1. Feynman Rules for N-component ϕ^4. Introducing a species index $i = 1, \cdots, N$, the D-dimensional (Euclidean) action is $S = \int d^D x \left[\frac{1}{2} \big(\partial_\mu \phi_i\big)^2 + \frac{1}{2} m^2 \phi_i \phi_i + \frac{1}{8} \lambda \big(\phi_i \phi_i\big)^2 \right]$.

   (a) Except for the specifics of the vertex factor, the coordinate space Feynman rules have the conventional form for a real scalar field theory:

   i. Draw diagrams composed of unoriented lines and quartic vertices. For the free energy, these are connected vacuum diagrams. For the K-point connected correlation function $\langle \phi_{i_1}(x_1) \cdots \phi_{i_K}(x_K) \rangle_{\rm conn}$, these are fully connected diagrams containing K external lines (i.e., lines whose endpoints do not both connect to some quartic vertex).

   ii. Label each end of every line with an independent species index $i, j, \cdots$. Label every vertex, and external line endpoint, with an independent D-dimensional spacetime coordinate $x, y, \cdots$.

   iii. Every line ${}^{i}_{x}$ —— ${}^{j}_{y}$ represents the free propagator $G_0^{ij}(x,y) = \delta^{ij} \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip\cdot(x-y)}}{p^2+m^2}$, which is the Green's function of the linear operator $\delta_{ij}(-\partial^2 + m^2)$ defining the covariance of the free theory.

   iv. Every vertex ${}^{i}_{l}\!\times\!{}^{j}_{k}$ represents the factor $P_{ijkl} \equiv -\lambda\,(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ (plus a spacetime coordinate label). This factor comes from minus the fourth variational derivative of the Lagrangian, and is symmetric under any permutation of the species indices because partial derivatives commute: $\frac{\delta}{\delta\phi_i(x_1)} \frac{\delta}{\delta\phi_j(x_2)} \frac{\delta}{\delta\phi_k(x_3)} \frac{\delta}{\delta\phi_l(x_4)} \left[\frac{\lambda}{8}\, \phi_a(x)\phi_a(x)\phi_b(x)\phi_b(x)\right]$ $= \lambda\,(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\, \delta^D(x{-}x_1) \cdots \delta^D(x{-}x_4)$.

   v. Integrate over the spacetime location of all vertices and sum over all repeated species indices (i.e., all indices except those labeling external endpoints).

   vi. Divide by the symmetry factor of the diagram, which is determined by examining the *unlabeled* diagram, regarding external endpoints as fixed, and considering what permutations of lines and/or vertices leave the diagram invariant.

   (b) The equivalent momentum space Feynman rules are:

   i. Draw appropriate diagrams composed of unoriented lines and quartic vertices.

   ii. Label endpoints of every line with independent species indices $i, j, \cdots$. In addition, give every line an orientation (an arrow) and label the line with an independent D-dimensional momentum $p, q, \cdots$.

   iii. Each labeled line $i \xrightarrow[p]{} j$ represents a factor of $\widetilde{G}_0^{ij}(p) \equiv \delta^{ij}/(p^2+m^2)$. (Continued back to Minkowski space, the denominator becomes $p^2 + m^2 - i\epsilon$.)

   iv. Each vertex represents a factor of $P_{ijkl} = -\lambda\,(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$.

