation theory is only reliable when $E_{\text{c.m.}} \ll E_{\text{Landau}}$.

(f) In the s-channel, a chain diagram with n loops (each of which also includes the coupling counterterm diagram necessary to remove UV cutoff dependence) contributes $(-\lambda)^{n+1} I(s)^n$ to $-\Gamma^{(4)}$. Since $I(s) \sim \kappa\ln(s/m^2)$, this shows explicitly that at n-loop order, there are contributions which grow with energy like n powers of $\ln(s/m^2)$. When summed over n, the bubble chain is just a geometric series. Adding the contributions of t and u-channel bubble chains, the total contribution to the vertex function from this class of diagrams is

$$\Gamma^{(4)}_{\text{chain}} = \lambda\sum_{n=0}^{\infty}(-\lambda I(s))^n + (s\to t) + (s\to u) = \frac{\lambda}{1+\lambda I(s)} + \frac{\lambda}{1+\lambda I(t)} + \frac{\lambda}{1+\lambda I(u)}\,.$$

Using the asymptotic form $I(s) \sim \kappa\ln(s/m^2)$ [with $\kappa = -1/(32\pi^2)$], valid for ultra-relativistic momenta $s \gg 4m^2$, and again assuming that the scattering angle is not extremely close to 0 or π, so that $t,u \sim -O(s)$, we have $\Gamma^{(4)}_{\text{chain}} \sim 3\lambda/\left[1 + \kappa\lambda\ln(s/m^2)\right]$. Because κ is negative, there is a real value of s, namely $m^2\, e^{1/(|\kappa|\lambda)}$, at which the denominator vanishes and the amplitude diverges. This cannot be physically sensible — either the theory itself, or our approximation, must break down at these energy scales.

5. Consider a $\lambda\phi^4/4!$ theory in $d = 4{-}\epsilon$ dimensions:

(a) The one-loop scalar self-energy is

$$\Pi^{(1)}(p) = \delta m^2_{(1)} + \tfrac{1}{2}\,\lambda\, I_d(m^2)\,, \tag{1}$$

where

$$I_d(m^2) \equiv \mu^\epsilon \int \frac{dq^d}{(2\pi)^d}\frac{1}{q^2+m^2}\,. \tag{2}$$

We have defined the renormalized coupling λ to be dimensionless by introducing the renormalization scale μ with dimensions of mass, so that $\lambda_0 \equiv \lambda\,\mu^\epsilon + \delta\lambda\,\mu^\epsilon$, with $\delta\lambda = O(\lambda^2)$. All physical results must be independent of μ. The self-energy includes (as it must) the mass

[2] Want a more rigorous approach? Use the representation $(AB)^{-1} = \int_0^1 dx\,[xA + (1{-}x)B]^{-2}$ to write $J(k)$ in terms of an integrand that is rotationally invariant about a single point in momentum space: $J(k) = \frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\int_0^1 dx\,\big[(1{-}x)(q^2{+}m^2) + x((q{-}k)^2{+}m^2)\big]^{-2} = \frac{1}{2}\int_0^1 dx\int\frac{d^4q}{(2\pi)^4}\big[(q{-}xk)^2{-}x^2k^2{+}xk^2\big]^{-2} = \frac{1}{2}\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\big[\ell^2 + x(1{-}x)k^2\big]^{-2} = \frac{1}{32\pi^2}\int_0^1 dx\int_0^{\Lambda^2} d\ell^2\,\ell^2\,\big[\ell^2 + x(1{-}x)k^2\big]^{-2} = \frac{1}{32\pi^2}\int_0^1 dx\,\Big[\ln\big(\frac{\Lambda^2}{x(1-x)k^2}+1\big) - 1\Big]$, up to terms vanishing as $\Lambda\to\infty$. Inside the x integral, one may rewrite the logarithm as $\ln(\Lambda^2/k^2)$ plus a function of x that integrates to a finite constant. Therefore, the leading singular behavior is $J(k) \sim \frac{1}{32\pi^2}\ln(\Lambda^2/k^2)$. In part (b) $J(0)$ was calculated exactly, and its leading singular part was $J(0) \sim \frac{1}{32\pi^2}\ln(\Lambda^2/m^2)$. These are precisely the results found with the quick-and-dirty approach.

