1. Non-relativistic free propagator.

   (a) $i\frac{d}{dt}\,\phi(\underset{\sim}{x}) = [\phi(\underset{\sim}{x}), H] = -\frac{1}{2m}\nabla^2\phi(\underset{\sim}{x})$.

   (b) Let $\phi(t,\underset{\sim}{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{ik\cdot x}\, a(t,\underset{\sim}{k})$. Then $i\frac{d}{dt}\, a(t,\underset{\sim}{k}) = \epsilon(k)\, a(t,\underset{\sim}{k})$ with $\epsilon(k) \equiv k^2/(2m)$. This integrates to $a(t,\underset{\sim}{k}) = e^{-i\epsilon(k)t}\, a(\underset{\sim}{k})$ with $a(\underset{\sim}{k})$ the time-zero annihilation operator. So

   $$\phi(t,\underset{\sim}{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{ik\cdot x - i\epsilon(k)t}\, a(\underset{\sim}{k})\,.$$

   (c) Inserting the spatial Fourier representations of $\phi(t,\underset{\sim}{x})$ and $\phi(0,0)^\dagger$ into the definition of the propagator gives

   $$\begin{aligned} G(t,\underset{\sim}{x}) &= \Theta(t) \int \frac{d^3k}{(2\pi)^3}\,\frac{d^3k'}{(2\pi)^3}\, e^{ik\cdot x - i\epsilon(k)t}\, \langle 0|a(\underset{\sim}{k})^\dagger a(\underset{\sim}{k}')|0\rangle \\ &= \Theta(t) \int \frac{d^3k}{(2\pi)^3}\, e^{ik\cdot x - ik^2t/(2m)} = \Theta(t)\, (\tfrac{m}{2\pi i t})^{3/2}\, e^{imx^2/(2t)}\,. \end{aligned}$$

   In the last step, the shifted Gaussian integral was evaluated, as usual, by completing the square in the exponent. One may regard the time t as having an infinitesimal negative imaginary part to ensure convergence of the integral. The spatial Fourier transform of the propagator can be read off from the penultimate form, $\tilde{G}(t,\underset{\sim}{k}) = \Theta(t)\, e^{-i\epsilon(k)t}$.

   (d) A temporal Fourier transform gives

   $$\widetilde{G}(\omega,\underset{\sim}{k}) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, \tilde{G}(t,\underset{\sim}{k}) = \int_0^\infty dt\, e^{i\omega t - i\epsilon(k)t}\,.$$

   To make sense of this integral, one must regard the frequency ω as having an infinitesimal positive imaginary part. Making this explicit, by writing $\omega + i\delta$ in place of ω, with δ a positive infinitesimal, we have

   $$\widetilde{G}(\omega,\underset{\sim}{k}) = \frac{i}{\omega - \epsilon(k) + i\delta}\,.$$

   (e) $\widetilde{G}(\omega,\underset{\sim}{k})$ is a meromorphic function, analytic in the upper half plane and having a simple pole in the lower half plane at $\omega = \epsilon(k) - i\delta$. Fourier transforming $\widetilde{G}(\omega,\underset{\sim}{k})$ gives

   $$\int \tfrac{d\omega}{2\pi}\, e^{-i\omega t}\, \widetilde{G}(\omega,\underset{\sim}{k}) = i\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \frac{e^{-i\omega t}}{\omega - \epsilon(k) + i\delta} = \begin{cases} 0\,, & t < 0; \\ e^{-i\epsilon(k)t}\,, & t > 0\,, \end{cases}$$

   where the contour integral is closed in the upper half plane when $t < 0$, giving zero, and closed in the lower half plane when $t > 0$, giving (minus) the residue of the pole at $\omega = \epsilon(k) - i\delta$. The result is precisely $\tilde{G}(t,\underset{\sim}{k})$ as given above. The infinetisimal δ serves to displace the pole off the contour of integration on the real axis, and its inclusion is essential to make this contour integral well-defined.

   (f) $\mathrm{Re}\,\widetilde{G}(\omega,\underset{\sim}{k}) = \frac{1}{2}\big(\widetilde{G}(\omega,\underset{\sim}{k}) + \widetilde{G}(\omega,\underset{\sim}{k})^*\big) = \frac{1}{2}\big(\frac{i}{\omega-\epsilon(k)+i\delta} - \frac{i}{\omega-\epsilon(k)-i\delta}\big) = \frac{\delta}{(\omega-\epsilon(k))^2+\delta^2} = \pi\,\delta(\omega - \epsilon(k))\,,$ where the implicit limit $\delta \to 0$ is taken in the last equality. The penultimate form is a standard representation of a delta function. (One may also recall and use the result that (for real x) $1/(x - i\delta) = \mathrm{PV}(\frac{1}{x}) + i\pi\,\delta(x)$, where PV stands for principal value.)

