1. Equations of motion. Use the Heisenberg picture definition of time evolution with a free Hamiltonian:

$$i\frac{\partial}{\partial t}\hat{\phi}(x) = -[\hat{H},\, \hat{\phi}(x)] = \int d^3y\; [\hat{\phi}(x),\, \hat{\phi}^\dagger(y)\{-(\nabla^2/2m) + U(x)\}\hat{\phi}(y)]$$
$$= \int d^3y\; \delta^3(x-y)\; \{-(\nabla^2/2m) + U(x)\}\hat{\phi}(y) = \Big\{-\frac{\nabla^2}{2m} + U(x)\Big\}\hat{\phi}(x)\,.$$

If a two-body interaction, $\frac{1}{2}\int d^3x\, d^3x'\; \phi^\dagger(x)\phi^\dagger(x')V(x{-}x')\phi(x')\phi(x)$, is added, then

$$i\frac{\partial}{\partial t}\hat{\phi}(x) = -[\hat{H},\, \hat{\phi}(x)]$$
$$= \Big\{-\frac{\nabla^2}{2m} + U(x)\Big\}\hat{\phi}(x) + \tfrac{1}{2}\int d^3y\, d^3y'\; \Big[\hat{\phi}(x),\, \hat{\phi}^\dagger(y)\hat{\phi}^\dagger(y')V(y{-}y')\hat{\phi}(y')\hat{\phi}(y)\Big]$$
$$= \Big\{-\frac{\nabla^2}{2m} + U(x)\Big\}\hat{\phi}(x) + \tfrac{1}{2}\int d^3y'\; V(x{-}y')\hat{\phi}^\dagger(y')\hat{\phi}(y')\hat{\phi}(x)$$
$$+ \tfrac{1}{2}\int d^3y\; V(y{-}x)\,\hat{\phi}^\dagger(y)\hat{\phi}(y)\,\hat{\phi}(x)$$
$$= \Big\{-\frac{\nabla^2}{2m} + U(x) + \int d^3y\; V(x{-}y)\,\hat{\phi}^\dagger(y)\hat{\phi}(y)\Big\}\hat{\phi}(x)\,.$$

Yes, theories of interacting particles must have non-linear equations of motion. If the basic field(s) satisfy linear equations of motion of the form $i(\partial\hat{\phi}(x)/\partial t) = h\,\hat{\phi}(x)$, for some c-number operator h, then necessarily the Hamiltonian has the quadratic form $\hat{H} = \int d^3x\; \hat{\phi}^\dagger(x)\, h\, \hat{\phi}(x)$ (up to an irrelevant additive constant). This is a free theory, in which particles may interact with arbitrary fixed background fields, but do not interact with each other.

2. (a) For convenience, let the finite volume $\mathcal{V}$ be a periodic cube of size L. The allowed values of momentum become discrete, so $\hat{H}(\mu) = \sum_{\underset{\sim}{p}} \Big[\frac{p^2}{2m} - \mu\Big]\, \hat{a}^\dagger_{\underset{\sim}{p}}\, \hat{a}_{\underset{\sim}{p}}$, where the sum runs over all values of momentum which are compatible with periodic boundary conditions, $\underset{\sim}{p} = \frac{2\pi}{L}\,\underset{\sim}{n}$, with $\underset{\sim}{n}$ integer-valued. For each allowed value of $\underset{\sim}{p}$, the operator $\hat{a}^\dagger_{\underset{\sim}{p}}\, \hat{a}_{\underset{\sim}{p}}$ counts the number of particles with the given momentum. The canonical anti-commutation relations, for spinless fermions, imply that this operator has only 0 and 1 as eigenvalues — *i.e.*, each momentum mode can be occupied by at most one fermion. Occupying a mode $\underset{\sim}{p}$ reduces $\tilde{E}(\mu)$ provided $\underset{\sim}{p}^2/(2m) < \mu$, and raises it if $\underset{\sim}{p}^2/(2m) > \mu$. Therefore, the minimal value is produced by occupying all momentum modes for which $\underset{\sim}{p}^2/(2m) < \mu$, and leaving all other modes empty. This is precisely the content of the conditions that $\hat{a}(\underset{\sim}{p})^\dagger|\text{g.s.}\rangle = 0$ for all $|\underset{\sim}{p}| < p_f$ (showing that all these modes are full), together with $\hat{a}(\underset{\sim}{p})|\text{g.s.}\rangle = 0$ for all $|\underset{\sim}{p}| > p_f$ (showing that all these modes are empty), if one defines $p_f \equiv \sqrt{2m\mu}$.