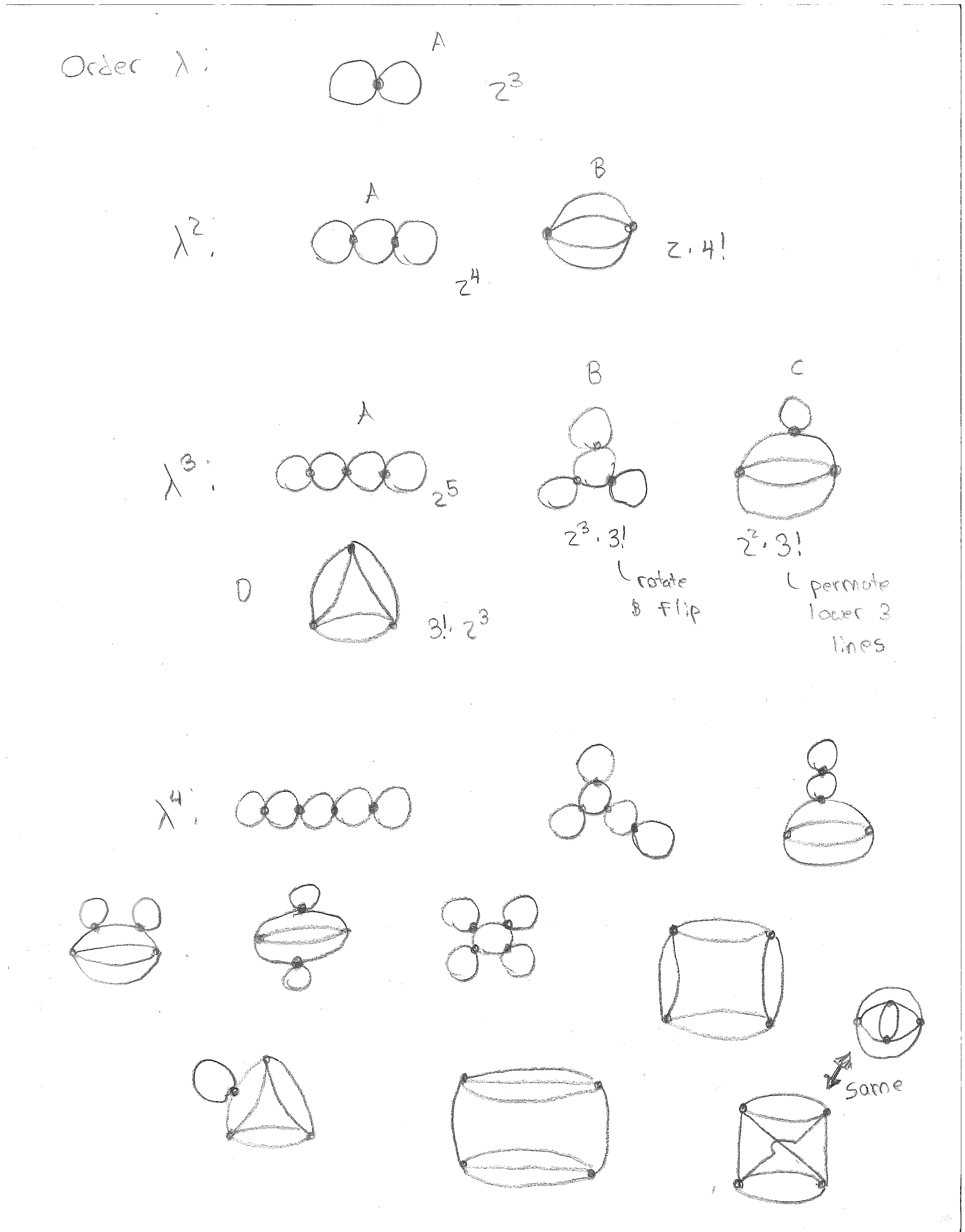
   v. Conserve momentum at every vertex. (That is, if $p_1 \cdots p_4$ are all incoming momenta at some vertex, regard the vertex factor as containing $(2\pi)^D\, \delta^D(p_1+p_2+p_3+p_4)$ and use this δ-function to determine the momenta on one of the lines meeting at this vertex in terms of the other momenta.)

   vi. Regard the momentum flowing in (or out) on external lines as fixed. Integrate over all other undetermined loop momenta with measure $\frac{d^D p}{(2\pi)^D}$.

   vii. Divide by the symmetry factor as usual (but remember that symmetry factors are determined by the unlabeled diagram with unoriented lines).

   These rules generate contributions to Fourier-transformed K-point connected correlators, $\widetilde{G}_{i_1\cdots i_K}(p_1 \cdots p_K) \equiv \int d^D x_1 \cdots d^D x_K\; e^{i(p_1\cdot x_1 + \cdots + p_K \cdot x_K)}\; \langle \phi_{i_1}(x_1) \cdots \phi_{i_K}(x_K) \rangle_{\rm conn}$ (or the free energy).

(c) Distinct connected vacuum diagrams up to five-loop order (or $O(\lambda^4)$), with their associated symmetry factors (through four-loop order), are:



The number of K-loop diagrams increases *factorially* as the order K increases. This can be understood by considering a recursive construction of K-loop diagrams starting from $K-1$ loop diagrams. To increase a diagram's order, one must add one more vertex and join it to the diagram by cutting one or more lines of the original diagram and joining these cut lines to the new vertex. The number of choices for which lines to cut scales with the number of vertices (or the order of the diagram). The resulting combinatorial growth in the number of diagrams is much faster than exponential or polynomial growth. If (a *big* if!) all diagrams make roughly comparable contributions, then this growth in the number of diagrams will overwhem the λ^K suppression factor no matter how small is the coupling λ, suggesting (correctly) that perturbation theory in powers of λ can only be an asymptotic expansion, not a convergent Taylor expansion.

