Recommended Reading: Chapters 9 & 10 of Srednicki.

1. Feynman Rules for N-component ϕ^4. Consider a theory in D spacetime dimensions containing an N-component real scalar field $\vec{\phi}$ with quartic self-interactions and Euclidean action:

$$S = \int d^D x \left[\tfrac{1}{2}(\partial\vec{\phi}) \cdot (\partial\vec{\phi}) + \tfrac{1}{2} m^2 \vec{\phi} \cdot \vec{\phi} + \tfrac{1}{8}\lambda\, (\vec{\phi} \cdot \vec{\phi})^2 \right].$$

   (a) State the coordinate-space Feynman rules for this theory. Regard the propagator as a $N \times N$ matrix-valued function with components $G^{(0)}_{ij}(x,y) \equiv \delta_{ij}\, G^{(0)}(x,y)$. Explain why each quartic vertex should represent an expression containing the factor $P_{ijkl} \equiv \frac{1}{3}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ which is *symmetric* under permutation of indices. What is $G^{(0)}$? What is the complete vertex factor? What are the resulting rules for drawing, labeling and evaluating diagrams?

   (b) What are the equivalent momentum-space Feynman rules?

   (c) How many (distinct) connected vacuum diagrams exist at orders λ, λ^2, and λ^3? (Feeling ambitious — how about λ^4?) How fast should you expect the number of diagrams to increase for asymptotically large order?

2. The *self-energy* $\Sigma(k)$ corresponding to a scalar field propagator $G(x,x') \equiv i\langle \mathcal{T}(\phi(x)\phi(x'))\rangle$ is defined as $\tilde{\Sigma}(k) \equiv \tilde{G}(k)^{-1} - \tilde{G}^0(k)^{-1}$, where $\tilde{G}^0(k)$ is the corresponding free propagator. Hence, the self-energy measures the change in the inverse propagator caused by interactions.

   (a) Suppose the self energy, for some theory, is known explicitly. Explain why the physical mass of particles does not equal the unperturbed mass m_0, but instead is given by $m^2_{\text{ph}} = -\bar{p}^2$ where the four-vector $\bar{p}$ is a root of the equation $\bar{p}^2 + m_0^2 + \tilde{\Sigma}(\bar{p}) = 0$. If the self energy is small, $|\tilde{\Sigma}(\bar{p})| \ll m_0^2$, show that $m^2_{\text{ph}} = m_0^2 + \tilde{\Sigma}(p)$ where p is an "on-shell" momenta of the unperturbed theory, $p^2 + m_0^2 = 0$. What is the appropriate physical interpretation if the self-energy is complex?

   (b) Let $\tilde{\Sigma}(k) \sim \tilde{\Sigma}^{(1)}(k) + \tilde{\Sigma}^{(2)}(k) + \cdots$ be the perturbative expansion of the self-energy in powers of the interaction. Let $\tilde{G}(k) \sim \tilde{G}^{(0)}(k) + \tilde{G}^{(1)}(k) + \cdots$ be the corresponding expansion of the full propagator. How is $\tilde{G}^{(n)}(k)$ related to the $\tilde{\Sigma}^{(m)}(k)$? Work out the first few relations explicitly.

   (c) The diagrammatic expansion of the self energy is *simpler* than that of the full propagator—fewer diagrams contribute. Work out the first few orders explicitly and show that this is true. Explain why it is true in general.

   (d) A formal statement of the previous result can be phrased as: "the self-energy is minus the sum of all amputated, one-particle-irreducible propagator diagrams". Explain this. In other words, can you give simple definitions of "amputated" and "one-particle-irreducible" diagrams which make this assertion valid? Does this result depend on the form of the interaction?