The corresponding eigenvalue of $\hat{H}(\mu)$ is $\tilde{E}_{\text{g.s.}} = \sum_{\underset{\sim}{p}} \Theta(\mu - \frac{p^2}{2m})\,\Big[\frac{p^2}{2m} - \mu\Big]$, with $\Theta(x)$ the unit step function. In the large volume limit the sum becomes an integral, $(2\pi/L)^3 \sum_{\underset{\sim}{p}} \to \int d^3p$ or $\sum_{\underset{\sim}{p}} \to \mathcal{V}\int d^3p/(2\pi)^3$. Hence $\tilde{E}_{\text{g.s.}}/\mathcal{V} = \int \frac{d^3p}{(2\pi)^3}\, \Theta(\mu - \frac{p^2}{2m})\,\Big[\frac{p^2}{2m} - \mu\Big] = \frac{4\pi}{2m(2\pi)^3}\int_0^{p_f} (p^4 - p_f^2\, p^2) = -p_f^5/(30\pi^2 m)$.

(b) The ground state particle density is just the number of filled levels divided by the volume, $n(\mu) = \mathcal{V}^{-1}\sum_{\underset{\sim}{p}} \Theta(\mu{-}\frac{p^2}{2m}) \to \int \frac{d^3p}{(2\pi)^3}\Theta(\mu{-}\frac{p^2}{2m}) = \frac{4\pi}{(2\pi)^3}\int_0^{p_f} dp p^2 = p_f^3/(6\pi^2) = (2m\mu)^{3/2}/(6\pi^2)$. The ground state energy density as measured by $\hat{H}$ [not the shifted $\hat{H}(\mu) = \hat{H} - \mu\hat{N}$] is $\epsilon(\mu) = (\tilde{E}_{\text{g.s.}}/\mathcal{V}) + \mu\, n(\mu) = -p_f^5/(30\pi^2 m) + \mu\, p_f^3/(6\pi^2) = p_f^5/(20\pi^2 m) = (2m\mu)^{5/2}/(20\pi^2 m)$.

(c) Inverting the particle density — chemical potential relation yields $\mu(n) = \frac{1}{2m}(6\pi^2 n)^{2/3}$. Inserting this into the ground state energy density $\epsilon(\mu)$ gives $\epsilon(n) = \frac{3}{10m}\,(6\pi^2)^{2/3}\,n^{5/3}$.