counterterm contribution. As usual, the integral I_d may be evaluated by converting it into a d-dimensional Gaussian integral:

$$\begin{aligned} I_d(m^2) &= \mu^\epsilon \int_0^\infty ds \int \tfrac{d^dq}{(2\pi)^d}\, e^{-s(q^2+m^2)} = \mu^\epsilon \int_0^\infty ds\, e^{-sm^2} \Big[\int \tfrac{dq}{2\pi}\, e^{-sq^2}\Big]^d \\ &= \mu^\epsilon \int_0^\infty ds\, e^{-sm^2} (4\pi s)^{-d/2} = \mu^\epsilon \tfrac{m^{d-2}}{(4\pi)^{d/2}} \int_0^\infty dt\, t^{-d/2}\, e^{-t} = \mu^\epsilon \tfrac{m^{d-2}}{(4\pi)^{d/2}} \Gamma(1-\tfrac{d}{2}) \\ &= \tfrac{m^2}{(4\pi)^2} \left(\tfrac{m^2}{4\pi\mu^2}\right)^{-\epsilon/2} \Gamma(-1+\tfrac{\epsilon}{2}) = \tfrac{m^2}{(4\pi)^2} \left[-\tfrac{2}{\epsilon} + \gamma_E - 1 + \ln \tfrac{m^2}{4\pi\mu^2} + \mathcal{O}(\epsilon) \right], \end{aligned}$$

where the last step expands the result in powers of ϵ using $\Gamma\left(-1+\frac{\epsilon}{2}\right) = -(\frac{2}{\epsilon}+1-\gamma_E)+\mathcal{O}(\epsilon)$ and $(m^2/4\pi\mu^2)^{-\epsilon/2} = 1 - \frac{\epsilon}{2}\ln(m^2/4\pi\mu^2) + \mathcal{O}(\epsilon^2)$. Hence

$$\Pi^{(1)}(p) = \delta m^2 + \frac{\lambda\, m^2}{32\pi^2} \left[-\frac{2}{\epsilon} + \gamma_E - 1 + \ln \frac{m^2}{4\pi\mu^2} + \mathcal{O}(\epsilon) \right]. \tag{3}$$

To produce a finite result as $\epsilon \to 0$ one must adjust the mass counterterm to cancel the $1/\epsilon$ pole. For the self-energy not to induce any shift in the tree-level mass, the mass counterterm must also cancel the finite $O(\epsilon^0)$ part of that self-energy. In other words, we want

$$\delta m^2_{(1)} = \lambda \frac{m^2}{(4\pi)^2} \left[\frac{1}{\epsilon} - \tfrac{1}{2}(\gamma_E - 1) - \tfrac{1}{2} \ln \frac{m^2}{4\pi\mu^2} \right],$$

so that the renormalized one-loop self-energy vanishes identically, $\lim_{\epsilon\to 0} \Pi^{(1)}(p) = 0$.

(b) In a 4D theory with a rotationally invariant momentum cutoff Λ, the one-loop self energy is

$$\begin{aligned} \Pi^{(1)}(p) &= \delta m^2_{(1)} + \tfrac{1}{2}\lambda \int^\Lambda \frac{dq^4}{(2\pi)^4} \frac{1}{q^2+m^2} = \delta m^2_{(1)} + \frac{\lambda}{32\pi^2} \int^\Lambda \frac{q^2 dq^2}{q^2+m^2} \\ &= \delta m^2_{(1)} + \frac{\lambda}{32\pi^2} \left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{m^2} + 1 \right) \right]. \\ &= \delta m^2_{(1)} + \frac{\lambda}{32\pi^2} \left[\Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2} \right] + \mathcal{O}\Big(\frac{\lambda m^4}{\Lambda^2} \Big). \end{aligned} \tag{4}$$

For this result to vanish, when $\Lambda \to \infty$, we need $\delta m^2_{(1)} = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \ln(\frac{\Lambda^2}{m^2})\right]$. Or, to facilitate comparison with the previous result, $\delta m^2_{(1)} = -\lambda \frac{\Lambda^2}{32\pi^2} + \lambda \frac{m^2}{(4\pi)^2} \left[\frac{1}{2} \ln \frac{\Lambda^2}{4\pi\mu^2} - \frac{1}{2} \ln \frac{m^2}{4\pi\mu^2}) \right]$. There is no analogue of the $O(\Lambda^2)$ quadratic divergence in the dimensional continuation result. In effect, dimensional continuation sets the coefficient of all power divergences to zero, leaving only logarithmic divergences.[3] The $m^2 \ln m^2$ term is exactly the same in the two results. The coefficient of $\ln \Lambda/\mu$ in the momentum cutoff result matches the residue of $1/\epsilon$ in the dim-reg result. The finite pieces are not related in any particularly simple form, although one can simply equate the $O(\lambda m^2)$ terms and conclude that ϵ and Λ/μ are effectively related via $\Lambda/\mu = e^{1/\epsilon}\sqrt{4\pi e^{1-\gamma_E}}$.