2. The self-energy $\tilde\Sigma(k) \equiv \tilde G(k)^{-1} - \tilde G^0(k)^{-1}$. Solving for $\tilde G(k)$ and inserting the explicit form of the relativistic free scalar propagator $\tilde G^0(k)$ gives $\tilde G(k) = 1/\big[\tilde G^0(k)^{-1}+\tilde\Sigma(k)\big] = 1/\big[k^2+m_0^2+\tilde\Sigma(k)-i\epsilon\big]$.

   (a) If the interacting theory describes a scalar particle of physical mass $m_{\rm ph}$, then there must exist single particle states $|\underset{\sim}{p}\rangle$ whose exact energy is $E_{\rm ph}(\underset{\sim}{p}) = (\underset{\sim}{p}^2 + m_{\rm ph}^2)^{1/2}$. The interacting propagator will have a contribution from these single particle intermediate states which varies with the phase e^{ipx} where $p^0 = E_{\rm ph}(\underset{\sim}{p})$:

$$
\begin{aligned}
G(x) &= i\langle 0|\mathscr{T}\big(\phi(x)\phi(0)\big)|0\rangle = i\Theta(x^0)\sum_n \langle 0|\phi(x)|n\rangle\langle n|\phi(0)|0\rangle + (x\to -x)\\
&= i\Theta(x^0)\int \frac{d^3p}{(2\pi)^3}\,\langle 0|\phi(x)|\underset{\sim}{p}\rangle\langle \underset{\sim}{p}|\phi(0)|0\rangle + \text{multi-particle stuff} + (x\to -x)\\
&= i\Theta(x^0)\int \frac{d^3p}{(2\pi)^3}\, e^{-iE_{\rm ph}(\underset{\sim}{p})x^0 + i\underset{\sim}{p}\cdot\underset{\sim}{x}}\big|\langle \underset{\sim}{p}|\phi(0)|0\rangle\big|^2 + \text{multi-particle stuff} + (x\to -x)\,.
\end{aligned}
$$

   So the Fourier transform $\tilde G(p)$ receives contributions from single particle states proportional to $i\int_0^\infty dx^0\,\Big(e^{i(p^0-E_{\rm ph}(\underset{\sim}{p}))x^0} + e^{-i(p^0+E_{\rm ph}(\underset{\sim}{p}))x^0}\Big) = [-p^0 + E_{\rm ph}(\underset{\sim}{p}) - i\epsilon]^{-1} + [p^0 + E_{\rm ph}(\underset{\sim}{p}) - i\epsilon]^{-1}$ $\propto [-(p^0)^2 + E_{\rm ph}(\underset{\sim}{p})^2 - i\epsilon]^{-1}$, and necessarily diverges when $p^0 = \pm E_{\rm ph}(\underset{\sim}{p})$. Therefore, a pole in the interacting propagator at $p^0 = \pm E_{\rm ph}$ is a signal that there exists, in the full theory, single particle states with energy $E_{\rm ph}$ and momentum $\underset{\sim}{p}$. From the above representation of $\tilde G(p)$ in terms of the self-energy, such a pole will occur when the denominator has a zero, which occurs whenever there exists a four-momentum $\bar p$ for which $\bar p^2 + m_0^2 + \tilde\Sigma(\bar p) = 0$. If $|\tilde\Sigma(\bar p)| \ll m_0^2$ then the self-energy is a small perturbation, so the root $\bar p$ will be close to an unperturbed on-shell momentum p satisfying $p^2 + m_0^2 = 0$. Therefore, one may expand $\tilde\Sigma(\bar p)$ about the unperturbed on-shell momentum p and find $-\bar p^2 = m_0^2 + \tilde\Sigma(p) + \tilde\Sigma'(p)\,(\bar p - p) + \cdots = m_0^2 + \tilde\Sigma(p) + O(\tilde\Sigma^2/m_0^2)$ [since $\bar p^2 - p^2 = O(\tilde\Sigma)$]. Therefore, $m_{\rm ph}^2 = m_0^2 + \tilde\Sigma(p) + O(\tilde\Sigma^2/m_0^2)$. If $\operatorname{Im}\tilde\Sigma(p) \neq 0$, then the physical mass is not real, and the "on-shell" energy of the excitation is complex,

$$
E_{\rm ph} = \sqrt{\underset{\sim}{p}^2 + m_0^2 + \tilde\Sigma(p)} = \sqrt{\underset{\sim}{p}^2 + \operatorname{Re} m_{\rm ph}^2} + \frac{\frac{i}{2}\operatorname{Im}\tilde\Sigma(p)}{\sqrt{\underset{\sim}{p}^2 + \operatorname{Re} m_{\rm ph}^2}} + O(\operatorname{Im}\tilde\Sigma)^2\,.
$$