(d) Explicit differentiation gives $\frac{d\epsilon}{dn} = \frac{1}{2m}(6\pi^2 n)^{2/3} = \mu(n)$. To see that this is a general result, start with the definition $\tilde{E}_{\text{g.s.}}(\mu) = \langle \text{g.s.}|\hat{H} - \mu\hat{N}|\text{g.s.}\rangle$ and let $\tilde{\epsilon}(\mu) \equiv \tilde{E}_{\text{g.s.}}(\mu)/\mathcal{V}$. Note that $\frac{d}{d\mu}\,\tilde{\epsilon}(\mu) = -\langle \text{g.s.}|\hat{N}|\text{g.s.}\rangle/\mathcal{V} = -n(\mu)$. [This equality is not quite as trivial as it may seem, because the ground state also depends on μ. Because the ground state is an eigenstate of $\hat{H}(\mu)$, the variations of the bar and ket combine to give $\tilde{\epsilon}(\mu)\,\frac{d}{d\mu}(\langle \text{g.s.}|\text{g.s.}\rangle) = 0$. This is the Feynman-Hellman theorem.] Therefore, using the chain rule, $\frac{d\tilde{\epsilon}}{dn} = \frac{d\tilde{\epsilon}}{d\mu}\,\frac{d\mu}{dn} = -n\,\frac{d\mu}{dn}$. Finally, use $\epsilon = \tilde{\epsilon} + \mu\, n$ and differentiate, $\frac{d\epsilon}{dn} = \frac{d\tilde{\epsilon}}{dn} + \frac{d\mu}{dn}\,n + \mu = \mu$. This shows that the chemical potential may be interpreted as the rate of change of the energy (as measured by the original Hamiltonian $\hat{H}$) as a function of the number of particles. In the limit of a large volume (so that a change in particle number by one is an infinitesimal change in particle or energy density), this may equivalently be phrased as the statement that the chemical potential equals the change in energy produced by adding a single particle to the system.

3. Density Correlations. Since the ground state of the degenerate Fermi gas satisfies the conditions $\hat{a}(\underset{\sim}{p})^\dagger|\text{g.s.}\rangle = 0$ for all $|\underset{\sim}{p}| < p_f$, and $\hat{a}(\underset{\sim}{p})|\text{g.s.}\rangle = 0$ for $|\underset{\sim}{p}| > p_f$, the basic strategy for evaluating correlation functions of the field operators will be to plug in the mode expansion $\phi(\underset{\sim}{x}) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}\,\hat{a}(\underset{\sim}{p})$, and its conjugate, and then commute to the right (so that they will eventually annihilate the ground state) both fermion creation operators below the Fermi surface and annihilation operators above the Fermi surface. Or, commute to the left fermion creation operators above the Fermi surface and annihilation operators below the Fermi surface, since the ground state bra satisfies $\langle \text{g.s.}|\hat{a}(\underset{\sim}{p}) = 0$ for all $|\underset{\sim}{p}| < p_f$ and $\langle \text{g.s.}|\hat{a}(\underset{\sim}{p})^\dagger = 0$ for $|\underset{\sim}{p}| > p_f$. (For notational simplifity, the fermions will treated as spinless.)

(a) Proceding as stated, we have $G(\underset{\sim}{x}) \equiv \langle\phi^\dagger(\underset{\sim}{x})\phi(0)\rangle = \int\frac{d^3p}{(2\pi)^3}\,e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}}\int\frac{d^3p'}{(2\pi)^3}\,\langle\hat{a}(\underset{\sim}{p})^\dagger\hat{a}(\underset{\sim}{p}')\rangle = \int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}\,e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}}\,\Theta(p_f - |\underset{\sim}{p}|)\Theta(p_f - |\underset{\sim}{p}'|)\langle\hat{a}(\underset{\sim}{p})^\dagger\hat{a}(\underset{\sim}{p}')\rangle$. And, for $\underset{\sim}{p}$ and $\underset{\sim}{p}'$ both below the Fermi surface, $\langle\hat{a}(\underset{\sim}{p})^\dagger\hat{a}(\underset{\sim}{p}')\rangle = (2\pi)^3\delta^3(\underset{\sim}{p}-\underset{\sim}{p}') - \langle\hat{a}(\underset{\sim}{p}')\hat{a}(\underset{\sim}{p})^\dagger\rangle = (2\pi)^3\delta^3(\underset{\sim}{p}-\underset{\sim}{p}')$. So the correlator $G(\underset{\sim}{x})$ is precisely the $3d$ Fourier transform of a ball, $G(\underset{\sim}{x}) = \int\frac{d^3p}{(2\pi)^3}\,e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}}\;\Theta(p_f - |\underset{\sim}{p}|) = \frac{2\pi}{(2\pi)^3}\int_0^{p_f} dp\,p^2\int_0^\pi d\theta\,e^{-ip|\underset{\sim}{x}|\cos\theta} = \frac{1}{2\pi^2|\underset{\sim}{x}|}\int_0^{p_f} dp\,p\,\sin p|\underset{\sim}{x}| = -\frac{p_f}{2\pi^2|\underset{\sim}{x}|^2}\left(\cos p_f|\underset{\sim}{x}| - \frac{\sin p_f|\underset{\sim}{x}|}{p_f|\underset{\sim}{x}|}\right)$. The $|\underset{\sim}{x}| \to 0$ limit is $G(0) = p_f^3/(6\pi^2)$ and agrees, as it must, with the average particle density $\bar{n}$.

