1. Free complex scalar field.

(a) Start with two equal mass real scalar fields, \( \phi_i \) with \( i = 1, 2 \), The standard mode expansions for these fields are \( \phi_i(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega(k)}} \left( \hat{a}_i(k) e^{ikx} + \hat{a}_i^\dagger(k) e^{-ikx} \right) \), and their corresponding conjugate momenta are \( \pi_i(x) = \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{2\omega(k)}}{2ik} \left( \hat{a}_i(k) e^{ikx} - \hat{a}_i^\dagger(k) e^{-ikx} \right) \). These fields satisfy the canonical equal time commutation relations \( [\pi_i(x), \pi_j(y)] = [\phi_i(x), \phi_j(y)] = 0 \) and \( i[\pi_i(x), \phi_j(y)] = \delta_{ij} \delta^3(x-y) \). Now build a complex scalar field by defining \( \Phi(x) = (\phi_1(x) + i\phi_2(x))/\sqrt{2} \), and \( \Pi(x) = (\pi_1(x) + i\pi_2(x))/\sqrt{2} \). These complex fields satisfy the equal time commutation relations

\[
i[\Pi^\dagger(x), \Phi(y)] = \frac{i}{2} \left( [\pi^\dagger_1(x), \phi_1(y)] + [\pi^\dagger_2(x), \phi_2(y)] \right) = \delta^3(x-y),
\]

\[
i[\Pi(x), \Phi(y)] = \frac{i}{2} \left( [\pi_1(x), \phi_1(y)] + [\pi_2(x), \phi_2(y)] \right) = 0,
\]

\[
[\Pi^\dagger(x), \Pi(y)] = [\Phi(x), \Pi(y)] = 0,
\]

(and their Hermitian conjugates). Inverting the mode expansion (to express the creation and annihilation operators in terms of the fields) gives \( \hat{a}_j(k) = \int d^3x \frac{e^{ikx}}{\sqrt{2\omega(k)}} \{ i\pi_j(x) + \omega(k) \phi_j(x) \} \). Insert this (and its conjugate) into the Hamiltonian and simplify:

\[
\hat{H} = \int \frac{d^3k}{(2\pi)^3} \omega(k) \left( \hat{a}^\dagger_1(k) \hat{a}_1(k) + \hat{a}^\dagger_2(k) \hat{a}_2(k) \right)
\]

\[
= \int d^3x d^3y \frac{d^3k}{(2\pi)^3} e^{ik(x-y)} \left[ \left( -i\pi_1(x) + \omega(k) \phi_1(x) \right)i\pi_1(y) + \omega(k) \phi_1(y) \right)
\]

\[
+ \left( -i\pi_2(x) + \omega(k) \phi_2(x) \right)i\pi_2(y) + \omega(k) \phi_2(y) \right)
\]

\[
= \int d^3x d^3y \frac{d^3k}{(2\pi)^3} e^{ik(x-y)} \left[ \pi_1(x) \pi_1(y) + \pi_2(x) \pi_2(y) \right.
\]

\[
+ \omega(k)^2 \left( \phi_1(x) \phi_1(y) + \phi_2(x) \phi_2(y) \right)
\]

\[
- i\omega(k) \left( \left[ \pi_1(x), \phi_1(y) \right] + \left[ \pi_2(x), \phi_2(y) \right] \right),
\]

\[
= \int d^3x d^3y \frac{1}{2} \left[ \pi_1(x) \pi_1(y) + \pi_2(x) \pi_2(y) \right] \delta^3(x-y)
\]

\[
+ \frac{1}{2} \left( \phi_1(x) \phi_1(y) + \phi_2(x) \phi_2(y) \right) \left( -\nabla^2 + m^2 \right) \delta^3(x-y) - V \int \frac{d^3k}{(2\pi)^3} \omega(k).
\]