2. Second quantized operators.

   (a) i. $N = \int d^3x \, \phi(x)^\dagger \phi(x)$.

   ii. $\underset{\sim}{P} = \int d^3x \, \phi(x)^\dagger (-i\underset{\sim}{\nabla})\phi(x)$.

   iii. $n(x) = \phi(x)^\dagger \phi(x)$

   iv. $\epsilon(x) = \frac{1}{2m} \underset{\sim}{\nabla}\phi(x)^\dagger \cdot \underset{\sim}{\nabla}\phi(x)$.

   v. $\underset{\sim}{\pi}(x) = -\frac{i}{2} \, \phi(x)^\dagger \underset{\sim}{\nabla}\phi(x) + \frac{i}{2} \, (\underset{\sim}{\nabla}\phi(x))^\dagger \phi(x) \equiv -\frac{i}{2}\phi(x)^\dagger \overset{\leftrightarrow}{\nabla} \phi(x)$.

   (b) One can work backwards and infer the form of the particle flux $\underset{\sim}{j}$ by writing out dn/dt, showing that the result can be wrtten as a divergence, and reading off what $\underset{\sim}{j}$ must be:

$$\begin{aligned}\dot{n}(x) &= \phi(x)^\dagger \dot{\phi}(x) + \dot{\phi}(x)^\dagger \phi(x) \\ &= \tfrac{i}{2m} \left[\phi(x)^\dagger \nabla^2 \phi(x) - (\nabla^2\phi(x))^\dagger \phi(x)\right] \\ &= \tfrac{i}{2m} \, \underset{\sim}{\nabla} \cdot \left[\phi(x)^\dagger \underset{\sim}{\nabla}\phi(x) - (\underset{\sim}{\nabla}\phi(x))^\dagger \phi(x)\right] \equiv -\underset{\sim}{\nabla} \cdot \underset{\sim}{j}(x) \, .\end{aligned}$$

Hence the particle flux $\underset{\sim}{j}(x) = -\frac{i}{m} \, \phi(x)^\dagger \overset{\leftrightarrow}{\nabla} \phi(x) = \underset{\sim}{\pi}(x)/m$. This makes sense: in non-relativistic dynamics the momentum of a particle is just m times its velocity, and hence momentum density is just m times flux density. It is also worthwhile to confirm that the expectation value of this operator produces the appropriate result in, say, a one particle state of definite momentum, $|\underset{\sim}{p}\rangle = a(\underset{\sim}{p})^\dagger |0\rangle$:

$$\begin{aligned}\langle \underset{\sim}{p}| \, \underset{\sim}{j}(x) \, |\underset{\sim}{p}\rangle &= -\frac{i}{2m} \int \frac{d^3k'}{(2\pi)^3} \, \frac{d^3k}{(2\pi)^3} \, \left(e^{-ik'\cdot x} \overset{\leftrightarrow}{\nabla} e^{ik\cdot x}\right) \langle 0|a(\underset{\sim}{p}) \, a(\underset{\sim}{k}')^\dagger \, a(\underset{\sim}{k}) \, a(\underset{\sim}{p})^\dagger |0\rangle \\ &= -\frac{i}{2m} \, \left[e^{-ip\cdot x} \, (\underset{\sim}{\nabla} \, e^{ip\cdot x}) - (\underset{\sim}{\nabla} e^{-ip\cdot x}) \, e^{ip\cdot x}\right] = \frac{\underset{\sim}{p}}{m} = \underset{\sim}{v} \, ,\end{aligned}$$

the particle velocity. This is correct: the number flux (number of particles passing a surface per unit area per unit time) in a plane wave state with a unit probability density is just the velocity of the particle dotted into the surface area element.