(b) The ground state is translationally invariant, so the expectation value of the particle density $\hat{n}(\underset{\sim}{x})$ is $\underset{\sim}{x}$-independent and just equals the average density, $\langle\hat{n}(\underset{\sim}{x})\rangle = \bar{n}$. Hence the second term in the correlator $G_{nn}(\underset{\sim}{x}-\underset{\sim}{y}) \equiv \langle\hat{n}(x)\hat{n}(y)\rangle - \langle\hat{n}(x)\rangle\langle\hat{n}(y)\rangle$ is merely $\bar{n}^2$. The first term, after plugging in mode expansions of the four field operators in $\hat{n}(\underset{\sim}{x})\hat{n}(\underset{\sim}{y}) = \hat{\phi}(\underset{\sim}{x})^\dagger\hat{\phi}(\underset{\sim}{x})\hat{\phi}(\underset{\sim}{y})^\dagger\hat{\phi}(\underset{\sim}{y})$ and using the ground state conditions to simplify the outermost operators, gives $\langle\hat{n}(\underset{\sim}{x})\hat{n}(\underset{\sim}{y})\rangle = \int\frac{d^3p'}{(2\pi)^3}\int\frac{d^3p}{(2\pi)^3}e^{i(\underset{\sim}{p}-\underset{\sim}{p}')\cdot\underset{\sim}{x}}\;\Theta(p_f - |\underset{\sim}{p}|)\int\frac{d^3k'}{(2\pi)^3}\int\frac{d^3k}{(2\pi)^3}e^{i(\underset{\sim}{k}-\underset{\sim}{k}')\cdot\underset{\sim}{y}}\;\Theta(p_f - |\underset{\sim}{k}|)\langle\hat{a}(\underset{\sim}{p}')^\dagger\hat{a}(\underset{\sim}{p})\hat{a}(\underset{\sim}{k}')^\dagger\hat{a}(\underset{\sim}{k})\rangle$. Split the $\underset{\sim}{k}'$ integral into separate contributions from above and below the Fermi surface. When $|\underset{\sim}{k}'| < p_f$, moving $\hat{a}(\underset{\sim}{k}')^\dagger$ to the right gives a delta-function from anti-commuting it with $\hat{a}(\underset{\sim}{k})$ plus zero (when it hits the ground state), exactly as in previous part. So if $\underset{\sim}{p}'$, $\underset{\sim}{k}'$ and $\underset{\sim}{k}$ are all below the Fermi surface, $\langle\hat{a}(\underset{\sim}{p}')^\dagger\hat{a}(\underset{\sim}{p})\hat{a}(\underset{\sim}{k}')^\dagger\hat{a}(\underset{\sim}{k})\rangle = (2\pi)^3\delta^3(\underset{\sim}{k}'-\underset{\sim}{k})\langle\hat{a}(\underset{\sim}{p}')^\dagger\hat{a}(\underset{\sim}{p})\rangle = (2\pi)^3\delta^3(\underset{\sim}{k}'-\underset{\sim}{k})\,(2\pi)^3\delta^3(\underset{\sim}{p}'-\underset{\sim}{p})$. After doing the now-trivial momentum integrals, this contribution is precisely the square of the average particle density — or in other words, the term which was subtracted off in the definition of $G_{nn}(\underset{\sim}{x}-\underset{\sim}{y})$.