\[
= \int d^3x \frac{1}{2} \left[ \pi_1(x) \pi_1(x) + \pi_2(x) \pi_2(x) + \nabla \phi_1(x) \cdot \nabla \phi_1(x) + \nabla \phi_2(x) \cdot \nabla \phi_2(x)
\]

\[
+ m^2 \phi_1(x) \phi_1(x) + m^2 \phi_2(x) \phi_2(x) \right] - \mathcal{K}.
\]

\[
= \int d^3x \left[ \Pi^\dagger(x) \Pi(x) + \nabla \Phi^\dagger(x) \cdot \nabla \Phi(x) + m^2 \Phi^\dagger(x) \Phi(x) \right] - \mathcal{K}.
\]

In order to write the cross terms as commutators, the change of variables \( x \leftrightarrow y \), and \( k \rightarrow -k \) was made in the \( \phi_i \pi_i \) terms. Then, in all but the last constant term, the Fourier integral over \( k \) was performed, yielding a delta function, or derivatives thereof. Lastly, the derivatives acting on the delta function were integrated by parts (once in \( x \) and once in \( y \)) to reach the final form. The (infinite) additive constant \( \mathcal{K} = V \int \frac{d^3k}{(2\pi)^3} \omega(k) \) precisely cancels the zero point energy of the infinite set of harmonic oscillator degrees of freedom.
(b) Comparing the initial mode expansions to the alternative mode expansion of the complex field,

\[
\Phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{i2\omega(k)}} (\hat{a}_+(k) e^{ikx} + \hat{a}_+^\dagger(k) e^{-ikx}),
\]

\[
\Pi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{i2\omega(k)}}{2i} (\hat{a}_+(k) e^{ikx} - \hat{a}_+^\dagger(k) e^{-ikx}),
\]

shows that \(\hat{a}_+(k) = [\hat{a}_1(k) + i\hat{a}_2(k)]/\sqrt{2}\) and \(\hat{a}_-(k) = [\hat{a}_1(k) - i\hat{a}_2(k)]/\sqrt{2}\). Simple algebra shows that the \(\hat{a}_\pm\) also satisfy canonical commutation relations, namely

\[
[\hat{a}_+(k), \hat{a}_+^\dagger(k')] = (2\pi)^3 \delta^3(k-k'),
\]

with all other commutators vanishing. Inserting the inverse relations, \(\hat{a}_1(k) = (\hat{a}_+(k) + \hat{a}_-(k))/\sqrt{2}, \hat{a}_2(k) = (\hat{a}_+(k) - \hat{a}_-(k))/\sqrt{2i}\) into the original Hamiltonian gives

\[
\hat{H} = \int \frac{d^3k}{(2\pi)^3} \omega(k) \left\{ \hat{a}_1^\dagger(k) \hat{a}_1(k) + \hat{a}_2^\dagger(k) \hat{a}_2(k) \right\}
\]

\[
= \int \frac{d^3k}{(2\pi)^3} \omega(k) \frac{1}{2} \left\{ (\hat{a}_+^\dagger(k) + \hat{a}_-^\dagger(k)) (\hat{a}_+(k) + \hat{a}_-(k)) + (\hat{a}_+^\dagger(k) - \hat{a}_-^\dagger(k)) (\hat{a}_+(k) - \hat{a}_-(k)) \right\}
\]

\[
= \int \frac{d^3k}{(2\pi)^3} \omega(k) \left\{ \hat{a}_1^\dagger(k) \hat{a}_1(k) + \hat{a}_2^\dagger(k) \hat{a}_2(k) \right\}.
\]

This looks just like the original form of the Hamiltonian, except for changing \(\hat{a}_i(k) \rightarrow \hat{a}_\pm(k)\). Hence, the Hamiltonian, which in its original form appeared to measure the number of excitations created by the \(\hat{a}_i^\dagger(k)\) operators (suitably weighted by the single particle energy \(\omega(k)\)), can also be interpreted as measuring the number of excitations created by the \(\hat{a}_\pm^\dagger(k)\) operators (weighted by the same single particle energy).