   Consequently, on-shell "plane-waves" will decay exponentially in time, $|e^{i\bar p\cdot x}| = \exp(-\frac{1}{2}\Gamma x^0)$ with a decay rate $\Gamma = -\operatorname{Im}\tilde\Sigma(p)/\operatorname{Re}E_{\rm ph}$, indicating that the interacting "particle" is not stable. To be physically sensible, the decay rate had better be positive, which means $\operatorname{Im}\tilde\Sigma(p)$ must be negative. Note that this is the same sign as the $-i\epsilon$ imaginary part of the free propagator.

   (b) Let the self-energy have a perturbative expansion $\tilde\Sigma(k) \sim \sum_{n=1}^\infty \tilde\Sigma^{(n)}(k)$ where $\tilde\Sigma^{(n)}(k) = O(\lambda^n)$. Note that $\tilde\Sigma^{(0)}(k) = 0$ since, by assumption, $G^0(k)$ is the correct propagator in the unperturbed theory. Start with the defining relation for $\tilde G(k)$ and expand in powers of $\tilde\Sigma$,

$$
\tilde G(k) = \Big[\tilde G^0(k)^{-1} + \tilde\Sigma(k)\Big]^{-1} = \tilde G^0(k)\,\Big[1 + \tilde\Sigma(k)\,\tilde G^0(k)\Big]^{-1} \sim \tilde G^0(k)\sum_{n=0}^{\infty}\Big(-\tilde\Sigma(k)\,\tilde G^0(k)\Big)^n\,.
$$

   Plug in the perturbative expansion of $\tilde\Sigma$ and collect terms at the same order to find

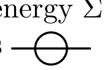
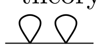
$$
\tilde G^{(n)} = \sum_{m=1}^{n}(-)^m\sum_{j_1=1}^{n}\cdots\sum_{j_m=1}^{n}\delta^n_{j_1+\cdots+j_m}\,\tilde G^0\,\tilde\Sigma^{(j_1)}\tilde G^0\,\tilde\Sigma^{(j_2)}\tilde G^0\cdots\tilde G^0\,\tilde\Sigma^{(j_m)}\,\tilde G^0\,,
$$

or

$$
\begin{aligned}
\tilde{G}^{(1)} &= -\tilde{G}^0\,\tilde{\Sigma}^{(1)}\,\tilde{G}^0\,,\\
\tilde{G}^{(2)} &= -\tilde{G}^0\,\tilde{\Sigma}^{(2)}\,\tilde{G}^0 + (-)^2\tilde{G}^0\,\tilde{\Sigma}^{(1)}\,\tilde{G}^0\,\tilde{\Sigma}^{(1)}\,\tilde{G}^0\,,\\
\tilde{G}^{(3)} &= -\tilde{G}^0\,\tilde{\Sigma}^{(3)}\,\tilde{G}^0 + (-)^2\tilde{G}^0\,\tilde{\Sigma}^{(1)}\,\tilde{G}^0\,\tilde{\Sigma}^{(2)}\,\tilde{G}^0\\
&\qquad + (-)^2\tilde{G}^0\,\tilde{\Sigma}^{(2)}\,\tilde{G}^0\,\tilde{\Sigma}^{(1)}\,\tilde{G}^0 + (-)^3\tilde{G}^0\,\tilde{\Sigma}^{(1)}\,\tilde{G}^0\,\tilde{\Sigma}^{(1)}\,\tilde{G}^0\,\tilde{\Sigma}^{(1)}\,\tilde{G}^0\,.
\end{aligned}
$$

