Due Monday, December 3, 2007

1. Dirac Fermions. Consider a theory of free Dirac fermions, with the usual Hamiltonian, $\hat{H} = \int d^3x\, \bar{\psi}(x)\, (\underset{\sim}{\gamma} \cdot \underset{\sim}{\nabla} + m)\, \psi(x)$. Using the standard mode expansion for a Dirac spinor, show that the free time-ordered propagator $S(x)_{\alpha\beta} \equiv \langle 0|\mathscr{T}\left(\psi_\alpha(x)\bar{\psi}_\beta(0)\right)|0\rangle$ is a Green's function for $i(\gamma^\mu\partial_\mu + m)$. What is its Fourier transform $\tilde{S}(p)$?

2. Hot Oscillators. Consider the canonical thermal ensemble describing an harmonic oscillator with frequency Ω at (inverse) temperature β.
   (a) The quantum statistical mechanical partition function $Z \equiv \mathrm{Tr}\, e^{-\beta\hat{H}}$ may be represented as a path integral, $Z = \int \mathscr{D}x(\tau)\, \exp\left\{-\int_0^\beta d\tau\, \left[\frac{1}{2}x(\tau)\left(-\partial_\tau^2 + \Omega^2\right)x(\tau)\right]\right\}$, where the integral extends over all *periodic* paths with period β [so that $x(\beta) = x(0)$]. Justify this.
   (b) Express Z in terms of a functional determinant and show that the free energy is given by $F = (2\beta)^{-1} \ln \det_+ (-\partial_\tau^2 + \Omega^2)$, where the subscript on $\det_+$ is a reminder that this is the determinant of an operator acting on the space of periodic functions.
   (c) For any (diagonalizable) linear operator M, explain why $\ln \det M = \mathrm{tr} \ln M$.
   (d) Express $\mathrm{tr} \ln \left(-\partial_\tau^2 + \Omega^2\right)$ as a sum over eigenvalues. Show that the sum doesn't converge (because of sloppiness in handling the overall normalization of the path integral measure). Express the derivative with respect to the oscillator frequency, $\partial F/\partial\Omega^2$, as a sum involving the eigenvalues of $-\partial_\tau^2 + \Omega^2$, and show that this improves the convergence.
   (e) Prove that $\sum_{n=-\infty}^{\infty} (n^2 + a^2)^{-1} = (\pi/a) \coth(\pi a)$, and use this result to evaluate $\partial F/\partial\Omega^2$.
   (f) Integrate with respect to Ω^2 to find the free energy. What (if anything) determines the integration constant?
   (g) Differentiate to find the expectation value of the energy, $\langle E\rangle = \langle\hat{H}\rangle = -\partial \ln Z/\partial\beta$. Do you find the correct result?

3. Consider a theory of two different real scalar fields, $\phi(x)$ and $\chi(x)$, described by the Lagrangian

$$-\mathscr{L} = \tfrac{1}{2}\left[(\partial\phi)^2 + m^2\phi^2 + (\partial\chi)^2 + M^2\chi^2\right] + \frac{g}{2}\,\phi^2\chi + \frac{\lambda}{4!}\,\phi^4 + \frac{\eta}{4!}\,\chi^4\,.$$

   With a small abuse of language, call the particle created by the field $\phi(x)$ a "ϕ"-particle, etc.
   (a) State the momentum space Feynman rules for this theory (in a form in which you have two different propagators, one for ϕ and one for χ).
   (b) Draw all tree graphs contributing to the elastic scattering process $\phi + \phi \to \phi + \phi$.
   (c) Let p_1 and p_2 be the 4-momenta of the incoming particles, and p_1' and p_2' the momenta of the outgoing particles. Define the "Mandelstam variables": $s \equiv -(p_1 + p_2)^2$, $t \equiv -(p_1 - p_1')^2$, and $u \equiv -(p_1 - p_2')^2$. Show that $s + t + u = 4m^2$.
   (d) Use the Feynman rules plus the LSZ reduction formula to evaluate the covariant scattering amplitude $\mathcal{M}$ for this process to lowest non-trivial order. Express the answer in terms of the Mandelstam variables.
   (e) Evaluate the total center-of-mass cross section (as a function of energy).
   (f) What is the domain of validity of your result?
   (g) Assume that $m \ll M$ and $g^2/M^2 \ll \lambda \ll 1$. Discuss (and sketch) the energy dependence of the result. Examine, in particular, the energy regions: (*i*) $E_{\text{c.m.}} - 2m \ll m$; (*ii*) $2m \ll E_{\text{c.m.}} \ll M$; (*iii*) $E_{\text{c.m.}} \approx M$; and (*iv*) $E_{\text{c.m.}} \gg M$. Which diagrams dominate in these different regimes? If the coupling constants g, λ and η are sufficiently small (what is sufficient?) is your result a good approximation in each of these regions?