(c) The states: \(\hat{a}_1^\dagger(k)|0\rangle, \hat{a}_2^\dagger(k)|0\rangle, \hat{a}_+^\dagger(k)|0\rangle, \hat{a}_-^\dagger(k)|0\rangle\) are all single particle states, but they are not linearly independent. For every momentum \(k\) there is a two-dimensional space of single particle states which are all degenerate eigenstates of the Hamiltonian. The two states, \(\hat{a}_1^\dagger(k)|0\rangle, \hat{a}_2^\dagger(k)|0\rangle\), provide one choice of orthonormal basis for the single particle eigenspace; the other pair of states \(\hat{a}_+^\dagger(k)|0\rangle, \hat{a}_-^\dagger(k)|0\rangle\) are simply a different, but equally valid, orthonormal basis. Only two (independent) types of particles exist in this theory, but because the particles are exactly degenerate in mass, there is no meaningful definition of which single particle states are “fundamental” and which are quantum superpositions. Note that this is completely analogous to photon polarization (or electron spin) states — for photons one can use linearly polarized basis states, or circularly polarized states (or for electrons, spin up and down with respect to the \(\hat{z}\) axis, or up and down with respect to \(\hat{x}\), or any other axis). Note that the total number of particles (as well as all other operators) may be written in either the 1, 2 basis, or the +, − basis,

\[
\hat{N} = \int \frac{d^3k}{(2\pi)^3} \left\{ \hat{a}_1^\dagger(k) \hat{a}_1(k) + \hat{a}_2^\dagger(k) \hat{a}_2(k) \right\} = \int \frac{d^3k}{(2\pi)^3} \left\{ \hat{a}_+^\dagger(k) \hat{a}_+(k) + \hat{a}_-^\dagger(k) \hat{a}_-(k) \right\}.
\]

2. Interacting complex scalar field.

(a) For a complex scalar field, local Lorentz invariant operators also invariant under \(U(1)\) phase rotations \((\phi \rightarrow e^{i\alpha}\phi)\) with dimensions up to 4 are:

<table>
<thead>
<tr>
<th>dimension</th>
<th>(U(1)) invariant operators</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>(\phi^*\phi)</td>
</tr>
<tr>
<td>4</td>
<td>((\phi^<em>\phi)^2, (\partial\phi)^</em> \cdot (\partial \phi), \phi^<em>(\partial^2 \phi), (\partial^2 \phi)^</em> \phi)</td>
</tr>
</tbody>
</table>

(b) For a free scalar field, one may evaluate the commutator \[ [\hat{D}(a), \hat{D}(b)] \]
separate the two terms of the commutator, insert projections onto eigenstates of definite energy, and evaluate the commutator in the eigenbasis. Hence, the Hermitian generator of a transformation which rotates the phase of \( \phi \) (and \( \Pi \)) is just \( Q = i \int d^3x \left[ \Pi(x) \phi(x) - \phi(x) \Pi(x) \right] + U(\alpha) \). The unitary operator which implements the \( U(1) \) transformation is \( U(\alpha) = e^{i\alpha Q} \). The Lagrangian form of this conserved charge is \( Q = i \int d^3x \left[ \langle \partial_0 \phi(x) \rangle^\dagger \phi(x) - \phi(x) \langle \partial_0 \phi(x) \rangle \right] \).

(c) With a single complex scalar field, any \( U(1) \) invariant theory is automatically charge conjugation invariant. With two or more complex scalar fields this is no longer the case. For example, if \( \chi \) and \( \phi \) are complex scalars with the same \( U(1) \) charge [so the \( U(1) \) symmetry is \( \chi \to e^{i\alpha} \chi \) and \( \phi \to e^{i\alpha} \phi \)], then \( i\chi^* \phi - i\chi \phi^* \) is a real, \( U(1) \) and Lorentz invariant term which is odd under charge conjugation.