Consequently, what survives is just the contribution when $\underset{\sim}{k}'$ is above the Fermi surface. For this piece, move the $\hat{a}(\underset{\sim}{k}')$ to the left, yielding $(2\pi)^3\delta^3(\underset{\sim}{p}-\underset{\sim}{k}')$ when it moves past $\hat{a}(\underset{\sim}{p})$ plus zero (when it hits the ground state bra). Using $\int_0^{p_f} d^3p$ as shorthand for $\int d^3p\, \Theta(p_f - |\underset{\sim}{p}|)$, etc., we thus have

$$
\begin{aligned}
G_{nn}(\underset{\sim}{x}-\underset{\sim}{y}) &= \int_0^{p_f} \frac{d^3p'}{(2\pi)^3} \int_0^{p_f} \frac{d^3k}{(2\pi)^3} \int_{p_f}^{\infty} \frac{d^3k'}{(2\pi)^3}\, e^{i(\underset{\sim}{k}'-\underset{\sim}{p}')\cdot \underset{\sim}{x}}\, e^{i(\underset{\sim}{k}-\underset{\sim}{k}')\cdot \underset{\sim}{y}}\, \langle \hat{a}(\underset{\sim}{p}')^\dagger \hat{a}(\underset{\sim}{k})\rangle \\
&= \int_0^{p_f} \frac{d^3k}{(2\pi)^3} \int_{p_f}^{\infty} \frac{d^3k'}{(2\pi)^3}\, e^{i(\underset{\sim}{k}'-\underset{\sim}{k})\cdot(\underset{\sim}{x}-\underset{\sim}{y})} \\
&= \int_0^{p_f} \frac{d^3k}{(2\pi)^3} \left[\delta^3(\underset{\sim}{x}-\underset{\sim}{y}) - \int_0^{p_f} \frac{d^3k'}{(2\pi)^3}\, e^{i(\underset{\sim}{k}'-\underset{\sim}{k})\cdot(\underset{\sim}{x}-\underset{\sim}{y})}\right] \\
&= \bar{n}\, \delta^3(\underset{\sim}{x}-\underset{\sim}{y}) - \int_0^{p_f} \frac{d^3k}{(2\pi)^3}\, e^{-i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{y})} \int_0^{p_f} \frac{d^3k'}{(2\pi)^3}\, e^{i\underset{\sim}{k}'\cdot(\underset{\sim}{x}-\underset{\sim}{y})} \\
&= \bar{n}\, \delta^3(\underset{\sim}{x}-\underset{\sim}{y}) - |G(\underset{\sim}{x}-\underset{\sim}{y})|^2 \\
&= \bar{n}\, \delta^3(\underset{\sim}{x}-\underset{\sim}{y}) - \frac{p_f^2}{4\pi^4 |\underset{\sim}{x}|^4} \left(\cos p_f|\underset{\sim}{x}| - \frac{\sin p_f|\underset{\sim}{x}|}{p_f|\underset{\sim}{x}|}\right)^2 .
\end{aligned}
$$

Since $\bar{n} = p_f^3/(6\pi^2)$ (for spinless degenerate fermions), this coincides with the stated answer, $G_{nn}(\underset{\sim}{x}) = \bar{n}\,\delta^3(\underset{\sim}{x}) - \frac{3\bar{n}}{2\pi^2} \frac{1}{p_f|\underset{\sim}{x}|^4} \left(\cos p_f|\underset{\sim}{x}| - \frac{\sin p_f|\underset{\sim}{x}|}{p_f|\underset{\sim}{x}|}\right)^2$. But this latter form, with both terms proportional to $\bar{n}$, remains correct after including fermion spin — the density n merely acquires a factor of the spin degeneracy.

