Read Chapters 10–13 of Srednicki.

1. Let \( \Theta = \hat{C} \hat{P} \hat{T} \) denote the product of charge conjugation, parity, and time-reversal. The combined transformation is an anti-unitary operator and is a symmetry of all Lorentz-invariant field theories. The transformation acts on an operator \( \hat{A}(x) \) as \( \Theta \hat{A}(x) \Theta^{-1} = \pm \hat{A}(\hat{-}x)^\dagger \), with a + sign for CPT-even operators, and – for CPT-odd operators. Assume that the vacuum state of a CPT (and translation) invariant theory is unique, and hence necessarily CPT invariant, \( \Theta |0\rangle = |0\rangle \).

   (a) Using the anti-unitarity of \( \Theta \), show that \( \langle 0| \hat{A}(x) |0\rangle = -\langle 0| \hat{A}(\hat{-}x)^\dagger |0\rangle \) if \( \hat{A} \) is CPT-odd, and explain why this implies that \( \langle 0| \hat{A}(x) |0\rangle \) vanishes. [Hint: regard the expectation value as the inner product between the bra \( |0\rangle \) and the ket \( \hat{A}(x)|0\rangle \].

   (b) If \( \hat{A} \) is either CPT even or odd, show that CPT plus translation invariance implies that \( \langle 0| [\hat{A}(y), \hat{A}(x)^\dagger] |0\rangle = -\langle 0| [\hat{A}(x), \hat{A}(y)^\dagger] |0\rangle \). Show that this implies that the spectral density of \( \hat{A} \) and \( \hat{A}^\dagger \) is an odd function, \( \chi(q) = -\chi(-q) \).

2. Let \( M \) be an arbitrary \( N \times N \) real symmetric matrix which is positive definite.

   (a) Consider the multi-dimensional Gaussian integral \( Z = \int d^N \phi \ e^{-\frac{1}{2} \phi^T M \phi} \), with \( \bar{\phi} \equiv (\phi_1, \cdots, \phi_N) \) an arbitrary real \( N \)-component vector. Prove that \( Z = [\det(M/(2\pi))]^{-1/2} \).

   (b) Consider the shifted Gaussian integral \( Z[J] \equiv \int d^N \phi \ e^{-\frac{1}{2} \phi^T M \phi + J^T \phi} \), with \( J \) an arbitrary \( N \)-component vector. Evaluate \( Z[J] \) (by suitably shifting integration variables). Does the validity of the result depend on whether \( J \) is real or complex?

   (c) Let \( \langle \cdots \rangle \) denote averages in the normalized Gaussian measure \( d\mu \equiv Z^{-1} d^N \phi \ e^{-\frac{1}{2} \phi^T M \phi} \), so for any function \( F(\phi) \), \( \langle F(\phi) \rangle = Z^{-1} \int d^N \phi \ e^{-\frac{1}{2} \phi^T M \phi} F(\phi) \). Averages of products of the components of \( \phi \) are referred to as moments. Show that \( Z[J] \) is a generating function for all such moments by explaining how to extract an arbitrary moment \( \langle \phi_{i_1} \phi_{i_2} \cdots \phi_{i_k} \rangle \) from \( Z[J] \).

   (d) For any Gaussian measure, the matrix \( M \) defining the measure is known as the covariance matrix. The inverse of this matrix, \( G \equiv M^{-1} \), is the variance matrix. Show that:

   i. All odd-order moments of a Gaussian measure vanish.

   ii. \( \langle \phi_i \phi_j \rangle = G_{ij} \).

   iii. \( \langle \phi_i \phi_j \phi_k \phi_l \rangle = G_{ij} G_{kl} + G_{ik} G_{jl} + G_{il} G_{jk} \).

   iv. \( \langle \phi_{i_1} \phi_{i_2} \cdots \phi_{i_{2k}} \rangle = \sum_{\text{pairings}} G_{\pi_1 \pi_2} G_{\pi_3 \pi_4} \cdots G_{\pi_{2k-1} \pi_{2k}} \), where the sum runs over all ways of grouping the set of indices \( \{i_1, i_2, \cdots, i_{2k}\} \) into distinct pairs \( \{\pi_1, \pi_2\}, \{\pi_3, \pi_4\}, \cdots \{\pi_{2k-1}, \pi_{2k}\} \). (For arbitrary \( k \), what is the number of distinct pairings?)