(d) In \( D = 3 \) spacetime dimensions, a scalar field \( \phi \) has dimension \( 1/2 \), so the operator \( (\phi^3) \) is now renormalizable. The most general (perturbatively) renormalizable \( U(1) \) invariant theory of a single scalar field can be put in the form \( \mathcal{L} = (\partial \phi)^* \cdot (\partial \phi) + m^2 |\phi|^2 + \lambda |\phi|^4 + \eta |\phi|^6 + \text{const.} \)


(a) Separate the two terms of the commutator, insert projections onto eigenstates of definite four-momentum, and use \( \langle 0 | \hat{A}(y) | n \rangle = a_n e^{ip_\nu y} \) and \( \langle 0 | \hat{A}^\dagger(x) | n \rangle = b_n e^{ip_\nu x} \), to find

\[
\int d^4 y e^{-iq \cdot (y - x)} \langle 0 | [\hat{A}(y), \hat{A}^\dagger(x)] | 0 \rangle = \int d^4 y e^{-ip \cdot (y - x)} \sum_n \left\{ |a_n|^2 e^{-i\omega_n (y - x)} - |b_n|^2 e^{-i\omega_n (y - x)} \right\} = \sum_n (2\pi)^4 \left\{ |a_n|^2 \delta (q - p_n) - |b_n|^2 \delta (q + p_n) \right\} \equiv \chi(q).
\]

(b) For a free scalar field, one may evaluate the commutator \( [\hat{\phi}(y), \hat{\phi}(x)] \) directly by inserting the mode expansion \( \hat{\phi}(x) = \int \frac{d^3 p}{(2\pi)^3} (e^{ip \cdot x} \hat{a}(p) + e^{-ip \cdot x} \hat{a}^\dagger(p))/\sqrt{2\omega(p)} \) and using the canonical commutation relations of \( \hat{a} \) and \( \hat{a}^\dagger \). This gives \( [\hat{\phi}(y), \hat{\phi}(x)] = \int \frac{d^3 p}{(2\pi)^3} (e^{ip \cdot (y - x)} - e^{-ip \cdot (y - x)})/(2\omega(p)) \). This is a four-dimensional plane wave [with \( p^0 = \omega(p) \)] inside a three-dimensional integral. Performing a spacetime Fourier transforms gives \( \chi(q) = \int d^4 y e^{-iq \cdot (y - x)} \int \frac{d^3 p}{(2\pi)^3} \left[ e^{ip \cdot (y - x)} - e^{-ip \cdot (y - x)} \right] = \int \frac{d\nu}{2\omega(q)} \left[ e^{i(q^0 - \omega(q))t} - e^{i(q^0 + \omega(q))t} \right] = \frac{2\pi}{2\omega(q)} \delta (q^0 - \omega(q)) \). One obtains the same result by noting that only single particle intermediate states can contribute to the representation in part (a), with \( a_n = b_n = 1/\sqrt{2\omega(p_n)} \) for the fundamental scalar field.

(c) Inserting the result from (b) for the spectral density into the correlator gives