(c) $(\Delta N_{\mathcal{B}})^2 = \left\langle \left(\hat{N}_{\mathcal{B}} - \langle \hat{N}_{\mathcal{B}}\rangle\right)^2 \right\rangle = \left\langle \hat{N}_{\mathcal{B}}^2 \right\rangle - \left\langle \hat{N}_{\mathcal{B}} \right\rangle^2$ is the mean-square fluctuation in the number of particles within a ball $\mathcal{B}$. Inserting $\hat{N}_{\mathcal{B}} \equiv \int_{\mathcal{B}} d^3x\, \hat{n}(x)$ gives

$$
(\Delta N_{\mathcal{B}})^2 = \int_{\mathcal{B}} d^3\underset{\sim}{x} \int_{\mathcal{B}} d^3\underset{\sim}{y}\, \langle \hat{n}(\underset{\sim}{x})\hat{n}(\underset{\sim}{y})\rangle - \langle \hat{n}(\underset{\sim}{x})\rangle\langle \hat{n}(\underset{\sim}{y})\rangle = \int_{\mathcal{B}} d^3\underset{\sim}{x} \int_{\mathcal{B}} d^3\underset{\sim}{y}\, G_{nn}(\underset{\sim}{x}-\underset{\sim}{y})\,.
$$

Calculating this integral, exactly, is very difficult — but this is not necessary. We merely want to understand how the result depends on the average density n and the radius R of the ball. If one inserts the above result for G_{nn} into the integral, then the first (delta-function) piece gives an easy contribution, which is precisely the result of Poisson statistics, $(\Delta N_{\mathcal{B}})^2\big|_{\text{first term}} = \langle N_{\mathcal{B}}\rangle = \bar{n}\, \frac{4}{3}\pi R^3$. The second term in G_{nn} is strictly negative and evidently must decrease the number fluctuations $\Delta N_{\mathcal{B}}$. But by how much? If these two contributions nearly cancel each other, then it will be necessary to compute the double volume integral in the second term very accurately — which is a problem. Its much better to rewrite $(\Delta N_{\mathcal{B}})^2$ in a form which avoids this cancellation. To do so, note that $\int d^3\underset{\sim}{y}\, G_{nn}(\underset{\sim}{x}-\underset{\sim}{y}) = \int d^3\underset{\sim}{y}\, \langle \hat{n}(\underset{\sim}{x})\hat{n}(\underset{\sim}{y})\rangle - \langle \hat{n}(\underset{\sim}{x})\rangle\langle \hat{n}(\underset{\sim}{y})\rangle = \langle \hat{n}(\underset{\sim}{x})\hat{N}\rangle - \langle \hat{n}(\underset{\sim}{x})\rangle\langle \hat{N}\rangle = 0$. The last step follows from the fact that the ground state, in which these expectation values are being taken, is an eigenstate of the total number operator $\hat{N}$. So $\int_{\mathcal{B}} d^3\underset{\sim}{y}\, G_{nn}(\underset{\sim}{x}-\underset{\sim}{y}) = -\int_{\mathbb{R}^3\setminus\mathcal{B}} d^3\underset{\sim}{y}\, G_{nn}(\underset{\sim}{x}-\underset{\sim}{y})$, where the second integral covers space exterior to the ball $\mathcal{B}$. Consequently $(\Delta N_{\mathcal{B}})^2 = \int_{\mathcal{B}} d^3\underset{\sim}{x} \int_{\mathbb{R}^3\setminus\mathcal{B}} d^3\underset{\sim}{y}\, \frac{p_f^2}{4\pi^4|\underset{\sim}{x}-\underset{\sim}{y}|^4} \left(\cos p_f|\underset{\sim}{x}-\underset{\sim}{y}| - \frac{\sin p_f|\underset{\sim}{x}-\underset{\sim}{y}|}{p_f|\underset{\sim}{x}-\underset{\sim}{y}|}\right)^2$. This is still somewhat tricky to estimate correctly. To do so, first note that when $|\underset{\sim}{x}-\underset{\sim}{y}| \gg p_f^{-1}$, which is the bulk of the integration domain, the $(\cos p_f|\underset{\sim}{x}-\underset{\sim}{y}| - \cdots)^2$ factor oscillates rapidly between 0 and 1, and may be replaced by 1/2. Next, introduce spherical coordinates for $\underset{\sim}{y}$, but with the polar axis chosen to coincide with the direction of $\underset{\sim}{x}$, and then use ordinary spherical coordinates for $\underset{\sim}{x}$. After integrating over the azimuthal angle of $\hat{y}$ and the direction $\hat{x}$, the remaining integrals have the form