   (e) Now consider complex Gaussian integrals: let \( M \) be an arbitrary complex Hermitian \( N \times N \) matrix which is positive definite, and define the complex Gaussian integral, \( Z = \int d\phi \ d\bar{\phi} \ e^{-\phi^T M \phi} \), and its shifted generalization, \( Z[J] = \int d\phi \ d\bar{\phi} \ e^{-\phi^T M \phi + J^T \phi} \). Here \( \bar{\phi} \equiv (\phi_1, \cdots, \phi_k) \) is an arbitrary complex vector, as is \( J \), and \( d\phi \ d\bar{\phi} \) is short-hand for independent integration over the real and imaginary parts of each component of \( \phi \): \( d\phi \ d\bar{\phi} \equiv \prod_{k=1}^N d(\text{Re}\phi_k) d(\text{Im}\phi_k) \). Generalize each of the previous parts to this case. You should find that:

   i. \( \langle \phi_{i_1} \cdots \phi_{i_m} \phi_{j_1}^* \cdots \phi_{j_n}^* \rangle = 0 \) if \( m \neq n \).

   ii. \( \langle \phi_i \phi_j^* \rangle = G_{ij} \), where \( G = ||G_{ij}|| \) is the inverse of \( M \).

   iii. \( \langle \phi_{i_1} \cdots \phi_i \phi_{j_1}^* \cdots \phi_{j_n}^* \rangle = \sum_{\text{permutations}} G_{i_1 \pi_1} G_{i_2 \pi_2} \cdots G_{i_k \pi_k} \), where the sum runs over all permutations \( \pi = \{\pi_1, \cdots, \pi_k\} \) of the indices \( \{j_1, \cdots, j_k\} \).

3. Multi-dimensional saddle-point integrals. Consider a multi-dimensional integral of the form \( I(\lambda) \equiv \int d^N x \ e^{-f(\vec{x})/\lambda} \), where \( f(\vec{x}) \) is a smooth function with a global minimum at \( \vec{x} = \vec{x}_0 \).
(a) Find the small $\lambda$ asymptotic expansion of $I(\lambda)$. Specifically, show that $I(\lambda) \sim e^{-f(\vec{x}_0)/\lambda} \times \det \left( f''(\vec{x}_0)/2\pi\lambda \right)^{-1/2} \left( 1 + \sum_{n=1}^{\infty} I^{(n)}(\lambda)n \right)$, with $f''(\vec{x}) \equiv \left\| \frac{\partial^2 f(\vec{x})}{\partial x_i \partial x_j} \right\|$ the curvature matrix of $f$.

(b) Evaluate $I^{(1)}$ and $I^{(2)}$ (in terms of derivatives of $f$). Collect equivalent terms and simplify as much as possible.

(c) See if you can construct an algorithm for computing the coefficients $I^{(n)}$, at arbitrary order, based on drawing a suitable set of graphs (or diagrams) and associating every line with a “propagator” equal to the inverse of the curvature matrix evaluated at $\vec{x}_0$, $G_{ij} \equiv \left( f''(\vec{x}_0)^{-1} \right)_{ij}$, and every vertex at which $K$ lines meet with some factor proportional to the $K$-th derivative of $f(\vec{x})$ evaluated at $\vec{x}_0$, $\Gamma_{i_1 \cdots i_K}^{(K)} \equiv \left. \frac{\partial^K f(\vec{x})}{\partial x_{i_1} \cdots \partial x_{i_K}} \right|_{\vec{x}=\vec{x}_0}$. For each such contribution, you will also need an overall factor equal to the inverse of some integer — what determines this integer?