\[
G(k) = \int \frac{d\nu}{2\pi} \frac{\chi(\nu, k)}{\nu - k^0 - i\epsilon \text{sgn}\nu} = \int \frac{d\nu}{2\pi} \frac{2\pi}{2\omega(k)} \frac{\nu - k^0}{\nu - k^0 - i\epsilon \text{sgn}\nu} = \frac{1}{2\omega(k)} \left[ \frac{1}{\omega(k) - k^0 - i\epsilon \text{sgn}\omega(k)} + \frac{1}{\omega(k) + k^0 - i\epsilon \text{sgn}\omega(k)} \right] = \frac{1}{-(k^0)^2 + (\omega(k) - i\epsilon \text{sgn}\omega(k))^2} = \frac{1}{(k^0)^2 + (k^2 + m^2 - i\epsilon) - i\epsilon}.
\]
In the step where \((\omega(k) - i\epsilon \text{sgn}(\omega(k)))^2\) is multiplied out, the essential point to note is that
\[-2i\epsilon\omega(k)\text{sgn}(\omega(k)) = -i \times (2|\omega(k)|\epsilon) = -i \times (\text{positive infinitesimal}).\]
You are perfectly free to call this resulting positive infinitesimal \(\epsilon\) (that is, you are free to ignore the rescaling of \(\epsilon\) by a factor of \(2|\omega(k)|\)) since the only thing which matters when you write a denominator of \(k^2 + m^2 - i\epsilon\) is the sign of the infinitesimal displacement of the pole away from the real axis.

(d) In the \(BB^\dagger\) spectral density, we see a delta-function spike at \(\omega = \pm 2\), which implies that the theory contains stable particles with mass 2, which can be created by the action of \(\hat{B}^\dagger\) on the vacuum. Call these \('b'\)-particles. States containing two \(b\) particles must appear starting at \(\omega = 4\), but no such threshold is visible in the \(BB^\dagger\) correlator, while this threshold is visible in the \(AA^\dagger\) correlator. Similarly, states containing three \(b\) particles must appear starting at \(\omega = 6\), and such a threshold is visible in the \(BB^\dagger\) correlator, while this threshold is not apparent in the \(AA^\dagger\) correlator.

Clearly visible in the \(AA^\dagger\) spectral density (but not \(BB^\dagger\)) is a resonance at \(\omega = 8\) with a width of roughly 1. This suggests that the operator \(A^\dagger\), acting on the vacuum, can create a distinct of type of excitation with energy about 8 — call it an \('a'\) — but this excitation is not a stable particle, rather it decays into lighter stable particles, namely the \(b\) particles. A two-body decay, \(a \rightarrow bb\), is energetically allowed and completely consistent with the form of the \(AA^\dagger\) spectral density. There is also a visible step, or additional threshold, in the \(BB^\dagger\) spectral density at \(\omega = \pm 10\). Given the above, this would be consistent with a contribution from production of \(ab\) states, which subsequently decay to \(bbb\).

What remains to be explained is why the \(AA^\dagger\) and \(BB^\dagger\) spectral densities are so different — why do \(bb\) states contribute to the \(AA^\dagger\) correlator but not to the \(BB^\dagger\) correlator, etc.? Evidently:

- \(B^\dagger\) has non-zero amplitudes to create a single \(b\) particle, create a \(bbb\) three-particle state, or create an \(ab\) two-particle state which decays to \(bbb\).
- \(A^\dagger\) has non-zero amplitudes to create a \(bb\) two-particle state, or create the \(a\) resonance which subsequently decays to \(bb\).

In each case, the operators can also have non-zero amplitudes to create yet more complicated multi-particle states whose contributions produce features at higher energy, or are too small to see in the “data”.

The absence of two \(b\)-particle contributions in the \(BB^\dagger\) spectral density, together with their presence in the \(AA^\dagger\) density, suggests the existence of some symmetry under which the stable \(b\)-particles and the operator \(\hat{B}\) are odd, while the \(a\)-resonance and the operator \(\hat{A}\) are even. Such a symmetry would imply that \(\hat{B}\) and \(\hat{B}^\dagger\) could only create odd numbers of \(b\)-particles when acting on the vacuum, while \(\hat{A}\) and \(\hat{A}^\dagger\) could only create even numbers of \(b\) particles. This nicely matches all the information above.

In summary, the two spectral densities imply the existence of one stable particle, of mass 2, plus an unstable resonance with mass 8 and decay width (or inverse lifetime) about 1, together with the existence of some symmetry under which the lightest stable particle is odd, while the resonance is even and decays to even numbers of stable particles.