   (c) Similarly,

$$\begin{aligned}\dot{\epsilon}(x) &= \tfrac{1}{2m} \left[\underset{\sim}{\nabla}\phi(x)^\dagger \cdot \underset{\sim}{\nabla}\dot{\phi}(x)\right] + \cdot \underset{\sim}{\nabla}\dot{\phi}(x)^\dagger \cdot \underset{\sim}{\nabla}\phi(x) \\ &= \tfrac{i}{(2m)^2} \left[(\underset{\sim}{\nabla}\phi(x))^\dagger \cdot \underset{\sim}{\nabla}\nabla^2\phi(x) - (\underset{\sim}{\nabla}\nabla^2\phi(x))^\dagger \cdot \underset{\sim}{\nabla}\phi(x)\right] \\ &= \tfrac{i}{(2m)^2} \, \underset{\sim}{\nabla} \cdot \left[(\underset{\sim}{\nabla}\phi(x))^\dagger \, \nabla^2\phi(x) - (\nabla^2\phi(x))^\dagger \, \underset{\sim}{\nabla}\phi(x)\right] \equiv -\underset{\sim}{\nabla} \cdot \underset{\sim}{q}(x) \, .\end{aligned}$$

Hence the energy flux $\underset{\sim}{q}(x) = -\frac{i}{(2m)^2} \left[(\underset{\sim}{\nabla}\phi(x))^\dagger \, \nabla^2\phi(x) - (\nabla^2\phi(x))^\dagger \, \underset{\sim}{\nabla}\phi(x)\right]$. This gives the correct result in a single particle state of momentum $\underset{\sim}{p}$,

$$\langle \underset{\sim}{p}| \, \underset{\sim}{q}(x) \, |\underset{\sim}{p}\rangle = -\frac{i}{(2m)^2} \, \left[(\underset{\sim}{\nabla} e^{-ip\cdot x}) \, (\nabla^2 \, e^{ip\cdot x}) - (\nabla^2 e^{-ip\cdot x}) \, (\underset{\sim}{\nabla} \, e^{ip\cdot x})\right] = (\tfrac{p^2}{2m}) \, \underset{\sim}{v} \, .$$

   (d) And likewise,

$$\begin{aligned}\dot{\pi}_j(x) &= -\tfrac{i}{2} \left[\phi(x)^\dagger \nabla_j \dot{\phi}(x) - (\nabla_j\phi(x))^\dagger \dot{\phi}(x) + \dot{\phi}(x)^\dagger \nabla_j \phi(x) - (\nabla_j \dot{\phi}(x))^\dagger \phi(x)\right] \\ &= \tfrac{1}{4m} \left[\phi(x)^\dagger \nabla_j \nabla^2 \phi(x) - (\nabla_j\phi(x))^\dagger \nabla^2\phi(x) - (\nabla^2\phi(x))^\dagger \nabla_j\phi(x) + (\nabla_j\nabla^2\phi(x))^\dagger \phi(x)\right] \\ &= \tfrac{1}{4m} \, \nabla_i \left[\phi(x)^\dagger \nabla_i \nabla_j \phi(x) - (\nabla_j\phi(x))^\dagger \nabla_i\phi(x) - (\nabla_i\phi(x))^\dagger \nabla_j\phi(x) + (\nabla_i\nabla_j\phi(x))^\dagger \phi(x)\right] \\ &\equiv -\nabla_i s_{ij}(x) \, .\end{aligned}$$

So the stress tensor is given by

$$
\begin{aligned}
s_{ij}(x) &= -\tfrac{1}{4m}\left[\phi(x)^\dagger\nabla_i\nabla_j\phi(x) - (\nabla_j\phi(x))^\dagger\nabla_i\phi(x) - (\nabla_i\phi(x))^\dagger\nabla_j\phi(x) + (\nabla_i\nabla_j\phi(x))^\dagger\phi(x)\right] \\
&= -\tfrac{1}{m}\,\phi(x)^\dagger\overleftrightarrow{\nabla}_i\overleftrightarrow{\nabla}_j\,\phi(x)\,.
\end{aligned}
$$

Its single particle plane wave expectation is

$$
\langle \underset{\sim}{p}|\, s_{ij}(x)\,|\underset{\sim}{p}\rangle = -\frac{1}{m}\,(e^{-ip\cdot x}\,\overleftrightarrow{\nabla}_i\,\overleftrightarrow{\nabla}_j\,e^{ip\cdot x}) = \frac{p_i\,p_j}{m} = p_i\,v_j\,,
$$

which is the correct result for $s_{ij}\,d\Sigma_j$ to give the amount of i'th component of momentum passing through the surface $d\underset{\sim}{\Sigma}$ per unit area per unit time in a plane wave state with unit probability density.