(c) The four-point vertex function, evaluated to one-loop order at vanishing external momenta, using dimensional continuation, is given by $\mu^{-\epsilon}\,\Gamma(\{0\}) = -\lambda + \frac{3}{2}\lambda^2 J_d(m^2) - \delta\lambda_{(1)}$, where the factor of three comes from the three different (s, t, and u channel) one-loop graphs related by permutations of external lines, and $J_d(m^2) \equiv \mu^\epsilon \int \frac{d^dq}{(2\pi)^d}\, (q^2+m^2)^{-2}$. This integral may

[3] Quadratic divergences come from the boundary regions of our UV cutoff, and therefore one can change the coefficient of Λ^2 by deforming the specific shape of the UV momentum space boundary. By suitably averaging over two different cutoff shapes one can arrange to remove the Λ^2 term. This, in effect, is what dimensional continuation does automatically. In contrast, logarithmic terms arise from integrals which are equally sensitive to each decade in the integration domain, and are not sensitive to the specific shape of the UV boundary.

be evaluated using the same approach as in part (a), but it is even easier to notice that $J_d(m^2) = -\frac{d}{dm^2}\, I_d(m^2)$. Plugging in the explicit result for I_d yields $J_d(m^2) = \frac{1}{(4\pi)^2}\big[\frac{2}{\epsilon} - \gamma_E - \ln\frac{m^2}{4\pi\mu^2} + \mathcal{O}(\epsilon)\big]$. To cancel the $1/\epsilon$ pole the coupling counterterm, to one-loop order, must be given by $\delta\lambda_{(1)} = 3\frac{\lambda^2}{(4\pi)^2}\,\frac{1}{\epsilon} + \text{finite}$.

(d) The four-point vertex function, to one-loop order, is

$$\mu^{-\epsilon}\,\Gamma_4^{(1)}(p_1,p_2;p_1',p_2') = -\lambda - \delta\lambda_{(1)} + K_d(s) + K_d(t) + K_d(u) + \mathcal{O}(\lambda^3)\,, \tag{5}$$

where s, t, and u are the usual Mandelstam variables, and

$$\begin{aligned}
K_d(k^2) &\equiv \tfrac{1}{2}\lambda^2\mu^\epsilon \int \frac{dq^{4-\epsilon}}{(2\pi)^{4-\epsilon}} \frac{1}{((q{+}k)^2+m^2)\,(q^2+m^2)} \\
&= \tfrac{1}{2}\lambda^2\mu^\epsilon \int_0^1 dx \int \frac{dq^{4-\epsilon}}{(2\pi)^{4-\epsilon}} \frac{1}{[x\,(q^2+k^2+2q\cdot k+m^2)+(1{-}x)(q^2+m^2)]^2} \\
&= \tfrac{1}{2}\lambda^2\mu^\epsilon \int_0^1 dx \int \frac{dq^{4-\epsilon}}{(2\pi)^{4-\epsilon}} \frac{1}{[q^2+m^2+2xk\cdot q+xk^2]^2} \\
&= \tfrac{1}{2}\lambda^2\mu^\epsilon \int_0^1 dx \int \frac{d\ell^{4-\epsilon}}{(2\pi)^{4-\epsilon}} \frac{1}{[\ell^2+m^2+x(1{-}x)k^2]^2} \\
&= \frac{\lambda^2}{32\pi^2}\int_0^1 dx \left[\frac{2}{\epsilon} - \gamma_E - \ln\left(\frac{m^2+x(1{-}x)k^2}{4\pi\mu^2}\right)\right] \\
&= \frac{\lambda^2}{32\pi^2}\left[\frac{2}{\epsilon} - \gamma_E - \int_0^1 dx\,\ln\left(\frac{m^2+x(1{-}x)k^2}{4\pi\mu^2}\right)\right].
\end{aligned} \tag{6}$$

The first steps use the Feynman trick, $(A_1A_2)^{-1} = \int_0^1 dx\,(xA_1+(1{-}x)A_2)^{-2}$, followed by the change of variables $\ell = q + xk$. The resulting spherically symmetric integral over ℓ is exactly the same as the integral J_d discussed above (but with m^2 replaced by $m^2 + x(1{-}x)k^2$).

As required (for a consistent theory), the residue of the $1/\epsilon$ pole is momentum independent; the one-loop coupling counterterm determined in the previous part will completely cancel the $1/\epsilon$ poles in the sum of the s, t, and u channel contributions in the complete result (5), leaving a finite physical answer.