(c) To get some idea about what's going on, first look at the inverses of the above explicit expressions. Solving for $\tilde{\Sigma}^{(1)}$, $\tilde{\Sigma}^{(2)}$, etc., gives

$$
\begin{aligned}
-\tilde{\Sigma}^{(1)} &= (\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}\,,\\
-\tilde{\Sigma}^{(2)} &= (\tilde{G}^0)^{-1}\,\tilde{G}^{(2)}\,(\tilde{G}^0)^{-1} - (\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}\,,\\
-\tilde{\Sigma}^{(3)} &= (\tilde{G}^0)^{-1}\,\tilde{G}^{(3)}\,(\tilde{G}^0)^{-1} - (\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}\,\tilde{G}^{(2)}\,(\tilde{G}^0)^{-1}\\
&\qquad - (\tilde{G}^0)^{-1}\,\tilde{G}^{(2)}\,(\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}\\
&\qquad + (\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}\,.
\end{aligned}
$$

Every diagram (beyond 0'th order) for the propagator $\tilde{G}(p)$ has two external lines which together contribute two powers of the free propagator, $\tilde{G}^0(p)^2$. The above result shows that if one simply omits these factors for the external lines when calculating diagrams which contribute to the first-order correction to the propagator, then the result is precisely (minus) the first-order self-energy $\tilde{\Sigma}^{(1)}$. Now consider second-order diagrams for the propagator. Some diagrams, such as  (in a ϕ^4 theory) consist of a single "blob" in between the two external lines. In contrast, the diagram  has two "blobs" connected by a single line. Momentum conservation forces the line connecting the two blobs to carry exactly the same momentum as the external lines, and therefore it simply contributes a factor $\tilde{G}^0(k)$ (where k is the external momentum). Each of the individual blobs represents exactly the same contribution which appeared in the first-order self-energy. Consequently, the entire contribution from this diagram, omitting the external lines, is $(-\tilde{\Sigma}^{(1)}(k))\,\tilde{G}^0(k)(-\tilde{\Sigma}^{(1)}(k)) = (\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}$. Therefore, the contribution of this diagram to the $(\tilde{G}^0)^{-1}\,\tilde{G}^{(2)}\,(\tilde{G}^0)^{-1}$ term in the expression for $\tilde{\Sigma}^{(2)}$ is exactly cancelled by the second term subtracting $(\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}\,\tilde{G}^{(1)}\,(\tilde{G}^0)^{-1}$. So if one calculates the second-order propagator diagrams which do *not* contain two "blobs" connected by a single line, and omits the free propagator factors corresponding to the external lines, then the result is precisely (minus) the second-order self-energy $\tilde{\Sigma}^{(2)}(k)$. Similarly, the terms appearing in the expression for the third-order self-energy involving products of lower-order propagator corrections serve to exactly cancel the contribution from any third-order diagram which consists of multiple "blobs" connected by single lines. Therefore, (minus) the third order self-energy is obtained by omitting the propagators representing external lines in third-order propagator diagrams which contain only one "blob".

This is a general result. To see why, it is easiest to work backwards. Assume that the perturbative expansion of $-\tilde{\Sigma}(k)$ is given by the sum of all propagator diagrams with only one "blob" in which the free propagators for external lines are omitted. Then the n-th term in the expansion of the propagator in powers of the self-energy, $\tilde{G}(k) = \tilde{G}^0(k)\sum_{n=0}^{\infty}\left[-\tilde{\Sigma}(k)\,\tilde{G}^0(k)\right]^n$, is precisely the sum of all propagator diagrams consisting of n "blobs" connected by single lines. Consequently, every propagator diagram appears once (and only once) in this expansion, and makes the correct contribution to the full propagator. So the suggested diagrammatic interpretation of the self-energy must be correct.

(d) "Amputating" a diagram is defined to mean omitting the factors of free propagators for *external* lines, or equivalently multiplying by the inverse free propagator for each external line, so $\tilde{G}_{\rm amp}(p_1 \cdots p_n) \equiv \left[\prod_{j=1}^{n}(p_j^2 + m^2)\right] \tilde{G}(p_1 \cdots p_n)$. In coordinate space, this is a derivative operation, $G_{\rm amp}(x_1 \cdots x_n) \equiv \left[\prod_{j=1}^{n}(-\partial_j^2 + m^2)\right] G(x_1 \cdots x_n)$, where ∂_j is the gradient with respect to coordinate x_j.

"One-particle irreducible" diagrams (commonly abbreviated as 1PI) are diagrams which remain connected when any single line is cut. Any propagator diagram consisting of two (or more) "blobs" connected by a single line is "one-particle reducible" since it may be separated into two disconnected pieces by cutting the single line between blobs. Therefore, the result of part (c) may be stated more formally as "the perturbative expansion of (minus) the self-energy is given by the sum of all two-point amputated one-particle irreducible diagrams". Nothing in this discussion depends on the particular form of the interaction.