$$
(\Delta N_{\mathcal{B}})^2 \simeq \frac{p_f^2}{8\pi^4}\, (4\pi) \int_0^R r^2\, dr\, (2\pi) \int_R^\infty r'^2\, dr' \int_{-1}^{1} dz\, [r^2 + r'^2 - 2rr'z]^{-2}
$$

$$
\begin{aligned}
&= \frac{p_f^2}{\pi^2} \int_0^R r^2\, dr \int_R^\infty r'^2\, dr' \, \frac{2}{(r^2 - r'^2)^2} \\
&= \frac{p_f^2}{\pi^2} \int_0^R r^2\, dr \left(\frac{R}{R^2 - r^2} + \frac{1}{2r} \, \ln \frac{R+r}{R-r} \right) && (1) \\
&= \frac{p_f^2 R^2}{\pi^2} \int_0^1 dy \left(\frac{y^2}{1-y^2} + \frac{y}{2} \, \ln \frac{1+y}{1-y} \right) . && (2)
\end{aligned}
$$

So we get a result which equals $(p_f R)^2$ times some dimensionalless integral. There's just one problem — this integral doesn't exist! The first term is not integrable as $y \to 1$. However, we made an approximation (replacing the oscillatory factor by 1/2) which was only a good approximation when $|\underset{\sim}{x} - \underset{\sim}{y}| \gtrsim p_f^{-1}$. That inequality is always satisfied if $R - |\underset{\sim}{x}| \gtrsim p_f^{-1}$, but it can be violated when $R - |\underset{\sim}{x}| \lesssim p_f^{-1}$. Properly including the oscillating factor would tame the divergence we just found, since $\frac{1}{z^4} \left(\cos z - \frac{\sin z}{z} \right)^2$ remains finite as $z \to 0$. Including this factor would only significantly change the integrand of (1) when r is within p_f^{-1} of the upper endpoint, and the net effect is equivalent to just changing the upper limit from R to $R - p_f^{-1}$. This is the same as changing the upper limit in (2) to $1 - 1/(p_f R)$. Doing the final integral then yields $(\Delta N_{\mathcal{B}})^2 \simeq \frac{p_f^2 R^2}{2\pi^2} \left[\ln(p_f R) + C \right]$, for some (undetermined) constant C. Therefore, $(\Delta N_{\mathcal{B}})^2$ scales as $p_f^2 R^2 \sim n^{2/3} R^2$ times a logarithm of this quantity.

(d) For Poisson statistics, $(\Delta N_{\mathcal{B}})^2 = \langle N_{\mathcal{B}} \rangle = \frac{4}{3}\pi(R^3 n)$. So the mean square particle number fluctuation grows cubically with the radius of the ball for classical particles, but only quadratically for fermions. When $n^{1/3} R \gg 1$, this means that the particle number fluctuations are much much smaller for degenerate fermions. This reflects the "rigidity" which is caused by the effective repulsion between fermions that is a consequence of Fermi statistics — even for non-interacting fermions.